HOMOTOPY PRODUCTS FOR H-SPACES

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1. INTRODUCTION

In this note we consider two products in the generalized homotopy groups of an H-space. The first is a commutator or generalized Samelson product. It assigns to each $\alpha \in \pi(A, Y)$ and each $\beta \in \pi(B, Y)$ an element $\langle\alpha, \beta\rangle \in \pi(A \# B, Y)$, where A and B are polyhedra and Y is an H-space (definitions and notation are presented below). The definition is given by means of a commutator, and thus this product is closely related to the homotopy-commutativity of Y. Proposition 6 asserts that if $A = B = Y$ and $\alpha = \beta = \iota$, where ι denotes the homotopy class of the identity map, then $\langle\alpha, \beta\rangle$ is trivial if and only if Y is homotopy-commutative. From this we obtain Stasheff's axial theorem [10, Theorem 1.10], which gives necessary and sufficient conditions for a loop space to be homotopy-commutative. The second product that we consider is the associator product. It assigns to α and β as above and $\gamma \in \pi(C, Y)$ an element $\langle\alpha, \beta, \gamma\rangle \in \pi(A \# B \# C, Y)$, where A, B, and C are polyhedra and Y is an H-space. This product is defined by means of an associator and is related to the homotopy-associativity of Y. In fact, if $A = B = C = Y$ and $\alpha = \beta = \gamma = \iota$, $\langle\alpha, \beta, \gamma\rangle$ vanishes if and only if Y is homotopy-associative (Proposition 10). We show that if A, B, and C are suspensions, the commutator and associator products are multiplicative in each variable. Thus if A, B, and C are spheres, these products provide homomorphisms on homotopy groups,

$$\pi_p(Y) \otimes \pi_q(Y) \rightarrow \pi_{p+q}(Y) \quad \text{and} \quad \pi_p(Y) \otimes \pi_q(Y) \otimes \pi_r(Y) \rightarrow \pi_{p+q+r}(Y).$$

In Proposition 12, the primary obstruction (in the ordinary cohomological sense) to the homotopy-commutativity of Y and to the homotopy-associativity of Y is computed. The preceding homomorphisms give cohomology coefficient homomorphisms which enter into the computation of these obstruction elements.

For the case where A, B, and C are spheres, the commutator and associator products seem to be similar to James' obstructions to homotopy-commutativity and to homotopy-associativity, respectively, both of which are defined as separation elements [6].

2. PRELIMINARIES

We consider only path-connected topological spaces with base points. The base point is generically denoted by $*$. All maps and homotopies will respect base points. If R and S are spaces, $\pi(R, S)$ denotes the collection of homotopy classes of maps from R to S. If $f, g: R \rightarrow S$ are maps, $f \simeq g$ means that f is homotopic to g. The constant map from R to S (mapping all of R onto $* \in S$) is written $*: R \rightarrow S$. If $f \simeq *$, we say that f is *nullhomotopic*. The homotopy class of a map $g: R \rightarrow S$ is written $\{g\} \in \pi(R, S)$. Maps $h: R \rightarrow R'$ and $k: S \rightarrow S'$ induce transformations $h^*: \pi(R', S) \rightarrow \pi(R, S)$ for all S and $k_*: \pi(R, S) \rightarrow \pi(R, S')$ for all R in the obvious way.

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Next we introduce notation for certain functors. For more details see [1], [4], or [10]. We let Σ denote the (reduced) suspension functor, and we let Ω denote the loop space functor. These functors can be iterated so that, for example, $\Sigma^2 R$ is $\Sigma(\Sigma R)$. We shall also consider the cartesian product (written $\times$) and the wedge (written $\vee$). Recall that

$$R \vee S = R \times * \cup * \times S \subset R \times S.$$

If A is a subspace of X that contains the base point, then X/A denotes the (quotient) space obtained from X by identifying A to the base point. In particular, we set $R \# S = R \times S / R \vee S$. Another useful quotient space is $R \circ S$, the join of R and S. This is the space obtained from $R \times S \times I$ (I is the closed unit interval [0, 1]) by "factoring out" the relations

(1) $(r, s, 0) \sim (r, s', 0)$ for all $s, s' \in S$,

(2) $(r, s, 1) \sim (r', s, 1)$ for all $r, r' \in R$, and

(3) $(*, *, t) \sim *$ for all $t \in I$.

Next let Y be a space, let $j: Y \vee Y \to Y \times Y$ be the inclusion map, and let $\nabla: Y \vee Y \to Y$ be the folding map $(\nabla(y, *) = y = \nabla(*, y))$. We call Y an *H-space* if there is an $m: Y \times Y \to Y$ such that $mj \simeq \nabla: Y \vee Y \to Y$. The map m is called the *multiplication* in Y, and m(x, y) is written $x \cdot y$ or xy, where $x, y \in Y$. An H-space is *homotopy-commutative* if the maps $x, y \to xy$ and $x, y \to yx$ from $Y \times Y$ to Y are homotopic. It is *homotopy-associative* if $x, y, z \to x(yz)$ and $x, y, z \to (xy)z$ from $Y \times Y \times Y$ to Y are homotopic. A loop space with the usual multiplication of loops is an example of a homotopy-associative H-space. If $f, g: R \to Y$, the multiplication in Y induces a product $f \cdot g$ or $fg: R \to Y$. Thus if Y is an H-space, there exists a multiplication in $\pi(R, Y)$ which has $\{*\}$ as unit. It is also possible to multiply maps $f, g: \Sigma R \to S$ to get $f \cdot g$ or $fg: \Sigma R \to S$, where R and S are any spaces. This multiplication induces group structure in $\pi(\Sigma R, S)$. If Y is an H-space, the two multiplications in $\pi(\Sigma R, Y)$ coincide.

In a CW-complex [12] a 0-cell will be the base point. A space that is both a CW-complex and an H-space is called an *H-complex*. If R and S are CW-complexes, then $R \times S$ is not necessarily a CW-complex. However, it is known that if R and S are countable or if R or S is locally finite, then $R \times S$ is a CW-complex. We shall call a single CW-complex R *productive*, if $R \times R$ is a CW-complex.

3. LEMMAS FOR H-SPACES

First we recall the definition of a loop. A set π with a binary operation, written multiplicatively, is called a *loop* if:

(1) There exists a two-sided identity, denoted by e.

(2) The equations $x \cdot a = b$ and $a \cdot y = b$, where $a, b \in \pi$, admit a unique pair of solutions $x, y \in \pi$.

In particular, every element of π has a unique left inverse and a unique right inverse.

LEMMA 1 (James [6]). *If* Y *is an* H-*space and* K *is a* CW-*complex, then* $\pi(K, Y)$ *is a loop with multiplication induced by the multiplication in* Y *and with unit* $e = \{*\}$.

The left inverse of $\alpha \in \pi(K, Y)$ is denoted by $L(\alpha)$ and the right inverse of α is denoted by $R(\alpha)$.

LEMMA 2. *If* Y *is an* H-*complex, then there exist a space* S *and a map* $r: Y \to \Omega S$ *such that* $r_*: \pi(K, Y) \to \pi(K, \Omega S)$ *is one-to-one for all* CW-*complexes* K.

Remark. In [5] James has shown that if Y is a countable H-complex, then the inclusion $Y \to \Omega\Sigma Y$ is a retract.

We sketch the proof of Lemma 2. From the work of Dold and Lashof [2] (with modifications indicated in [10, p. 739]) there exist spaces and maps $Y \xrightarrow{i} E \xrightarrow{p} S$ such that (1) i is an inclusion map, (2) $pi = *$, (3) $i \simeq *$, and (4) $p_*: \pi_j(E, Y) \to \pi_j(S)$ is an isomorphism for all j. If $k_t: Y \to E$ is a nullhomotopy of i, define $r: Y \to \Omega S$ by $r(y)(t) = pk_t(y)$. The remainder of the proof is an adaptation of an argument due to Sugawara [11, Proposition 1]. We let $\pi_1(K; E, Y)$ denote the homotopy classes of maps of the pair CK, K into the pair E, Y, where CK denotes the (reduced) cone over K. We consider the diagram

$$\begin{array}{ccc} \pi_1(K; E, Y) & \xrightarrow{\partial} & \pi(K, Y) \\ \downarrow p_* & & \downarrow r_* \\ \pi(\Sigma K, S) & \xleftarrow{\kappa_*} & \pi(K, \Omega S), \end{array}$$

where κ_* is the canonical isomorphism [4, Section 2], ∂ is the transformation obtained by restriction, and p_* is the transformation induced by $p: E, Y \to S, *$. It follows from (2), (4), and the fact that K is a CW-complex that p_* is an isomorphism. But it is easy to verify that

$$\partial p_*^{-1} \kappa_* r_*(\alpha) = \alpha \qquad (\alpha \in \pi(K, Y)).$$

Thus $r_*: \pi(K, Y) \to \pi(K, \Omega S)$ is one-to-one.

Let R be any space, and let PR denote the geometric realization of the singular complex of R [7]. We call PR the *singular polyhedron* of R, and we note that PR is always a CW-complex. A mapping that induces isomorphisms of homotopy groups is called a *weak homotopy equivalence.* We recall that there exists a canonical mapping from PR to R which is a weak homotopy equivalence [7].

LEMMA 3. *If* Y *is an* H-*space, then* PY *is an* H-*space.*

Proof. Let $f: PY \to Y$ be the weak homotopy equivalence. We consider the element $\{m(f \times f)\} \in \pi(PY \times PY, Y)$ and the commutative diagram

$$\begin{array}{ccc} \pi(PY \times PY, PY) & \xrightarrow{j^*} & \pi(PY \vee PY, PY) \\ \downarrow f_* & & \downarrow f_* \\ \pi(PY \times PY, Y) & \xrightarrow{j^*} & \pi(PY \vee PY, Y). \end{array}$$

We observe that $PY \times PY$ and $PY \vee PY$ each have the homotopy type of a CW-complex [8]. Since f is a weak homotopy equivalence, it then follows that both of the maps f_* in the diagram are one-to-one and onto. Thus there exists a $\mu \in \pi(PY \times PY, PY)$ with $f_*(\mu) = \{m(f \times f)\}$. But

$$f_*j^*(\mu) = j^*f_*(\mu) = j^*\{m(f \times f)\} = \{\nabla_Y(f \vee f)\} = f_*\{\nabla_{PY}\},$$

where ∇_Y and ∇_{PY} are the folding maps of Y and PY, respectively. Therefore, $j^*(\mu) = \{\nabla_{PY}\}$, and so μ determines a multiplication in PY.

4. THE COMMUTATOR PRODUCT

We now define the commutator product. We assume that A, B, and $A \times B$ are CW-complexes. We let Y be an H-space, and we choose an $\alpha \in \pi(A, Y)$ and a $\beta \in \pi(B, Y)$. These determine elements $p_A^*(\alpha)$, $p_B^*(\beta) \in \pi(A \times B, Y)$, where $p_A\colon A \times B \to A$ and $p_B\colon A \times B \to B$ are the projections. By Lemma 1, there exists a commutator

$$(p_A^*(\alpha), p_B^*(\beta)) = L(p_B^*(\beta) \cdot p_A^*(\alpha)) \cdot (p_A^*(\alpha) \cdot p_B^*(\beta)) \quad \text{in} \quad \pi(A \times B, Y),$$

where L denotes left inverse. If $j\colon A \vee B \to A \times B$ is the inclusion, then $j^*(p_A^*(\alpha), p_B^*(\beta)) = e$, where e denotes the unit in $\pi(A \vee B, Y)$. To see this, consider the isomorphism

$$\theta\colon \pi(A \vee B, Y) \xrightarrow{\approx} \pi(A, Y) \oplus \pi(B, Y)$$

given by $\theta(\gamma) = i_A^*(\gamma) \oplus i_B^*(\gamma)$, where $i_A\colon A \to A \vee B$ and $i_B\colon B \to A \vee B$ are the injections and the symbol $\oplus$ denotes cartesian product. Clearly,

$$\theta j^*(p_A^*(\alpha), p_B^*(\beta)) = i_A^* j^*(p_A^*(\alpha), p_B^*(\beta)) \oplus i_B^* j^*(p_A^*(\alpha), p_B^*(\beta)).$$

However, $i_A^* j^*(p_A^*(\alpha), p_B^*(\beta))$ equals (α, e), the commutator of α and e in $\pi(A, Y)$. Since $(\alpha, e) = e$, $i_A^* j^*(p_A^*(\alpha), p_B^*(\beta)) = e$. Similarly, $i_B^* j^*(p_A^*(\alpha), p_B^*(\beta)) = e$. Thus,

$$\theta j^*(p_A^*(\alpha), p_B^*(\beta)) = e \oplus e,$$

and so $j^*(p_A^*(\alpha), p_B^*(\beta)) = e$, as asserted. Since $A \vee B$ is a subcomplex of $A \times B$, there exists an exact sequence

$$\pi(A \# B, Y) \xrightarrow{q^*} \pi(A \times B, Y) \xrightarrow{j^*} \pi(A \vee B, Y)$$

(see [4, Section 4] or [3]), where $q\colon A \times B \to A \# B$ is the projection. Hence there exists an element, written $\langle\alpha, \beta\rangle$, in $\pi(A \# B, Y)$ such that $q^*\langle\alpha, \beta\rangle$ is the commutator $(p_A^*(\alpha), p_B^*(\beta))$. By the following proposition this element is unique. (Compare with Lemma (2.1) of [1].)

PROPOSITION 4. *If* A, B, *and* $A \times B$ *are* CW-*complexes and* Y *is an* H-*space, then* $q^*\colon \pi(A \# B, Y) \to \pi(A \times B, Y)$ *is one-to-one.*

Proof. Consider first the case where Y is a loop space, $Y = \Omega X$. By using the natural isomorphism $\pi(\Sigma R, S) \approx \pi(R, \Omega S)$ which is valid for all spaces R and S, we see that it suffices to show that

$$(\Sigma q)^*\colon \pi(\Sigma(A \# B), X) \to \pi(\Sigma(A \times B), X)$$

is one-to-one. Let $A \circ B$ denote the join of A and B (defined in Section 2), and let $\lambda\colon A \circ B \to \Sigma(A \times B)$ be the map that identifies all points (a, b, 0) or (a, b, 1) in $A \circ B$ to the base point $*$ $(a \in A,\ b \in B)$. The composition $\Sigma q \lambda\colon A \circ B \to \Sigma(A \# B)$ identifies all points $(a, *, t)$ and $(*, b, u)$ to $*$, where $t, u \in I$. These latter points form a subcomplex of $A \circ B$ that consists of two cones joined at a single point, and hence is contractible. Thus $\Sigma q \lambda$ is a homotopy equivalence [12, p. 238]. Therefore Σq has a right homotopy inverse, and so $(\Sigma q)^*$ is one-to-one. This proves the proposition in the case where Y is a loop space. For any H-space Y, the singular polyhedron PY is an H-space by Lemma 3. Therefore, by Lemma 2, there exists an S and an $r: PY \to \Omega S$ such that r_* is one-to-one. Now consider the commutative diagram

$$\begin{array}{ccccc} \pi(A \# B, Y) & \xleftarrow[\approx]{f_*} & \pi(A \# B, PY) & \xrightarrow{r_*} & \pi(A \# B, \Omega S) \\ \downarrow q^* & & \downarrow q^* & & \downarrow q^* \\ \pi(A \times B, Y) & \xleftarrow[\approx]{f_*} & \pi(A \times B, PY) & \xrightarrow{r_*} & \pi(A \times B, \Omega S), \end{array}$$

where $f: PY \to Y$ is the weak homotopy equivalence. We have seen that q^* on the right is one-to-one, and so it follows that q^* on the left is one-to-one.

Definition 5. The *commutator product* of $\alpha \in \pi(A, Y)$ and $\beta \in \pi(B, Y)$ is the element $<\alpha, \beta> \in \pi(A \# B, Y)$ uniquely defined by

$$q^*<\alpha, \beta> = (p_A^*(\alpha), p_B^*(\beta)).$$

This definition is made under the assumption that (1) Y is an H-space and (2) A, B, and $A \times B$ are CW-complexes. It is clear that we can define a commutator product $<\alpha, \beta> \in \pi(A \# B, Y)$ of $\alpha \in \pi(A, Y)$ and $\beta \in \pi(B, Y)$ under the following slightly more general conditions: (1) Y is an H-space (2) A and B are spaces having the same homotopy type as A' and B', respectively, where A', B', and $A' \times B'$ are CW-complexes. We shall consider the commutator product in this more general form.

When A and B are spheres and Y is homotopy-associative, then $<\alpha, \beta>$ is the Samelson product [9].

If Y is homotopy-commutative, then the commutator $(p_A^*(\alpha), p_B^*(\beta)) = e$. Thus $<\alpha, \beta> = e$ in $\pi(A \# B, Y)$ for all α and β.

PROPOSITION 6. *Let* Y *have the homotopy type of a productive* CW-*complex* (*that is, a* CW-*complex whose cartesian product with itself is a* CW-*complex*) *and let* $\iota \in \pi(Y, Y)$ *be the homotopy class of the identity map* ($\iota = \{1\}$). *Then* $<\iota, \iota> = e$ *if and only if* Y *is homotopy-commutative.*

The proof follows from the definition and the fact that Y is homotopy-commutative if and only if the commutator $(\{p_1\}, \{p_2\})$ is e, where p_1 and p_2 are the two projections of $Y \times Y$ onto Y.

Next we relate the generalized Whitehead product of [1] to the commutator product. Recall that the generalized Whitehead product assigns to $\overline{\alpha} \in \pi(\Sigma A, X)$ and $\overline{\beta} \in \pi(\Sigma B, X)$ an element $[\overline{\alpha}, \overline{\beta}] \in \pi(\Sigma(A \# B), X)$, where A, B, and $A \times B$ are CW-complexes and X is any space. If $K_*: \pi(R, \Omega S) \to \pi(\Sigma R, S)$ denotes the canonical isomorphism [4, Section 2], then it easily follows that $K_*<\alpha, \beta> = [K_*(\alpha), K_*(\beta)]$. If the loop space ΩX has the same homotopy type as a productive CW-complex, then, by Proposition 6, ΩX is homotopy-commutative if and only if $[K_*(\iota), K_*(\iota)] = e$, where $\iota = \{1\} \in \pi(\Omega X, \Omega X)$. The following proposition, which is well known in the case where A and B are spheres, was proved in [1, Proposition 5.1] by an elementary argument: For any $\overline{\alpha} \in \pi(\Sigma A, X)$ and $\overline{\beta} \in \pi(\Sigma B, X)$, $[\overline{\alpha}, \overline{\beta}] = e$ if and only if there exists a map $\ell: \Sigma A \times \Sigma B \to X$ such that $\{\ell | \Sigma A\} = \overline{\alpha}$ and $\{\ell | \Sigma B\} = \overline{\beta}$. (Here $\ell | \Sigma A$ denotes the map obtained by restricting ℓ to $\Sigma A = \Sigma A \times * \subset \Sigma A \times \Sigma B$.) Thus a necessary and sufficient condition for ΩX to be homotopy-commutative is that there exist an $\ell: \Sigma\Omega X \times \Sigma\Omega X \to X$ such that ℓ on each factor represents $K_*(\iota) \in \pi(\Sigma\Omega X, X)$. But it is easy to see that the map $d: \Sigma\Omega X \to X$ defined by $d(\omega, t) = \omega(t)$ ($\omega \in \Omega X$, $t \in I$) has $K_*(\iota)$ as its homotopy class. Hence we have deduced Stasheff's theorem:

PROPOSITION 7 [10, Theorem 1.10]. *If* ΩX *has the same homotopy type as a productive* CW-*complex, then* ΩX *is homotopy-commutative if and only if there exists a map* $\ell: \Sigma\Omega X \times \Sigma\Omega X \to X$ *whose restriction to each factor is homotopic to* d.

We note that Stasheff proves his result under the hypothesis that X has the homotopy type of a countable CW-complex. It is known [8] that this hypothesis implies that ΩX has the homotopy type of a productive CW-complex.

5. THE ASSOCIATOR PRODUCT

Next we consider the associator product. Here, unless it is otherwise stated, we shall assume that Y is an H-complex such that $y \cdot * = y = * \cdot y$ for all $y \in Y$. If Y is a productive H-complex, this last property follows from the definition of an H-space. The homotopy extension property of the pair $Y \times Y$, $Y \vee Y$ enables one to replace the multiplication by a homotopic multiplication that has the desired property.

By Lemma 1, there are maps $\lambda, \rho: Y \to Y$ such that $\{\lambda\} \cdot \iota = e$ and $\iota \cdot \{\rho\} = e$, where $\iota = \{1\}$ and $e = \{*\}$ in $\pi(Y, Y)$. Now let

$$\{f\} = \alpha \in \pi(A, Y), \quad \{g\} = \beta \in \pi(B, Y), \quad \text{and } \{h\} = \gamma \in \pi(C, Y),$$

where A, B, and C are CW-complexes such that $A \times B \times C$ is a CW-complex. If

$$p_1: A \times B \times C \to A, \quad p_2: A \times B \times C \to B, \quad \text{and} \quad p_3: A \times B \times C \to C$$

are the projections, we obtain maps $fp_1 = f'$, $gp_2 = g'$, $hp_3 = h': A \times B \times C \to Y$. We consider the associator

$$a = (\lambda \circ (f'g')h') \cdot (f'(g'h')): A \times B \times C \to Y$$

and note that

$$a | A \times B \times * \simeq *, \quad a | A \times * \times C \simeq *, \quad \text{and} \quad a | * \times B \times C \simeq *.$$

Moreover, since each of these nullhomotopies comes from the nullhomotopy $\lambda \cdot 1 \simeq *: Y \to Y$, they are all compatible. Thus we get a nullhomotopy

$$a | T \simeq *: T \to Y,$$

where

$$T = A \times B \times * \cup A \times * \times C \cup * \times B \times C \subset A \times B \times C.$$

If $j: T \to A \times B \times C$ is the inclusion, $A \# B \# C$ the quotient space $A \times B \times C/T$, and $p: A \times B \times C \to A \# B \# C$ the projection, then there exists an exact sequence ([4, Section 4] or [3])

$$\pi(A \# B \# C, Y) \xrightarrow{p^*} \pi(A \times B \times C, Y) \xrightarrow{j^*} \pi(T, Y).$$

Since $j^*\{a\} = e$ there exists an element $<\alpha, \beta, \gamma>$ in $\pi(A \# B \# C, Y)$ such that $p^*<\alpha, \beta, \gamma> = \{a\}$. The following proposition shows that $<\alpha, \beta, \gamma>$ is unique and is independent of the representatives f, g, and h of α, β, and γ.

PROPOSITION 8. *If* Y *is any* H-*space and* A, B, C, *and* $A \times B \times C$ *are* CW-*complexes, then*

$$p^*: \pi(A \# B \# C, Y) \to \pi(A \times B \times C, Y)$$

is one-to-one.

Proof. First, let $Y = \Omega X$. In this case it suffices to prove that

$$(\Sigma p)^*: \pi(\Sigma(A \# B \# C), X) \to \pi(\Sigma(A \times B \times C), X)$$

is one-to-one. Set

$$V = \Sigma A \vee \Sigma B \vee \Sigma C \vee \Sigma(A \# B) \vee \Sigma(A \# C) \vee \Sigma(B \# C) \vee \Sigma(A \# B \# C)$$

and let p_{12}, p_{13}, and p_{23} denote the projections of $A \times B \times C$ onto $A \# B$, $A \# C$, and $B \# C$, respectively. It is proved in [4, p. 301], by a homological argument, that the map

$$\sigma = \overline{\Sigma p_1} \cdot \overline{\Sigma p_2} \cdot \overline{\Sigma p_3} \cdot \overline{\Sigma p_{12}} \cdot \overline{\Sigma p_{13}} \cdot \overline{\Sigma p_{23}} \cdot \overline{\Sigma p}: \Sigma(A \times B \times C) \to V$$

is a homotopy equivalence, where the multiplication is obtained from $\Sigma(A \times B \times C)$ and the bar over a map signifies the map followed by its inclusion into V. Let η be a homotopy inverse of σ. Then, with $i: \Sigma(A \# B \# C) \to V$ denoting the inclusion and $r: V \to \Sigma(A \# B \# C)$ denoting the projection,

$$r\sigma \simeq (r\overline{\Sigma p_1}) \cdot (r\overline{\Sigma p_2}) \cdot (r\overline{\Sigma p_3}) \cdot (r\overline{\Sigma p_{12}}) \cdot (r\overline{\Sigma p_{13}}) \cdot (r\overline{\Sigma p_{23}}) \cdot (r\overline{\Sigma p}) \simeq \Sigma p.$$

Thus $(\Sigma p)\eta i \simeq r\sigma\eta i \simeq ri$, and this is 1, the identity of $\Sigma(A \# B \# C)$. Therefore $(\eta i)^* (\Sigma p)^* = 1$, and so $(\Sigma p)^*$ is one-to-one. This proves the proposition if Y is a loop space. The general case now follows as in Proposition 4.

Definition 9. The *associator product* of

$$\{f\} = \alpha \in \pi(A, Y), \quad \{g\} = \beta \in \pi(B, Y), \quad \text{and} \quad \{h\} = \gamma \in \pi(C, Y)$$

is the unique element $<\alpha, \beta, \gamma>$ in $\pi(A \# B \# C, Y)$ such that $q^*<\alpha, \beta, \gamma> = \{a\}$, where a is the associator of fp_1, gp_2, and hp_3.

Clearly, if Y is homotopy-associative, then $<\alpha, \beta, \gamma> = e$ for all α, β, and γ.

PROPOSITION 10. Y *is homotopy-associative if and only if* $<\iota, \iota, \iota> = e$ *in* $\pi(Y \# Y \# Y, Y)$.

Here it is assumed that Y is an H-complex such that (1) $y \cdot * = y = * \cdot y$ and (2) $Y \times Y \times Y$ is a CW-complex. We omit the proof.

6. AN APPLICATION

First we show that the commutator and associator products are multiplicative in each variable.

PROPOSITION 11. *Let* A, B, *and* C *be suspensions, and let* $\alpha, \hat{\alpha} \in \pi(A, Y)$, $\beta, \hat{\beta} \in \pi(B, Y)$, *and* $\gamma, \hat{\gamma} \in \pi(C, Y)$. *Then*

(i) $<\alpha\hat{\alpha}, \beta> = <\alpha, \beta> \cdot <\hat{\alpha}, \beta>$, $<\alpha, \beta\hat{\beta}> = <\alpha, \beta> \cdot <\alpha, \hat{\beta}>$,

(ii) $<\alpha\hat{\alpha}, \beta, \gamma> = <\alpha, \beta, \gamma> \cdot <\hat{\alpha}, \beta, \gamma>$, *and so forth.*

We sketch the proof of the first part of (i). By naturality, it suffices to prove that

$$<\iota, \iota> \circ (\alpha\hat{\alpha} \# \beta) = (<\iota, \iota> \circ (\alpha \# \beta)) \cdot (<\iota, \iota> \circ (\hat{\alpha} \# \beta)).$$

Since $A = \Sigma\bar{A}$ and $B = \Sigma\bar{B}$, there exists a homeomorphism $\theta: \Sigma^2(\bar{A} \# \bar{B}) \to \Sigma\bar{A} \# \Sigma\bar{B}$. It is easily seen that

$$(\alpha \# \beta)\theta \cdot (\hat{\alpha} \# \beta)\theta = ((\alpha\hat{\alpha}) \# \beta)\theta,$$

and thus

$$<\iota, \iota> \circ (\alpha\hat{\alpha} \# \beta) \circ \theta = ((<\iota, \iota> \circ (\alpha \# \beta)) \cdot (<\iota, \iota> \circ (\hat{\alpha} \# \beta))) \circ \theta.$$

The other parts of the proposition are proved similarly.

Thus if A, B, and C are spheres, $A = S^p$, $B = S^q$, and $C = S^r$ $(p, q, r \geq 1)$, then the commutator product and the associator product provide homomorphisms

$$c: \pi_p(Y) \otimes \pi_q(Y) \to \pi_{p+q}(Y) \quad \text{and} \quad a: \pi_p(Y) \otimes \pi_q(Y) \otimes \pi_r(Y) \to \pi_{p+q+r}(Y).$$

If Y is a productive H-complex, then Y is homotopy-commutative if and only if $\phi \simeq *: Y \# Y \to Y$, where ϕ is some map in the class $<\iota, \iota>$. Similarly, Y is homotopy-associative if and only if $\psi \simeq *: Y \# Y \# Y \to Y$, where ψ is a map in the class $<\iota, \iota, \iota>$. If Y is an $(n-1)$-connected H-complex $(n \geq 2)$, then it is easily seen that $Y \# Y$ is $(2n-1)$-connected and $Y \# Y \# Y$ is $(3n-1)$-connected. Thus there exists a primary obstruction to a nullhomotopy of ϕ, $o_\phi \in H^{2n}(Y \# Y; \pi_{2n}(Y))$, and a primary obstruction to a nullhomotopy of ψ, $o_\psi \in H^{3n}(Y \# Y \# Y; \pi_{3n}(Y))$. The element o_ϕ (respectively, o_ψ) can be regarded as the primary obstruction to the homotopy-commutativity (respectively, the homotopy-associativity) of Y. Let $b \in H^n(Y, *; \pi_n(Y))$ denote the basic class, let

$$q: Y \times Y, Y \vee Y \to Y \# Y, * \quad \text{and} \quad p: Y \times Y \times Y, T \to Y \# Y \# Y, *$$

denote the projections, and let $\times$ denote the cohomology cross product.

PROPOSITION 12. (i) $q^{*-1} c_*(b \times b) = o_\phi$,

(ii) $p^{*-1} a_*(b \times b \times b) = o_\psi$,

where

$$c_*: H^{2n}(Y \times Y, Y \vee Y; \pi_n(Y) \otimes \pi_n(Y)) \to H^{2n}(Y \times Y, Y \vee Y; \pi_{2n}(Y)) \quad \textit{and}$$

$$a_*: H^{3n}(Y \times Y \times Y, T; \pi_n(Y) \otimes \pi_n(Y) \otimes \pi_n(Y)) \to H^{3n}(Y \times Y \times Y, T; \pi_{3n}(Y))$$

are induced by the coefficient homomorphisms c *and* a *mentioned above.*

The proof is a consequence of elementary obstruction theory, several commutativity relations, and the definitions.

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