METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 13, No. 4, pp. 387–410, December 2006

SEMICLASSICAL FORMS OF CLASS $s = 2$: THE SYMMETRIC CASE, WHEN $\Phi(0) = 0$*

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Abstract. A regular linear form u is said to be semiclassical, if there exist two polynomials Φ monic and Ψ, $\deg(\Psi) \geq 1$ such that $(\Phi u)' + \Psi u = 0$. Recently, all the symmetric semiclassical linear forms of class $s \leq 1$ are determined. In this paper, by considering the inverse problem of the product of a form by a polynomial in the square case, we carry out the complete description of the symmetric semiclassical linear forms of class $s = 2$, when $\Phi(0) = 0$ which generalize those of class $s = 1$. Essentially, three canonical cases appear. Some particular cases refer to well-known orthogonal sequences. Representations of these linear forms are given.

Key words. Orthogonal polynomials, symmetric forms, semiclassical forms, integral representations.

AMS subject classifications. 42C05, 33C45

Introduction. Semiclassical orthogonal polynomials were introduced in [23]. They are natural generalization of the classical polynomials. Maroni [19,21] has worked on the linear form of moments and has given a unified theory of this kind of polynomials. A semiclassical linear form u satisfies the distributional equation $(\Phi u)' + \psi u = 0$ where $\Phi(x)$ is a monic polynomial and $\Psi(x)$ is a polynomial with $\deg(\Psi) \geq 1$. In [2], the authors determine all the symmetric semiclassical linear forms of class $s = 1$. See also [3,4] for some special cases. It is natural to consider the problem of determining all the symmetric semiclassical linear forms of class $s = 2$. In this paper, we are interested in the case when $\Phi(0) = 0$. For this, we consider the inverse problem of the product of a linear form by a polynomial by studying the following problem: given a symmetric semiclassical linear form v, find the symmetric linear form u defined by $x^2 u = -\lambda v \Leftrightarrow u = -\lambda x^{-2} v + \delta_0$, $\lambda \in \mathbb{C}^*$ in a different way than [17,1]. This kind of problem is an interesting process to construct certain families of semiclassical polynomials as treated in many recent works ([1], [5], [16], [17], [22]). The first section is devoted to the preliminary results and notations used in the sequel. In the second section, We found a relation between the symmetric semiclassical linear forms of class $s = 2$ and those of class $s \leq 1$ (Theorem 2.3.). Using this relation, we give, in Section 2, all the linear forms which we look for. Three canonical cases for the polynomial Φ arise: $\Phi(x) = x^2$, $\Phi(x) = x^4$ and $\Phi(x) = x^2(x^2 - 1)$. Representations of the new linear forms are obtained. As it turned out, we obtained explicitly three nonsymmetric semiclassical linear forms of class $s = 1$.

1. Notations and preliminary results. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle, n \geq 0$, the moments of u. For any linear form u and any polynomial h let $Du = u'$, hu, δ_0, and $x^{-1}u$ be the linear forms defined by: $\langle u', f \rangle := -\langle u, f' \rangle$, $\langle hu, f \rangle := \langle u, hf \rangle$, $\langle \delta_c, f \rangle := f(c)$, and $\big\langle (x-c)^{-1}u, f \big\rangle := \langle u, \theta_c f \rangle$ where $\big(\theta_c f\big)(x) = \dfrac{f(x) - f(c)}{x - c}$, $c \in \mathbb{C}$,

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$f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and for $u \in \mathcal{P}'$, we have

$$x(x^{-1}u) = u\ , \tag{1.1}$$

$$x^{-1}(xu) = u - (u)_0\delta_0\ , \tag{1.2}$$

$$x^{-2}(x^2u) = u - (u)_0\delta_0 + (u)_1\delta_0'\ , \tag{1.3}$$

$$(fu)' = f'u + fu'\ . \tag{1.4}$$

Let us define the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ by $(\sigma f)(x) := f(x^2)$. Then, we define the even part σu of u by $\langle \sigma u, f\rangle := \langle u, \sigma f\rangle$.
Therefore, we have [20]

$$f(x)(\sigma u) = \sigma(f(x^2)u)\ , \tag{1.5}$$

$$\sigma u' = 2(\sigma(xu))'\ . \tag{1.6}$$

The linear form v will be called regular if we can associate with it a sequence $\{S_n\}_{n\geq 0}$ $\big(\deg(S_n) \leq n\ \big)$ such that

$$\langle v, S_nS_m\rangle = r_n\delta_{n,m}\ , \quad n,m \geq 0\ , \quad r_n \neq 0\ , \quad n \geq 0\ .$$

Then $\deg(S_n) = n\ ,\ \ n \geq 0$, and we can always suppose each S_n monic (i.e. $S_n(x) = x^n + \cdots$). The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v.

A form v is regular if and only if $\triangle_n =: \det\Big((v)_{i+j}\Big)_{i,j=0}^{n} \neq 0\ ,\ \ n \geq 0$ $\Big($Hankel determinants$\Big)$ [8].

It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation $\Big($see, for instance, the monograph by Chihara [8]$\Big)$

$$\begin{cases} S_{n+2}(x) = (x - \xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x)\ , \quad n \geq 0\ , \\ S_1(x) = x - \xi_0\ , \qquad S_0(x) = 1\ , \end{cases} \tag{1.7}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C}^*\ ,\quad n \geq 0$, by convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation

$$\begin{cases} S_{n+2}^{(1)}(x) = (x - \xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x)\ , \quad n \geq 0, \\ S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1\ , \quad \big(S_{-1}^{(1)}(x) = 0\big)\ , \end{cases} \tag{1.8}$$

and $\{S_n(.,\mu)\}_{n\geq 0}$ the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [9]

$$S_{n+1}(x,\mu) = S_{n+1}(x) - \mu S_n^{(1)}(x)\ , \quad n \geq 0\ . \tag{1.9}$$

A linear form v is called symmetric if $(v)_{2n+1} = 0, n \geq 0$. The conditions $(v)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding sequence of orthogonal polynomials (OPS) $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (1.7) with $\xi_n = 0, n \geq 0$ [8]. The linear form $x^2 v$ is also symmetric.

In all this paper, except opposite mention, the linear form v will be supposed normalized, (i.e: $(v)_0 = 1$), symmetric, and regular.

Let us consider the quadratic decomposition of $\{S_n\}_{n\geq 0}$ and $\{S_n^{(1)}\}_{n\geq 0}$ (see [8], [20])

$$S_{2n}(x) = \tilde{P}_n(x^2)\ , \quad S_{2n+1}(x) = x\tilde{R}_n(x^2)\ , \tag{1.10}$$

$$S_{2n}^{(1)}(x) = \tilde{R}_n\left(x^2, -\rho_1\right)\ , \quad S_{2n+1}^{(1)}(x) = x\tilde{P}_n^{(1)}(x^2)\ . \tag{1.11}$$

The sequences $\{\tilde{P}_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ are respectively orthogonal with respect to σv and $x\sigma v$. We have for instance:

$$\begin{aligned} &\tilde{P}_{n+2}(x) = \left(x - \xi_{n+1}^{\tilde{P}}\right)\tilde{P}_{n+1}(x) - \rho_{n+1}^{\tilde{P}}\tilde{P}_n(x)\ , \quad n \geq 0\ , \\ &\tilde{P}_1(x) = x - \xi_0^{\tilde{P}}\ , \quad \tilde{P}_0(x) = 1\ , \end{aligned} \tag{1.12}$$

with

$$\xi_0^{\tilde{P}} = \rho_1\ , \quad \xi_{n+1}^{\tilde{P}} = \rho_{2n+2} + \rho_{2n+3}\ , \quad \rho_{n+1}^{\tilde{P}} = \rho_{2n+1}\rho_{2n+2}\ , \quad n \geq 0\ . \tag{1.13}$$

Proposition 1.1. [17] *We have*

$$S_{2n+1}(0) = 0\ , \quad S_{2n+2}(0) = (-1)^{n+1}\prod_{v=0}^{n}\rho_{2v+1}\ , \quad n \geq 0\ , \tag{1.14}$$

$$S_{2n+1}^{(1)}(0) = 0, \quad S_{2n}^{(1)}(0) = (-1)^n\prod_{v=0}^{n}\rho_{2v}\ , \quad n \geq 0\ . \tag{1.15}$$

Proposition 1.2.[8, 21] *v is regular if and only if σv and $x\sigma v$ are regular.*

The study of the linear form $u = -\lambda x^{-2}v + \delta_0\ ,\ \lambda \in \mathbb{C}^*$**.** For a $\lambda \in \mathbb{C}^*$, we can define a new linear form u as following:

$$u = -\lambda x^{-2}v + \delta_0\ . \tag{1.16}$$

From (1.16), and (1.1), we have

$$x^2 u = -\lambda v\ . \tag{1.17}$$

Remark. The above problem was treated by the second author and Maroni in [1, 17] and we are going to handle it differently using the quadratic decomposition to have new applications.

Proposition 1.3. *The functional u is regular if and only if $\tilde{P}_n(0,\lambda) \neq 0\ ,\ n \geq 0$, where $\tilde{P}_n$ is defined by* (1.10).

Proof. Applying the operator σ for (1.17), and using (1.5), we obtain

$$x\sigma u = -\lambda\sigma v\ . \tag{1.18}$$

From (1.18), and (1.2), we get

$$\sigma u = -\lambda x^{-1}\sigma v + \delta_0 \ . \tag{1.19}$$

From (1.16), we deduce that u is symmetric linear form. Then, according to Proposition 1.2. u is regular if and only if $x\sigma u$ and σu are regular. But $x\sigma u = -\lambda\sigma v$ is regular because $\lambda \neq 0$ and σv is regular. So u is regular if and only if $\sigma u = -\lambda x^{-1}\sigma v + \delta_0$ is regular.
Or, $\{\tilde{P}_n\}_{n\geq 0}$ is the corresponding orthogonal sequence to σv, and it was shown in [17] that $\sigma u = -\lambda x^{-1}\sigma v + \delta_0$ is regular if and only if $\lambda \neq 0$, and $\tilde{P}_n(0,\lambda) \neq 0$, $n \geq 0$. Then we deduce the desired result. □

Remark. In fact, using the well known identity (see [8], page 86)

$$\tilde{P}_{n+1}(0)\tilde{P}^{(1)}_{n+1}(0) - \tilde{P}_{n+2}(0)\tilde{P}^{(1)}_{n}(0) = \prod_{v=0}^{n}\rho^{\tilde{P}}_{v+1} \ , \quad n \geq 0 \ . \tag{1.20}$$

Dividing the above equation by $\tilde{P}_{n+2}(0)\tilde{P}_{n+1}(0)$, and using (1.10), (1.13), (1.14), we get

$$\frac{\tilde{P}^{(1)}_{n+1}(0)}{\tilde{P}_{n+2}(0)} - \frac{\tilde{P}^{(1)}_{n}(0)}{\tilde{P}_{n+1}(0)} = -\prod_{v=0}^{n+1}\frac{\rho_{2v}}{\rho_{2v+1}} \ , \quad n \geq 0 \ .$$

This leads to

$$\tilde{P}^{(1)}_{n}(0) = -\tilde{P}_{n+1}(0)\sum_{k=0}^{n}\prod_{v=0}^{k}\frac{\rho_{2v}}{\rho_{2v+1}} \ , \quad n \geq 0 \ . \tag{1.21}$$

Using (1.9), and (1.21) we can easily find the result given in [17] according to Proposition 1.3.. When the linear form v is symmetric, then u is regular if and only if $\lambda \neq \left(\sum_{k=0}^{n}\prod_{v=0}^{k}\frac{\rho_{2v}}{\rho_{2v+1}}\right)^{-1}$.

When u is regular let $\{Z_n\}_{n\geq 0}$ be its corresponding sequence of polynomials satisfying the recurrence relation

$$\begin{aligned} &Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1}Z_n(x) \ , \quad n \geq 0 \ , \\ &Z_1(x) = x \ , \quad Z_0(x) = 1 \ . \end{aligned} \tag{1.22}$$

Since $\{Z_n\}_{n\geq 0}$ is symmetric, let us consider its quadratic decomposition:

$$Z_{2n}(x) = P_n(x^2) \ , \quad Z_{2n+1}(x) = xR_n(x^2) \ . \tag{1.23}$$

From (1.18), we have

$$R_n(x) = \tilde{P}_n(x) \ , \quad n \geq 0 \ . \tag{1.24}$$

Remark. From (1.13), and (1.22), the sequence $\{P_n\}_{n\geq 0}$ satisfies the recurrence relation (1.12) with

$$\beta^P_0 = \gamma_1 \ , \quad \beta^P_{n+1} = \gamma_{2n+2} + \gamma_{2n+3} \ , \quad \gamma^P_{n+1} = \gamma_{2n+1}\gamma_{2n+2} \ . \tag{1.25}$$

From (1.19), we can deduce the following result

PROPOSITION 1.4.[22] *The sequence of polynomials* $\{P_n\}_{n\geq 0}$ *satisfy the relation*

$$P_{n+1}(x) = \tilde{P}_{n+1}(x) + \tilde{a}_n \tilde{P}_n(x) , \quad n \geq 0 \tag{1.26}$$

where

$$\tilde{a}_n = -\frac{\tilde{P}_{n+1}(0,\lambda)}{\tilde{P}_n(0,\lambda)} , \quad n \geq 0 . \tag{1.27}$$

LEMMA 1.5. *We have*

$$Z_{n+2}(x) = S_{n+2}(x) + a_n S_n(x) , \quad n \geq 0 \quad \text{with}$$

$$a_{2n} = \tilde{a}_n , \quad a_{2n+1} = \rho_{2n+2} , \quad n \geq 0 . \tag{1.28}$$

Proof. According to formula (1.16) of [1, p.14], we have
$a_{2n+1} = \rho_{2n+2}$, $n \geq 0$, and

$$Z_{2n+2}(x) = S_{2n+2}(x) + a_{2n} S_{2n}(x) , \quad n \geq 0 .$$

In (1.26) replace x by x^2 and compare the obtained equation by the above one, we obtain $a_{2n} = \tilde{a}_n$, $n \geq 0$, according to (1.23). □

PROPOSITION 1.6. *We may write*

$$\gamma_1 = -\lambda , \quad \gamma_{2n+2} = \tilde{a}_n , \quad \gamma_{2n+3} = \frac{\rho_{2n+1}\rho_{2n+2}}{\tilde{a}_n} , \quad n \geq 0 . \tag{1.29}$$

DEFINITION 1.7. (*see* [19],[21]) *A linear form v is called semiclassical when it is regular and there exist two polynomials $\tilde{\Phi}$ and $\tilde{\Psi}$ such that:*

$$\left(\tilde{\Phi} v\right)' + \tilde{\Psi} v = 0, \quad \deg(\tilde{\Psi}) \geq 1, \quad \tilde{\Phi} \text{ monic.} \tag{1.30}$$

PROPOSITION 1.8. [19] *The semiclassical linear form v verifying equation* (1.30) *is of class* $\tilde{s} = \max\big(\deg\tilde{\Psi} - 1, \deg\tilde{\Phi} - 2\big)$ *if and only if*

$$\prod_c \left(|\tilde{\Phi}'(c) + \tilde{\Psi}(c)| + \left| \left\langle u, \theta_c \tilde{\Psi} + \theta_c^2 \tilde{\Phi} \right\rangle \right| \right) > 0 , \tag{1.31}$$

where c goes over the roots set of $\tilde{\Phi}$.

The semiclassical character is kept by shifting. Indeed, the shifted linear form $\hat{v} = (h_{a^{-1}} o \tau_{-b}) v$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ satisfies

$$\left(\hat{\tilde{\Phi}} \hat{v}\right)' + \hat{\tilde{\Psi}} \hat{v} = 0 \tag{1.32}$$

with

$$\hat{\tilde{\Phi}}(x) = a^{-t} \tilde{\Phi}(ax+b) , \quad \hat{\tilde{\Psi}}(x) = a^{1-t} \tilde{\Psi}(ax+b) , \quad t = deg(\tilde{\Phi}) . \tag{1.33}$$

Where the linear forms $\tau_{-b} v$ (translation of v) and $h_a v$ (dilatation of u) are defined by

$$\langle \tau_b v, f \rangle := \langle v, \tau_{-b} f \rangle := \langle v, f(x+b) \rangle , \quad \langle h_a v, f \rangle := \langle v, h_a f \rangle := \langle v, f(ax) \rangle , \quad f \in \mathcal{P} .$$

The sequence $\{\hat{S}_n(x) = a^{-n}S_n(ax+b)\}_{n\geq 0}$ is orthogonal with respect to $\hat{v}$ and fulfills (1.7) with

$$\hat{\xi}_n = \frac{\xi_n - b}{a}\ , \quad \hat{\rho}_{n+1} = \frac{\rho_{n+1}}{a^2}\ , \quad n \geq 0\ . \tag{1.34}$$

In the sequel, the linear form v will be supposed symmetric semiclassical of class $\tilde{s}$ and satisfying (1.30).

From (1.17), and (1.30), it is clear that when the linear form u is regular it is also semiclassical and satisfies

$$(\Phi u)' + \Psi u = 0 \tag{1.35}$$

with

$$\Phi(x) = x^2\tilde{\Phi}(x)\ , \quad \Psi(x) = x^2\tilde{\Psi}(x)\ . \tag{1.36}$$

The class s of u is at most $\tilde{s}+2$.

Proposition 1.9. *The class of u depends only on the zero $x=0$ of* Φ.

Proof. Let c be a root of Φ such that $c \neq 0$, then $\tilde{\Phi}(c) = 0$.
If $\tilde{\Phi}'(c) + \tilde{\Psi}(c) \neq 0$, using (1.36) we have $\Phi'(c) + \Psi(c) = c^2\big(\tilde{\Phi}'(c) + \tilde{\Psi}(c)\big) \neq 0$.
If $\tilde{\Phi}'(c) + \tilde{\Psi}(c) = 0$, we have $c^2\big(\tilde{\Phi}'(c) + \tilde{\Psi}(c)\big) = x^2\big(\tilde{\Phi}'(c) + \tilde{\Psi}(c)\big) = 0$, which leads to $\theta_c^2\Phi + \theta_c\Psi = x^2\big(\theta_c^2\tilde{\Phi} + \theta_c\tilde{\Psi}\big)$. Then, using (1.17) and the above result, we get $\left\langle u, \theta_c^2\Phi + \theta_c\Psi\right\rangle = -\lambda\left\langle v, \theta_c^2\tilde{\Phi} + \theta_c\tilde{\Psi}\right\rangle \neq 0$, according to (1.31). □

Proposition 1.10. *We have*
1) *If* $\tilde{\Phi}(0) - \lambda\left\langle v, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\right\rangle \neq 0$ *then* $s = \tilde{s}+2$.
2) *If* $\tilde{\Phi}(0) = \left\langle v, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\right\rangle = 0$ *and* $2\tilde{\Phi}'(0) + \tilde{\Psi}(0) - \lambda\left\langle v, \theta_0^2\tilde{\Psi} + 2\theta_0^3\tilde{\Phi}\right\rangle \neq 0$ *then* $s = \tilde{s}+1$.

Proof. For 1) see formula (2.3) of [1, p.14] .
According to Proposition 1.9., the class of u depends only on the zero $x=0$ of Φ. If $\tilde{\Phi}(0) = \left\langle v, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi}\right\rangle = 0$, then it is possible to simplify by x (1.35)-(1.36), and u fulfils (1.35) with

$$\Phi(x) = x\tilde{\Phi}(x)\ , \quad \Psi(x) = \tilde{\Phi}(x) + x\tilde{\Psi}(x)\ . \tag{1.37}$$

Here, we have

$\Phi'(0) + \Psi(0) = 2\tilde{\Phi}(0) = 0,\quad$ and $\quad\left\langle u, \theta_0\Psi + \theta_0^2\Phi\right\rangle = \left\langle u, \tilde{\Psi} + 2\theta_0\tilde{\Phi}\right\rangle$.

From (1.17), we get

$\left\langle u, \theta_0\Psi + \theta_0^2\Phi\right\rangle = 2\tilde{\Phi}'(0) + \tilde{\Psi}(0) - \lambda\left\langle v, \theta_0^2\tilde{\Psi} + 2\theta_0^3\tilde{\Phi}\right\rangle$. Hence 2) follows. □

A differential recurrence relation. Note that the OPS relatively to a semiclassical linear form has a differential recurrence relation [21]. Then, if we consider that the linear form v is semiclassical, its OPS $\{S_n\}_{n\geq 0}$ fulfills the following differential recurrence relation

$$\tilde{\Phi}(x)S'_{n+1}(x) = \frac{\tilde{C}_{n+1}(x) - \tilde{C}_0(x)}{2} S_{n+1}(x) - \rho_{n+1}\tilde{D}_{n+1}(x)S_n(x)\ ,\ n \geq 0 \tag{1.38}$$

where

$$\begin{cases} \tilde{C}_0(x) = -\tilde{\Phi}'(x) - \tilde{\Psi}(x)\ ,\ \tilde{D}_0(x) = -\left(v\theta_0\tilde{\Psi}\right)(x) - \left(v\theta_0\tilde{\Phi}\right)'(x)\ ,\ \tilde{D}_{-1}(x) = 0\ , \\ \tilde{C}_{n+1}(x) = -\tilde{C}_n(x) + 2x\tilde{D}_n(x)\ , \quad n \geq 0\ , \\ \rho_{n+1}\tilde{D}_{n+1}(x) = -\tilde{\Phi}(x) + \rho_n\tilde{D}_{n-1}(x) - x\tilde{C}_n(x) + x^2\tilde{D}_n(x)\ ,\ n \geq 0\ , \end{cases} \tag{1.39}$$

with $\quad \left(v\theta_0\tilde{\Psi}\right)(x) = \left\langle v, \dfrac{\tilde{\Psi}(x) - \tilde{\Psi}(\zeta)}{x-\zeta}\right\rangle .$

According to (1.16), and (1.35)-(1.36), the linear form u is also symmetric and semiclassical and its OPS $\{Z_n\}_{n\geq 0}$ satisfied a differential recurrence relation. We have the following result (see [1] , [5]):

Proposition 1.11. *The sequence $\{Z_n\}_{n\geq 0}$ fulfills*

$$\Phi(x)Z'_{n+1}(x) = \frac{C_{n+1}(x) - C_0(x)}{2} Z_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)Z_n(x)\ ,\ n \geq 0 \tag{1.40}$$

with

$$\begin{cases} C_{n+2}(x) = x^2\tilde{C}_{n+1}(x) + 2(a_n - \rho_{n+1})x\tilde{D}_n(x) - 2(a_{n+1} - \rho_{n+2})x\tilde{D}_{n+1}(x),\ n \geq 0, \\ C_1(x) = x^2\tilde{C}_0(x) - 2\lambda x\tilde{D}_0(x)\ , \quad C_0(x) = -2x\tilde{\Phi}(x) + x^2\tilde{C}_0(x)\ , \\ D_{n+2}(x) = x^2\tilde{D}_{n+1}(x) + (a_n - \rho_{n+1})\left(\tilde{D}_n(x) - \dfrac{a_{n+1}}{a_n}\tilde{D}_{n+2}(x)\right)\ , \quad n \geq 0\ , \\ D_1(x) = x^2\tilde{D}_0(x)\ , \quad D_0(x) = -\tilde{\Phi}(x) + x\tilde{C}_0(x) - \lambda\tilde{D}_0(x)\ . \end{cases}$$

Corollary 1.12. *Each polynomial of $\{Z_n\}_{n\geq 0}$ satisfies a second order differential equation of Laguerre-type, (or holonomic second order differential equation)*

$$J(x,n)Z''_{n+1}(x) + K(x,n)Z'_{n+1}(x) + L(x,n)Z_{n+1}(x) = 0, \quad n \geq 0, \tag{1.41}$$

with

$$\begin{cases} J(x,n) = x^2\tilde{\Phi}(x)\big(x^2\tilde{D}_n(x) + v_n(x)\big), \\ K(x,n) = x^2\big(\tilde{C}_0(x) + \tilde{\Phi}'(x)\big)\big(x^2\tilde{D}_n(x) + v_n(x)\big) - x^2\tilde{\Phi}(x)\big(2x\tilde{D}_n(x) + x^2\tilde{D}'_n(x) + v'_n(x)\big), \\ L(x,n) = \frac{1}{2}\big(\tilde{C}_n(x) - \tilde{C}_0(x)\big)\big(x^4\tilde{D}'_n(x) + x^2v'_n(x) - 2xv_n(x)\big) + (\theta_n u_n(x) - \tilde{\theta}_{n+1}u_{n+1}(x))\times \\ \qquad \frac{1}{2}\big(2x\tilde{D}_n(x) + x^2\tilde{D}'_n(x) + v'_n(x)\big) - \big(x^2\tilde{D}_n(x) + v_n(x)\big)\big\{\frac{1}{2}\big(\tilde{C}'_n(x) - \tilde{C}'_0(x) + \\ \qquad\qquad + \theta_n u'_n(x) - \tilde{\theta}_{n+1}u'_{n+1}(x)\big) - x\Psi(x) - \lambda\tilde{D}_0(x) + x^2\nabla_n + \tilde{\nabla}_n\big\}\ , \end{cases}$$

and

$$\begin{cases} u_n(x) = (a_{n-1} - \rho_n)x\tilde{D}_{n-1}(x),\ v_n(x) = (a_{n-1} - \rho_n)\left(\tilde{D}_{n-1}(x) - \frac{a_n}{a_{n-1}}\tilde{D}_{n+1}(x)\right), \\ \theta_n = 1 - (-1)^n,\ \tilde{\theta}_n = (-1)^{n+1} - 1,\ \nabla_n = \displaystyle\sum_{k=0}^{n} D_k,\ \tilde{\nabla}_n = \sum_{k=0}^{[\frac{n-2}{2}]} v_{2k+1}(x),\ \sum_{0}^{-1} = 0,\ a_{-1} = 1. \end{cases}$$

Proof. It is well known that a semiclassical OPS fulfils a second order differential equation [21]. For the sequence $\{Z_n\}_{n\geq 0}$, we have

$$\Phi D_{n+1} Z''_{n+1} + \{C_0 D_{n+1} - W(\Phi, D_{n+1})\} Z'_{n+1} + \\ + \left\{ W\left(\frac{C_{n+1}-C_0}{2}, D_{n+1}\right) - D_{n+1}\sum_{k=0}^{n} D_k \right\} Z_{n+1} = 0\ , \quad n\geq 0\ ,$$

where $W(f,g) = fg' - f'g$. Then, substituting for C_n and D_n , $n \geq 0$, from the previous proposition into the above equation and taking into account (1.28), (1.36), and (1.39) we get the desired result. □

PROPOSITION 1.13. *We have*

$$2x^2\Phi(x)P'_{n+1}(x^2) = \left(\frac{1}{2}x\big(C_{2n+2}(x) - C_0(x)\big) - \gamma_{2n+2}D_{2n+2}(x)\right) P_{n+1}(x^2) - \\ -\gamma^P_{n+1} D_{2n+2}(x) P_n(x^2)\ , \qquad n\geq 0\ . \tag{1.42}$$

Proof. In the relation (1.40), replace n by $2n+1$ and then multiply it by x, so that $x\Phi(x)Z'_{2n+2}(x) = x\dfrac{C_{2n+2}(x)-C_0(x)}{2}Z_{2n+2}(x) - \gamma_{2n+2}xD_{2n+2}(x)Z_{2n+1}(x)$, but $xZ_{2n+1}(x) = Z_{2n+2}(x) + \gamma_{2n+1}Z_{2n}(x)$ according to (1.22), then

$$x\Phi(x)Z'_{2n+2}(x) = \left(x\frac{C_{2n+2}(x)-C_0(x)}{2} - \gamma_{2n+2}D_{2n+2}(x)\right) Z_{2n+2}(x) \\ -\gamma_{2n+2}\gamma_{2n+1}D_{2n+2}(x)Z_{2n}(x).$$

Finally, from (1.23), and (1.25) we get (1.42). □

Remark. The relation (1.42) enables us to obtain the differential recurrence relation satisfied by the sequence $\{P_n\}_{n\geq 0}$ $\big($ see the second section $\big)$.

Finally, if we suppose that the form v has the following integral representation:
$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx\ , \quad f\in\mathcal{P}, \quad$ with $(v)_0 = \int_{-\infty}^{+\infty} V(x)dx = 1$
where V is a locally integrable function with rapid decay, then the form u is represented by [1]

$$\langle u, f\rangle = f(0)\left\{1 + \lambda Pf\int_{-\infty}^{+\infty}\frac{V(x)}{x^2}dx\right\} - \lambda Pf\int_{-\infty}^{+\infty}\frac{V(x)}{x^2}f(x)dx\ , \tag{1.43}$$

where

$$Pf\int_{-\infty}^{+\infty}\frac{V(x)}{x^2}f(x)dx = \\ \lim_{\epsilon\longrightarrow 0}\left\{\int_{-\infty}^{-\epsilon}\frac{V(x)}{x^2}f(x)dx + \int_{+\epsilon}^{+\infty}\frac{V(x)}{x^2}f(x)dx - \frac{2}{\epsilon}V(0)f(0)\right\}.$$

2. Symmetric semiclassical forms of class $s = 2$: Case $\Phi(0) = 0$. Let us recall that a regular linear form u is called a Laguerre-Hahn form of class s $\big($ see [2]$\big)$, if it satisfies the function equation

$$(\Phi u)' + \Psi u + B(x^{-1}u^2) = 0\ , \tag{2.1}$$

where Φ, Ψ, and B are polynomials, (Φ monic) and $s = \max(p-1, d-2)$, $t = \deg(\Phi)$, $p = \deg(\Psi)$, $r = \deg(B)$, $d = \max(t, r)$.

When $B = 0$, we meet the semiclassical linear forms.

PROPOSITION 2.1. (*see* [2]) *Let u be a symmetric Laguerre-Hahn form of class s, satisfying* (2.1)*, then if s is even, Φ and B are even and Ψ is odd ; if s is odd, Φ and B are odd and Ψ is even.*

In the semiclassical case ($B = 0$), we obtain

COROLLARY 2.2. *Let u be a symmetric semiclassical form of class s, satisfying* (1.35)*. If s is even then Φ is even and Ψ is odd. If s is odd then Φ is odd and Ψ is even.*

In the sequel, we suppose $s = 2$, u symmetric, and $\Phi(0) = 0$. Then, according to the above corollary u satisfies (1.35) with

$$\Phi(x) = c_4x^4 + c_2x^2 \ , \quad \Psi(x) = a_3x^3 + a_1x \ , \quad |c_4| + |a_3| \neq 0 \ . \tag{2.2}$$

In this particular case, it is possible to characterize the involved semiclassical forms of class $s = 2$ as following:

THEOREM 2.3. *The following statements are equivalent*

(a) *u is a symmetric semiclassical form of class $s = 2$ satisfying* (1.35) *with* $\Phi(0) = 0$ $\big($ *i.e. with* (2.2) $\big)$.

(b) *There exist a symmetric semiclassical normalized linear form v of class $\tilde{s} \leq 1$, and $\big(\tilde{a}_0,\ \tilde{a}_2,\ \tilde{c}_1,\ \tilde{c}_3\big) \in \mathbb{C}^4$ such that :*

$$u = -\lambda x^{-2} v + \delta_0 \ , \quad \lambda = -(u)_2 \ , \tag{2.3}$$

$$\begin{cases} \big((\tilde{c}_3x^3 + \tilde{c}_1x)v\big)' + (\tilde{a}_2x^2 + \tilde{a}_0)v = 0 \ , \\ \quad |\tilde{c}_3| + |\tilde{c}_1| \neq 0 \ , \quad \tilde{a}_2 \neq 0 \ , \quad \tilde{a}_0 \neq 0 \ , \end{cases} \tag{2.4}$$

$$-\big(2\tilde{c}_3 + \tilde{a}_2\big)\lambda + \big(2\tilde{c}_1 + \tilde{a}_0\big) \neq 0 \ . \tag{2.5}$$

For the proof, we need the following lemma

LEMMA 2.4. *When u is a symmetric semiclassical linear form of class $s = 2$ satisfying* (1.35) *with $\Phi(0) = 0$, then $x\sigma u$ and $x^2\sigma u$ are regular.*

Proof. Since u is a regular and symmetric linear form , then $x\sigma u$ is regular according to Proposition 1.2.

From (1.35) and (2.2), we have

$$\big((c_4x^4 + c_2x^2)u\big)' + (a_3x^3 + a_1x)u = 0 \ .$$

Multiplication by x gives

$$\big((c_4x^5 + c_2x^3)u\big)' + \big((a_3 - c_4)x^4 + (a_1 - c_2)x^2\big)u = 0 \ . \tag{2.6}$$

Applying the operator σ for the previous equation and using (1.5)-(1.6), we obtain

$$\big(\Phi^R(x)(x\sigma u)\big)' + \Psi^R(x)(x\sigma u) = 0$$

where $\Phi^R(x) = c_4x^2 + c_2x$, and $\Psi^R(x) = \dfrac{1}{2}\big((a_3 - c_4)x + a_1 - c_2\big)$.
Assume $a_3 - c_4 = 0$. Then, from (2.6),we obtain

$$\Big\langle \big((c_4x^5 + c_2x^3)u\big)' + \big(a_1 - c_2\big)x^2u \;,\; x^{2n} \Big\rangle = 0 \;, \quad n \geq 0 \;, \tag{2.7}$$

By (2.7) with $n = 0$, it is easy to see that $a_1 - c_2 = 0$ since $(u)_2 = \gamma_1 \neq 0$ and then, (2.7) becomes

$$c_4(u)_{2n+4} + c_2(u)_{2n+2} = 0 \;, \quad n \geq 1 \;. \tag{2.7$'$}$$

If $c_4 = 0$ then $(u)_{2n+2} = 0$, $n \geq 1$ (since Φ is monic) and if $c_4 \neq 0$ then $(u)_{2n+4} = k(u)_4$, $n \geq 1$ where $k = -\dfrac{c_2}{c_4}$. Therefore, we can deduce that the Hankel determinant $\triangle_5 =: \det\Big((u)_{i+j}\Big)_{i,j=0}^{5} = 0$ which is contradictory with the regularity of u. Hence

$$a_3 - c_4 \neq 0 \;. \tag{2.8}$$

Thus, $\deg(\Psi^R) = 1$ and $\deg(\Phi^R) \leq 2$. Then, $x\sigma u$ is a classical form. So, there exist parameters $\lambda_{n+1} \in \mathbb{C}^*$, $n \geq 0$ such that its corresponding OPS $\{R_n\}_{n\geq 0}$ satisfies the following differential equation

$$\Phi^R(x)R''_{n+1}(x) - \Psi^R(x)R'_{n+1}(x) = \lambda_{n+1}R_{n+1}(x) \;, \quad n \geq 0 \;,$$

since, the classical orthogonal polynomials are solutions of a second order differential equation of hypergeometric type according to [7].
Substituting x by 0 in the above equation, we obtain

$$\Psi^R(0)R'_{n+1}(0) = \lambda_{n+1}R_{n+1}(0) \;, \quad n \geq 0 \;. \tag{2.9}$$

We have necessarily

$$2\Psi^R(0) = a_1 - c_2 \neq 0 \;. \tag{2.10}$$

In fact, if $\Psi^R(0) = 0$, then $R_{n+1}(0) = 0$, $n \geq 0$. Taking into account (1.24), (1.20) becomes $\displaystyle\prod_{v=0}^{n} \rho_{v+1}^{\tilde{P}} = 0$, $n \geq 0$, which is contradictory with the regularity of $x\sigma u$.
Now, assume that there exists $n_0 \geq 1$ such that $R_{n_0}(0) = 0$. Then, according to (2.9)-(2.10), we get $R'_{n_0}(0) = 0$ which is a contradiction because it is well known (see [7],[22]) that the zeros of the classical OPS are simple. Hence $R_n(0) \neq 0 \quad n \geq 0$. So, by [8], the linear form $x^2\sigma u = x(x\sigma u)$ is regular. □
Now, we are able to give the proof of Theorem 2.3.

Proof. (a)⇒ (b). Let $v = -\dfrac{1}{\lambda}x^2u$ which is equivalent to (2.3). Since v is symmetric, $\sigma v = -\dfrac{1}{\lambda}x\sigma u$ and $x\sigma v = -\dfrac{1}{\lambda}x^2\sigma u$, then v is regular according to Proposition

1.2. and Lemma 2.4. .

From (1.17), (2.2), (2.6), (2.8), and (2.10), we obtain (2.4) where

$$\tilde{c}_3 = c_4\ ,\ \tilde{c}_1 = c_2\ ,\ \tilde{a}_2 = a_3 - c_4\ \ ,\text{ and } \tilde{a}_0 = a_1 - c_2\ \ .$$

From (2.4), it is clear that the linear form v is semiclassical of class $\tilde{s} \leq 1$.

Finally, we have $\Phi'(0) + \Psi(0) = 0$ and $\left\langle u, \theta_0\Psi + \theta_0^2\Phi \right\rangle = -\lambda(c_4 + a_3) + (c_2 + a_1)$.

Then, using Proposition 1.8., and the above results, we deduce (2.5).

(b)$\Rightarrow$ (a). From (2.4), v satisfies (1.30) with $\tilde{\Phi}(x) = \tilde{c}_3x^3 + \tilde{c}_1x$, $\quad \tilde{\Psi}(x) = \tilde{a}_2x^2 + \tilde{a}_0$.

Here, we have $\left\langle v, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi} \right\rangle = 0$ and $\tilde{\Phi}'(0) + \tilde{\Psi}(0) = \tilde{a}_0 + \tilde{c}_1$.

Then, using the standard criterion of simplification (1.31), we obtain the two different cases :

i) $\tilde{a}_0 + \tilde{c}_1 \neq 0$, then v is of class $\tilde{s} = 1$.

In this case, we have $\quad \tilde{\Phi}(0) = \left\langle v, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi} \right\rangle = 0,\quad$ and from (2.5), we obtain

$$2\tilde{\Phi}'(0) + \tilde{\Psi}(0) - \lambda \left\langle v, \theta_0^2\tilde{\Psi} + 2\theta_0^3\tilde{\Phi} \right\rangle = -(2\tilde{c}_3 + \tilde{a}_2)\lambda + (2\tilde{c}_1 + \tilde{a}_0) \neq 0\ .$$

Then, the class of u is $s = 2$ according to Proposition 1.10., 2).

ii) $\tilde{a}_0 + \tilde{c}_1 = 0$, then it is possible to simplify (2.4) by x. Thus, v satisfies (1.30) with $\tilde{\Phi}(x) = \tilde{c}_3x^2 + \tilde{c}_1,\quad \tilde{\Psi}(x) = (\tilde{a}_2 + \tilde{c}_3)x$.

It is clear that the linear form v is of class $\tilde{s} = 0$. In this case, we have

$$\tilde{\Phi}(0) - \lambda \left\langle v, \theta_0\tilde{\Psi} + \theta_0^2\tilde{\Phi} \right\rangle = -(2\tilde{c}_3 + \tilde{a}_2)\lambda + \tilde{c}_1 \neq 0\ .$$

Then, the class of u is $s = 2$ according to Proposition 1.10., 1). □

From (1.32), Corollary 2.2 and Theorem 2.3. , we distinguish three canonical cases for Φ:

$$\Phi(x) = x^2\ ,\quad \Phi(x) = x^4\ ,\quad \Phi(x) = x^2(x^2 - 1)$$

which correspond respectively to the three canonical cases for $\tilde{\Phi}$:

$$\tilde{\Phi}(x) = x\ ,\quad \tilde{\Phi}(x) = x^3\ ,\quad \tilde{\Phi}(x) = x(x^2 - 1)\ .$$

The last one was mentioned in [2], the authors gave all the symmetric semiclassical linear forms v of class $s = 1$. They are obtained as particular cases of symmetric Laguerre-Hahn forms of class $s = 1$, when $B = 0$.

2.1. $\Phi(x) = x^2$**.** In this case , v is the symmetric semiclassical form with $\tilde{\Phi}(x) = x$. Indeed, $v = \mathcal{H}(2\alpha + 1)$: the generalized Hermite form (see [2,8]).

We have [2]

$$\rho_{2n+1} = n + \alpha + 1\ ,\quad \rho_{2n+2} = n + 1\ , \tag{2.11}$$

$$\begin{cases} \tilde{\Psi}(x) = 2x^2 - 2(\alpha + 1)\ , \\ \tilde{C}_n(x) = -2x^2 + (-1)^n(2\alpha + 1)\ ,\quad \tilde{D}_n(x) = -2x\ ,\quad n \geq 0\ . \end{cases} \tag{2.12}$$

In addition, $\{S_n\}_{n\geq 0}$ verifies (1.10) with

$$\tilde{P}_n(x) = L_n^{\alpha}(x)\ ,\quad \tilde{R}_n(x) = L_n^{\alpha+1}(x)\ ,\quad n \geq 0\ , \tag{2.13}$$

where $L_n^{\alpha}(x)$ denotes the classical Laguerre polynomials which are orthogonal with respect to $\sigma v = \mathcal{L}(\alpha)$.
Using (1.10), (1.11), (2.11), and Proposition 1.1., we get successively

$$\tilde{P}_n(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \ , \quad n \geq 0 \ , \tag{2.14}$$

$$S_{2n+2}^{(1)}(0) = \tilde{R}_{n+1}(0, -\rho_1) = (-1)^{n+1}\Gamma(n+2) \ , \quad n \geq 0 \ . \tag{2.15}$$

From (2.15) and (1.9), we obtain

$$(\alpha+1)\tilde{R}_n^{(1)}(0) = (-1)^{n+1}\Gamma(n+2) - \tilde{R}_{n+1}(0) \ , \quad n \geq 0 \ .$$

Then, by replacing $\alpha+1$ by α in the above equation and using (2.13)-(2.14), we get if $\alpha \neq 0$

$$\tilde{P}_n^{(1)}(0) = \frac{(-1)^{n+1}}{\alpha}\left(\Gamma(n+2) - \frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)}\right) \ , \quad n \geq 0 \ . \tag{2.16}$$

And if $\alpha = 0$, from (1.21) and (2.11), we obtain

$$\tilde{P}_n^{(1)}(0) = (-1)^n\Gamma(n+2)\sum_{k=0}^{n}\frac{1}{k+1} \ , \quad n \geq 0 \ . \tag{2.17}$$

So, from (1.9), (2.14), (2.16), and (2.17) we deduce

$$\tilde{P}_n(0,\lambda) = \frac{(-1)^n\Gamma(n+\alpha+1)d_{\alpha,n}}{(\alpha+\delta_{\alpha,0})\,\Gamma(\alpha+1)} \ , \quad n \geq 0 \ , \tag{2.18}$$

where

$$d_{\alpha,n} = \begin{cases} (\alpha+\lambda) - \dfrac{\lambda\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)} \ , \quad \alpha \neq 0 \ , \quad n \geq 0 \ , \\ 1 + \lambda\displaystyle\sum_{k=0}^{n-1}\frac{1}{k+1} \ , \quad \alpha = 0 \ , \quad n \geq 0 \ . \qquad \left(\displaystyle\sum_{0}^{-1} = 0\right) \end{cases} \tag{2.19}$$

The regularity conditions are $\quad \alpha \neq -(n+1) \ , \ d_{\alpha,n} \neq 0 \ , \quad n \geq 0 \ .$
(1.27) and (2.18)-(2.19) give

$$\tilde{a}_n = \frac{(n+\alpha+1)d_{\alpha,n+1}}{d_{\alpha,n}} \ , \quad n \geq 0 \ . \tag{2.20}$$

Using (2.11), (2.20), and Proposition 1.6., we get

$$\gamma_1 \quad = -\lambda \ , \quad \gamma_{2n+2} = \tilde{a}_n \ , \quad \gamma_{2n+3} = \frac{(n+1)d_{\alpha,n}}{d_{\alpha,n+1}} \ , \quad n \geq 0 \ . \tag{2.21}$$

From (1.37), we have $\quad \Psi(x) = 2x^3 - (2\alpha+1)x.$
Using Proposition 1.11., (1.28), (2.12), (2.19), and (2.20), we obtain after division by x, for $n \geq 0$

$$\begin{cases} C_0(x) \quad = -2x^3 + (2\alpha - 1)x \ , \ C_1(x) = -2x^3 + (2\alpha + 4\lambda + 1)x \ , \\ C_{2n+2}(x) = -2x^3 - X_n \ , \quad C_{2n+3}(x) = -2x^3 + X_{n+1} \ , \\ D_0(x) \quad = -2x^2 + 2(\alpha + \lambda) \ , \quad D_{2n+1}(x) = -2x^2 \ , \\ D_{2n+2}(x) = -2x^2 - \dfrac{2\left(\alpha^2 + \delta_{0,\alpha}\right)(\alpha + \lambda)\lambda\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 2)d_{\alpha,n}d_{\alpha,n+1}} \ , \end{cases} \tag{2.22}$$

where $X_n = \left(2\alpha + 1 + \dfrac{4\lambda\left(\alpha + \delta_{0,\alpha}\right)\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 1)d_{\alpha,n}}\right)x$.

Next, the focus will be put on σu: the even part of u .
The linear form u verifies the functional equation

$$\left(x^2 u\right)' + \left(2x^3 - (2\alpha + 1)x\right)u = 0 \ .$$

Multiplication by x gives

$$\left(x^3 u\right)' + \left(2x^4 - 2(\alpha + 1)x^2\right)u = 0 \ .$$

Applying the operator σ for the above equation and using (1.5)-(1.6), we obtain

$$2\left(x^2 \sigma u\right)' + 2\left(x^2 - (\alpha + 1)x\right)\sigma u = 0 \ .$$

Then σu is semiclassical form and satisfies the functional equation

$$\left(\Phi^P(x)\sigma u\right)' + \Psi^P(x)\sigma u = 0 \tag{2.23}$$

where $\qquad \Phi^P(x) = x^2 \ , \quad \Psi^P(x) = x^2 - (\alpha + 1)x$.
We have $\Psi^P(0) + (\Phi^P)'(0) = 0$ and $\left\langle \sigma u, \theta_0 \Psi^P + \theta_0^2 \Phi^P \right\rangle = -(\lambda + \alpha)$.
Then, using Proposition 1.8., we obtain the two different cases:
i) $\lambda \neq -\alpha$, the class of σu is equal to 1.
ii) $\lambda = -\alpha$, σu is a Laguerre form with parameter value of $\alpha - 1$.
From (1.25) and (2.22), the coefficients $\left\{\beta_n^P, \gamma_{n+1}^P\right\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P = \gamma_1 \ , \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} \ , \quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2} \ ,$$

where γ_n , $n \geq 1$ are given by (2.21).

According to Proposition 1.13. and (2.22) where $x^2 \to x$, the sequence $\{P_n\}_{n\geq 0}$ satisfies the following differential recurrence relation (for $n \geq 0$)

$$\Phi^P(x)P_{n+1}'(x) = \frac{C_{n+1}^P(x) - C_0^P(x)}{2}P_{n+1}(x) - \gamma_{n+1}^P D_{n+1}^P(x)P_n(x) \tag{2.24}$$

with

$$\frac{C_{n+1}^P(x) - C_0^P(x)}{2} = \left(-\alpha - \frac{\lambda\left(\alpha + \delta_{0,\alpha}\right)\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 1)d_{\alpha,n}} + \frac{(n + \alpha + 1)d_{\alpha,n+1}}{d_{\alpha,n}}\right)x$$
$$+ \frac{\left(\alpha^2 + \delta_{0,\alpha}\right)(\alpha + \lambda)\lambda\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 1)(d_{\alpha,n})^2} \ ,$$

$$D_{n+1}^P(x) = -x - \frac{\left(\alpha^2 + \delta_{0,\alpha}\right)(\alpha + \lambda)\lambda\Gamma(\alpha + 1)\Gamma(n + 1)}{\Gamma(n + \alpha + 2)d_{\alpha,n}d_{\alpha,n+1}} \ ,$$

$C_0^P(x) = -x^2 + (\alpha-1)x, \quad D_0^P(x) = -x + \lambda + \alpha \ .$

Finally, we give the integral representations of u and σu.
The linear form $v = \mathcal{H}(2\alpha+1)$ has the following integral representation [8, p. 157]

$$\langle v, f\rangle = \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{+\infty} |x|^{2\alpha+1} e^{-x^2} f(x)dx \ , \ \ \Re(\alpha) > -1 \ , \ \ f \in \mathcal{P} \ . \tag{2.25}$$

Therefore, (1.43) becomes (see also [15])

$$\langle u, f\rangle = \left(1 + \frac{\lambda}{\alpha}\right) f(0) - \frac{\lambda}{\Gamma(\alpha+1)} \int_{-\infty}^{+\infty} |x|^{2\alpha-1} e^{-x^2} f(x)dx \ , \Re(\alpha) > 0 \ . \tag{2.26}$$

From (2.26)), we get

$$\begin{aligned} \langle \sigma u, f(x)\rangle &= \langle u, f(x^2)\rangle \\ &= \left(1 + \frac{\lambda}{\alpha}\right) f(0) - \frac{2\lambda}{\Gamma(\alpha+1)} \int_{0}^{+\infty} |x|^{2\alpha-1} e^{-x^2} f(x^2)dx \ . \end{aligned}$$

Then, we obtain after a change of variables

$$\langle \sigma u, f\rangle = \left(1 + \frac{\lambda}{\alpha}\right) f(0) - \frac{\lambda}{\Gamma(\alpha+1)} \int_{0}^{+\infty} x^{\alpha-1} e^{-x} f(x)dx \ , \Re(\alpha) > 0 \ . \tag{2.27}$$

Remarks. 1. From (2.26)-(2.27), we deduce that the linear form u is the symmetrized of a Laguerre-type linear form (see [3,11,12,13,14]).
2. A remarkable particular case is $\lambda = -\alpha$, the linear form u is the generalized Hermite form corresponding to the parameter value $2\alpha - 1$.

In the two other cases, we are going to proceed with the same stages and techniques.

2.2. $\Phi(x) = x^4$ **.** Let us keep the same notations of [2] where it was shown that the symmetric semiclassical $\hat{v}$ of class $s = 1$ with $\tilde{\Phi}(x) = x^3$ satisfies

$$\left(x^3 \hat{v}\right)' + \left(-2(\nu+1)x^2 - \frac{1}{2}\right) \hat{v} = 0 \ .$$

Putting $\nu = 2\alpha - 1$ and using (1.33)-(1.34), then the regular linear form $v = h_{2\sqrt{2}}\hat{v}$ satisfies

$$\left(x^3 v\right)' - 4\left(\alpha x^2 + 1\right) v = 0 \tag{2.28}$$

and

$$\begin{aligned} &\rho_1 = -\frac{1}{\alpha} \ , \quad \rho_{2n+2} = \frac{n+1}{(n+\alpha)(2n+2\alpha+1)} \ , \\ &\rho_{2n+3} = -\frac{n+2\alpha}{(n+\alpha+1)(2n+2\alpha+1)} \ , \quad n \geq 0 \ . \end{aligned}$$

The regularity condition is $\alpha \neq -\dfrac{n}{2}$, $\quad n \geq 0$, and we have

$$\begin{aligned} &\tilde{\Psi}(x) = -4(\alpha x^2 + 1) \ , \\ &\tilde{C}_n(x) = (2n + 4\alpha - 3)x^2 + 4(-1)^n \ , \qquad n \geq 0 \ , \\ &\tilde{D}_n(x) = 2(n + 2\alpha - 1)x \ , \qquad n \geq 0 \ . \end{aligned}$$

Applying the operator σ for (2.28) and using (1.5)-(1.6), we obtain

$$\big(x^2\sigma v\big)' - 2\big(\alpha x + 1\big)\sigma v = 0 \; . \tag{2.29}$$

Multiplication by x gives

$$\big(x^2(x\sigma v)\big)' - 2\left(\left(\alpha + \frac{1}{2}\right) x + 1\right)(x\sigma v) = 0 \; . \tag{2.30}$$

From (2.29)-(2.30), we deduce that $\{S_n\}_{n\geq 0}$ satisfies (1.10) with

$$\tilde{P}_n(x) = B_n^{\alpha}(x) \; , \quad \tilde{R}_n(x) = B_n^{\alpha+\frac{1}{2}}(x) \; , \quad n \geq 0 \; , \tag{2.31}$$

where $B_n^{\alpha}(x)$ denotes the classical Bessel polynomials which are orthogonal with respect to $\sigma v = \mathcal{B}(\alpha)$ (see [21]).

Remark.[3] From (2.31), v is the symmetrized linear form associated with the linear form $\mathcal{B}(\alpha)$ $\big($ i.e. v is symmetric and $\sigma v = \mathcal{B}(\alpha)$ $\big)$, with the notation $v = w\big(\mathcal{B}(\alpha)\big)$.

By applying the same process as we did to obtain (2.18)-(2.19) and using the above results, we get

$$P_n(0,\lambda) = \frac{2^n\Gamma(n+2\alpha-1)\left(1-2\alpha+2\delta_{\alpha,\frac{1}{2}}\right)d_{\alpha,n}}{2\Gamma(2n+2\alpha-1)} \; , \quad n \geq 0 \; , \tag{2.32}$$

where

$$d_{\alpha,n} = \begin{cases} \lambda - \dfrac{2}{2\alpha-1} - \dfrac{(-1)^n\lambda\Gamma(2\alpha-1)\Gamma(n+1)}{\Gamma(n+2\alpha-1)} \; , & n \geq 0 \; , \; \alpha \neq \dfrac{1}{2} \; , \\ 1 + (-1)^n \dfrac{n\lambda}{2} \; , & n \geq 0 \; , \; \alpha = \dfrac{1}{2} \; . \end{cases} \tag{2.33}$$

The regularity conditions are $\quad d_{\alpha,n} \neq 0 \; , \; \alpha \neq -\dfrac{n}{2} \; , \; n \geq 0.$

From (1.27) and (2.32)-(2.33), we get

$$\tilde{a}_n = -\frac{\big(n+2\alpha-1\big)d_{\alpha,n+1}}{\big(n+\alpha\big)\big(2n+2\alpha-1\big)d_{\alpha,n}} \; , \quad n \geq 0 \; . \tag{2.34}$$

So, from Proposition 1.6 we get

$$\begin{cases} \gamma_1 \quad = -\lambda \; , \quad \gamma_{2n+2} = \tilde{a}_n \; , \quad n \geq 0 \; , \\ \gamma_{2n+3} = \dfrac{(n+1)d_{\alpha,n}}{(2n+2\alpha+1)(n+\alpha)d_{\alpha,n+1}} \; , \quad n \geq 0 \; . \end{cases} \tag{2.35}$$

From (1.28), (1.37), (2.34), and Proposition 1.11., we obtain for $n \geq 0$

$$\begin{cases} \Psi(x) \quad = (1-4\alpha)x^3 - 4x \ , \\ C_0(x) \quad = (4\alpha-5)x^3 + 4x \ , \ \ C_1(x) = (4a-3)x^3 + 4\left(1 - \lambda(2\alpha-1)\right)x \ , \\ C_{2n+2}(x) = (4n+4\alpha-1)x^3 - Y_n \ , \\ C_{2n+3}(x) = (4n+4\alpha+1)x^3 + Y_{n+1} \ , \\ D_0(x) \quad = 4(\alpha-1)x^2 + 4 - 2\lambda(2\alpha-1) \ , \\ D_{2n+1}(x) = 2(2n+2\alpha-1)x^2 \ , \\ D_{2n+2}(x) = 4(n+\alpha)\left(x^2 - \dfrac{2(-1)^n\lambda\big(\lambda(2\alpha-1)-2\big)\Gamma(2\alpha)\Gamma(n+1)}{\Big(1-2\alpha-2\delta_{\frac{1}{2},\alpha}\Big)^2\Gamma(n+2\alpha)d_{\alpha,n}d_{\alpha,n+1}}\right) \ , \end{cases} \tag{2.36}$$

where $Y_n = \left(4 + \dfrac{(-1)^{n+1}8\lambda\Gamma(2\alpha)\Gamma(n+1)}{\Big(1-2\alpha-2\delta_{\frac{1}{2},\alpha}\Big)\Gamma(n+2\alpha-1)d_{\alpha,n}}\right)x$.

σu is a semiclassical linear form and satisfies (2.23) with

$$\Phi^P(x) = x^3 \ , \quad \Psi^P(x) = -2\alpha x^2 - 2x.$$

For the class of σu, we distinguish the two following cases:
i) $2\lambda^{-1} \neq 2\alpha - 1$, the class of σu is equal to 1.
ii) $2\lambda^{-1} = 2\alpha - 1$, σu is a Bessel form with parameter value of $\alpha - \dfrac{1}{2}$.
From (1.25) and (2.35) the coefficients $\left\{\beta_n^P, \gamma_{n+1}^P\right\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P = \gamma_1 \ , \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} \ , \quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2} \ ,$$

where γ_n , $n \geq 1$ are given by (2.35).

From Proposition 1.13. and (2.36), the sequence $\{P_n\}_{n\geq 0}$ satisfies (2.24) with

$$\frac{C_{n+1}^P(x) - C_0^P(x)}{2} = (n+1)x^2 -$$

$$-2\left(1 - \frac{(n+2\alpha-1)d_{\alpha,n+1}}{(2n+2\alpha-1)d_{\alpha,n}} + \frac{(-1)^{n+1}\lambda\Gamma(n+1)\Gamma(2\alpha)}{\Big(1-2\alpha+2\delta_{\alpha,\frac{1}{2}}\Big)\Gamma(n+2\alpha-1)d_{\alpha,n}}\right)x$$

$$-4\frac{(-1)^n\lambda\big(\lambda(2\alpha-1)-2\big)\Gamma(2\alpha)\Gamma(n+1)}{\Big(1-2\alpha-2\delta_{\alpha,\frac{1}{2}}\Big)^2\Gamma(n+2\alpha-1)(d_{\alpha,n})^2} \ , \quad n \geq 0 \ ,$$

$$D_{n+1}^P(x) = 2(n+\alpha)x - 4\frac{(-1)^n\lambda\big(\lambda(2\alpha-1)-2\big)(n+\alpha)\Gamma(2\alpha)\Gamma(n+1)}{\Big(1-2\alpha-2\delta_{\alpha,\frac{1}{2}}\Big)^2\Gamma(n+2\alpha)d_{\alpha,n}d_{\alpha,n+1}} \ , \ \ n \geq 0 \ ,$$

$$C_0^P(x) = (2\alpha-3)x^2 + 2x \ , \quad D_0^P(x) = 2(\alpha-1)x + (1-2\alpha)\lambda + 2 \ .$$

Integral representations. First, let us recall some results which are useful to obtain the integral representations of v, u and σu.
A solution of (1.30) has the integral representation

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} U(x)f(x)dx \ , \qquad f \in \mathcal{P} \ ,$$

where we suppose that the function U is absolutely continuous on $\mathbb{R}$ and its derivative U', if the following conditions hold [18]

$$(\tilde{\Phi}(x)U(x))' + \Psi(x)U(x) = \eta g \ , \tag{2.37}$$

$$\tilde{\Phi}(x)U(x)f(x)]_{-\infty}^{+\infty} = 0 \ , \qquad f \in \mathcal{P} \ , \tag{2.38}$$

$$\int_{-\infty}^{+\infty} U(x)dx \neq 0 \ , \tag{2.39}$$

where $\eta \neq 0$ is arbitrary ; g is locally integrable function with rapid decay representing the null form: $\displaystyle\int_{-\infty}^{+\infty} x^n g(x) = 0 \ , \quad n \geq 0 \ .$

The fundamental example representing the null linear form is given by the Stieltjes function [21]

$$s(x) = \begin{cases} 0 \ , & x \leq 0 \ , \\ e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}} \ , & x > 0 \ . \end{cases}$$

Proposition 2.4. [18] *Let* $h_\alpha(t) = \displaystyle\int_0^t x^{2\alpha} e^{-\frac{2}{x}} dx$*; we have the following expression*

$$J(\alpha) := 4 \int_0^{+\infty} t^{3-8\alpha} e^{\frac{2}{t^4}} e^{-t} \sin(t) h_{\alpha-1}(t^4) dt \tag{2.40}$$

$$= \frac{1}{2^{2m}} \prod_{\mu=0}^{2m+1} (2\alpha + \mu) \int_0^{+\infty} t^{3-8\alpha} e^{\frac{2}{t^4}} e^{-t} \sin(t) h_{\alpha+m}(t^4) dt \ , \quad m \geq 0 \ , \ \alpha \in \mathbb{C} \ .$$

Corollary 2.5. [18] *We have* $J\big((-\frac{n}{2})\big) = 0, n \geq 0.$

This result is consistent with the fact that the Bessel form is not regular for these values of α.

Conjecture 2.6. [18] *The unique zeros of* $J(\alpha)$ *are* $\alpha_n = -\dfrac{n}{2} \ , \quad n \geq 0.$

Proposition 2.7. [18] *For* $\alpha \geq 1$*, we have* $J(\alpha) > 0.$

Proposition 2.8.

$\star$) *For* α *such that* $J(\alpha) \neq 0$

$$\langle v, f \rangle = J(\alpha)^{-1} \int_{-\infty}^{+\infty} |x|^{4\alpha-3} e^{-\frac{2}{x^2}} \left(\int_{x^2}^{+\infty} \xi^{-2\alpha} e^{\frac{2}{\xi}} s(\xi) d\xi \right) f(x) dx \ . \tag{2.41}$$

$\star$) *For* α *such that* $J\left(\alpha - \dfrac{1}{2}\right) \neq 0$ and $\alpha \neq \dfrac{1}{2}$

$$\langle v, f \rangle = -\frac{2\alpha - 1}{2J\left(\alpha - \frac{1}{2}\right)} \int_{-\infty}^{+\infty} |x|^{4\alpha-3} e^{-\frac{2}{x^2}} \left(\int_{x^2}^{+\infty} \xi^{-2\alpha+1} e^{\frac{2}{\xi}} s(\xi) d\xi \right) f(x) dx \ . \tag{2.42}$$

Proof. From (2.37) a solution of (2.28) has the integral representation

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx\ ,\ f\in\mathcal{P} \quad \text{if} \quad \big(x^3V(x)\big)' - 4\big(\alpha x^2+1\big)V(x) = \eta g(x).$$

With the choice of $g(x) = sgn(x)s(x^2)$, we obtain the following solution:

$$V(x) = \begin{cases} 0\ , \quad x=0\ , \\ \eta|x|^{4\alpha-3}e^{-\frac{2}{x^2}}\displaystyle\int_{x^2}^{+\infty}\xi^{-2\alpha}e^{\frac{2}{\xi}}s(\xi)\ ,\ x\neq 0\ . \end{cases} \tag{2.43}$$

It is evident when $|x|\to+\infty$ [18]

$$|V(x)| \leq |\eta||x|^{4\Re(\alpha)-3}\int_{x^2}^{+\infty}\xi^{-2\Re(\alpha)}e^{-\xi^{\frac{1}{4}}}d\xi = o(e^{\frac{-|x|^{\frac{1}{2}}}{2}})\ .$$

So, the condition (2.38) is fulfilled. It remains just to prove (2.39). From (2.43), using (2.40), we get

$$\begin{aligned}\int_{-\infty}^{+\infty} V(x)dx &= 2\eta\int_0^{+\infty} x^{4\alpha-3}e^{-\frac{2}{x^2}}\left(\int_{x^2}^{+\infty}\xi^{-2\alpha}e^{\frac{2}{\xi}}s(\xi)d\xi\right)dx \\ &= \eta\int_0^{+\infty} y^{2\alpha-2}e^{-\frac{2}{y}}\left(\int_{y}^{+\infty}\xi^{-2\alpha}e^{\frac{2}{\xi}}s(\xi)d\xi\right)dy \\ &= \eta J(\alpha)\ .\end{aligned}$$

Hence (2.41).

Using the same process described above with $g(x) = x|x|s(x^2)$ instead of $g(x) = sgn(x)s(x^2)$, we get (2.42). ∎

Remark. If we start from (2.42) and apply the same process as we did for (2.26), we obtain the following new integral representation of $\mathcal{B}(\alpha)$ for α such that $J\left(\alpha-\frac{1}{2}\right)\neq 0$ and $\alpha\neq\frac{1}{2}$

$$\langle\mathcal{B}(\alpha), f\rangle = -\frac{2\alpha-1}{2J\left(\alpha-\frac{1}{2}\right)}\int_0^{+\infty}x^{2\alpha-2}e^{-\frac{2}{x}}\left(\int_x^{+\infty}\xi^{-2\alpha+1}e^{\frac{2}{\xi}}s(\xi)d\xi\right)f(x)dx\ . \tag{2.44}$$

From (1.43) and (2.42), we obtain

$$\begin{aligned}\langle u, f\rangle &= \frac{(2\alpha-1)\lambda}{2J\left(\alpha-\frac{1}{2}\right)}\int_{-\infty}^{+\infty}|x|^{4\alpha-5}e^{-\frac{2}{x^2}}\left(\int_{x^2}^{+\infty}\xi^{-2\alpha+1}e^{\frac{2}{\xi}}s(\xi)d\xi\right)f(x)dx+ \\ &+\left(1-\frac{(2\alpha-1)\lambda}{2J\left(\alpha-\frac{1}{2}\right)}\int_{-\infty}^{+\infty}|x|^{4\alpha-5}e^{-\frac{2}{x^2}}\left(\int_{x^2}^{+\infty}\xi^{-2\alpha+1}e^{\frac{2}{\xi}}s(\xi)d\xi\right)dx\right)f(0)\ ,\end{aligned}$$

but according to (2.41), where $\alpha\to\alpha-\dfrac{1}{2}$, we have

$$-\frac{2\alpha-1}{2J(\alpha-\frac{1}{2})}\int_{-\infty}^{+\infty}|x|^{4\alpha-3}e^{-\frac{2}{x^2}}\left(\int_{x^2}^{+\infty}\xi^{-2\alpha+1}e^{\frac{2}{\xi}}s(\xi)d\xi\right)\frac{1}{x^2}dx = -\frac{2\alpha-1}{2}\ .$$

Therefore, for α such that $J(\alpha-\frac{1}{2})\neq 0$ and $\alpha\neq\dfrac{1}{2}$

$$\langle u, f\rangle = \left(1-\frac{(2\alpha-1)\lambda}{2}\right)f(0)+$$

$$+\frac{(2\alpha-1)\lambda}{2J\left(\alpha-\frac{1}{2}\right)}\int_{-\infty}^{+\infty}|x|^{4\alpha-5}e^{-\frac{2}{x^2}}\left(\int_{x^2}^{+\infty}\xi^{-2\alpha+1}e^{\frac{2}{\xi}}s(\xi)d\xi\right)f(x)dx\ . \tag{2.45}$$

If we start from (2.45) and apply the same process as we did for (2.26), we obtain for α such that $J\left(\alpha-\frac{1}{2}\right)\neq 0$ and $\alpha\neq\dfrac{1}{2}$

$$\langle\sigma u,f\rangle=\left(1-\frac{(2\alpha-1)\lambda}{2}\right)f(0)+$$

$$+\frac{(2\alpha-1)\lambda}{2J\left(\alpha-\frac{1}{2}\right)}\int_{0}^{+\infty}x^{2\alpha-3}e^{-\frac{2}{x}}\left(\int_{x}^{+\infty}\xi^{-2\alpha+1}e^{\frac{2}{\xi}}s(\xi)d\xi\right)f(x)dx\ . \tag{2.46}$$

Remarks. 1. From (2.45)-(2.46), we deduce that the linear form u is the symmetrized of a Bessel-type linear form (see [3]).
2. A remarkable particular case is $2\lambda^{-1}=2\alpha-1$, the form $u=w\left(\mathcal{B}\left(\alpha-\frac{1}{2}\right)\right)$.

2.3. $\Phi(x)=x^2(x^2-1)$**.** It was shown in [2] that the symmetric semiclassical v of class $s=1$ with $\tilde{\Phi}(x)=x(x^2-1)$ satisfies

$$\big(x(x^2-1)v\big)'+\big(-2(\alpha+\beta+2)x^2+2(\beta+1)\big)v=0\ .$$

Indeed, $v=GG(\alpha,\beta)$, the generalized Gegenbauer (see [6]).
Again in [2 , 8] , we have

$$\begin{cases}\rho_{2n+1}=\dfrac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\ ,\ \ n\geq 0\ ,\\ \rho_{2n+2}=\dfrac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}\ ,\ \ n\geq 0\ .\end{cases} \tag{2.47}$$

The regularity conditions are $\alpha\neq -n\ ,\ \beta\neq -n\ ,\ \alpha+\beta\neq-(n+1)\ ,\ n\geq 1$.
And we have

$$\begin{cases}\tilde{C}_n(x)=(2n+2\alpha+2\beta+1)x^2+(-1)^{n+1}(2\beta+1)\ ,\quad n\geq 0\ ,\\ \tilde{D}_n(x)=2(n+\alpha+\beta+1)x\ ,\quad n\geq 0\ .\end{cases} \tag{2.48}$$

In addition, $\{S_n\}_{n\geq 0}$ verifies (1.10) with

$$\tilde{P}_n(x)=\frac{1}{2^n}P_n^{\alpha,\beta}(2x-1)\ ,\quad \tilde{R}_n(x)=\frac{1}{2^n}P_n^{\alpha,\beta+1}(2x-1)\ ,\quad n\geq 0\ , \tag{2.49}$$

where $P_n^{\alpha,\beta}(x)$ denotes the classical Jacobi's polynomials which are orthogonal with respect to $\mathcal{J}(\alpha,\beta)$. This last linear form satisfies $\big((x^2-1)\mathcal{J}(\alpha,\beta)\big)'+\big(-(\alpha+\beta+2)x-\alpha+\beta\big)\mathcal{J}(\alpha,\beta)=0$ (see[6, 8]).
Then, $\sigma v=(h_{\frac{1}{2}}o\tau_1)\mathcal{J}(\alpha,\beta)$.
By applying the same process as we did to obtain (2.18)-(2.19) and using the above results, we can get for $n\geq 0$

$$P_n(0,\lambda)=\frac{(-1)^n(\alpha+\beta+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{(\beta+\delta_{\beta,0})\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}d_n^{\beta}\ , \tag{2.50}$$

with

$$d_{\beta,n}=\begin{cases} -\lambda\dfrac{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}+ \\ \qquad\qquad +\dfrac{\beta}{\alpha+\beta+1}+\lambda\ ,\ \beta\neq 0\ ,\ n\geq 0\ , \\ \dfrac{1}{\alpha+1}+\lambda\displaystyle\sum_{k=0}^{n-1}\dfrac{2k+\alpha+2}{(k+1)(k+\alpha+1)}\ ,\ \beta=0\ ,\ n\geq 0\ ,\ \left(\sum_{0}^{-1}=0\right)\ . \end{cases} \tag{2.51}$$

The regularity condition is $\quad d_n^{\beta}\neq 0\ ,\quad n\geq 0\ .$

Again from (1.27) and (2.50)-(2.51), we get

$$\tilde{a}_n=\frac{(n+\beta+1)(n+\alpha+\beta+1)d_{\beta,n+1}}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)d_n^{\beta}}\ ,\quad n\geq 0\ . \tag{2.52}$$

We deduce according to Proposition 1.6., (2.47), and (2.52)

$$\begin{cases} \gamma_1 \quad = -\lambda\ ,\quad \gamma_{2n+2}=\tilde{a}_n\ ,\quad n\geq 0\ , \\ \gamma_{2n+3}=\dfrac{(n+1)(n+\alpha+1)d_{\beta,n}}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)d_{\beta,n+1}}\ ,\ n\geq 0\ . \end{cases} \tag{2.53}$$

From (1.28), (1.37), (2.52), and Proposition 1.11., we get for $n\geq 0$

$$\begin{cases} \Psi(x)=(-2\alpha-2\beta-3)x^3+(2\beta+1)x\ , \\ C_0(x)=(2\alpha+2\beta-1)x^3-(2\beta-1)x\ , \\ C_1(x)=(2\alpha+2\beta+1)x^3-\big(2\beta+1+4\lambda(\alpha+\beta+1)\big)x\ , \\ C_{2n+2}(x)=(4n+2\alpha+2\beta+3)x^3+Z_n\ , \\ C_{2n+3}(x)=(4n+2\alpha+2\beta+5)x^3-Z_{n+1}\ , \\ D_0(x)=2(\alpha+\beta)x^2-2\big(\beta+\lambda(\alpha+\beta+1)\big)\ , \\ D_{2n+1}=2(2n+\alpha+\beta+2)x^2\ , \\ D_{2n+2}=2(2n+\alpha+\beta+2)\times \\ \left(x^2+\dfrac{\left(\beta^2+\delta_{0,\beta}\right)\lambda\left(\frac{\beta}{\alpha+\beta+1}+\lambda\right)\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)d_{\beta,n}d_{\beta,n+1}}\right)\ , \end{cases} \tag{2.54}$$

with $Z_n=\left(2\beta+1+\dfrac{4\left(\beta+\delta_{0,\beta}\right)\lambda\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)d_{\beta,n}}\right)x\ .$

σu is a semiclassical linear form and satisfies (2.23) with

$$\Phi^P(x)=x^2(x-1)\ ,\quad \Psi^P(x)=-\big(\alpha+\beta+2\big)x^2+\big(\beta+1\big)x\ .$$

Concerning the class of σu, we have the two different cases:

i) $\lambda\neq-\dfrac{\beta}{\alpha+\beta+1}$, the class of σu is equal to 1.

ii) $\lambda=-\dfrac{\beta}{\alpha+\beta+1}$, $\left(h_2 o\tau_{-\frac{1}{2}}\right)\sigma u=\mathcal{J}\ (\alpha,\beta-1)$.

From (1.25) and (2.53) the coefficients $\left\{\beta_n^P,\gamma_{n+1}^P\right\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ are given by

$$\beta_0^P=\gamma_1\ ,\quad \beta_{n+1}^P=\gamma_{2n+2}+\gamma_{2n+3}\ ,\quad \gamma_{n+1}^P=\gamma_{2n+1}\gamma_{2n+2}\ ,$$

where $\gamma_n\ , n\geq 1$ are given by (2.53).

The sequence $\{P_n\}_{n\geq 0}$ satisfies (2.24) with (for $n \geq 0$)

$$\frac{1}{2}\big(C_{n+1}^P(x) - C_0^P(x)\big) = (n+1)x^2 + \Lambda_n x + \Upsilon_n \ ,$$
$$D_{n+1}^P(x) = (2n+\alpha+\beta+2)\big(x+\Xi_n\big) \ ,$$
$$C_0^P(x) = (\alpha+\beta-1)x^2 - (\beta-1)x \ , \quad D_0^P(x) = (\alpha+\beta)x - \big(\lambda(\alpha+\beta+1)+\beta\big) \ ,$$

where

$$\Lambda_n = \beta + \frac{(\beta+\delta_{0,\beta})\,\lambda\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)d_n^\beta} - \\ - \frac{(n+\beta+1)(n+\alpha+\beta+1)d_{\beta,n+1}}{(2n+\alpha+\beta+1)d_{\beta,n}} \ ,$$

$$\Upsilon_n = -\frac{\big(\beta^2+\delta_{0,\beta}\big)\,\lambda\left(\lambda+\frac{\beta}{\alpha+\beta+1}\right)\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+1)}{(2n+\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)(d_{\beta,n})^2} \ ,$$

$$\Xi_n = \frac{\big(\beta^2+\delta_{0,\beta}\big)\,\lambda\left(\lambda+\frac{\beta}{\alpha+\beta+1}\right)\Gamma(\beta+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)d_{\beta,n}d_{\beta,n+1}} \ .$$

The form v has the following integral representation [8, p. 156], for $\Re(\alpha) > -1$, $\Re(\beta) > -1$, $f \in \mathcal{P}$

$$\langle v, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}|x|^{2\beta+1}(1-x^2)^\alpha f(x)dx \ . \tag{2.55}$$

Therefore, for $\Re(\alpha) > -1$, $\Re(\beta) > 0$, we obtain by (1.43)

$$\langle u, f\rangle = -\lambda\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}|x|^{2\beta-1}(1-x^2)^\alpha f(x)dx + \\ + \left(1+\frac{\lambda(\alpha+\beta+1)}{\beta}\right)f(0) \ . \tag{2.56}$$

Applying $\left(h_2 o\tau_{-\frac{1}{2}}\right)$ for (1.19), we get in this case

$$\hat{\sigma u} = \delta_{-1} - 2\lambda\big(x+1\big)^{-1}\mathcal{J}(\alpha,\beta) \tag{2.57}$$

where $\quad \hat{\sigma u} = \left(h_2 o\tau_{-\frac{1}{2}}\right)(\sigma u)$.

Remark. If $\{\hat{P}_n\}_{n\geq 0}$ is the corresponding orthogonal sequence to $\hat{\sigma u}$, then according to (1.34), we have $\quad \beta_n^{\hat{P}} = 2\beta_n^P - 1$, $\quad \gamma_{n+1}^{\hat{P}} = 4\gamma_{n+1}^P$, $\quad n \geq 0$.
We have for $\Re(\alpha) > -1$, $\Re(\beta > -1$[21]

$$\langle\mathcal{J}(\alpha,\beta), f\rangle = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta f(x)dx \ . \tag{2.58}$$

Then, from (2.57)-(2.58), we obtain

$$\langle\hat{\sigma u}, f\rangle = f(-1) \ - \ 2\lambda\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\frac{f(x)-f(-1)}{x+1}dx \ .$$

But, when $\Re(\beta) > 0$ we have

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\frac{1}{x+1}dx = \frac{\alpha+\beta+1}{2\beta} \ .$$

Therefore, for $\Re(\alpha) > -1$, $\Re(\beta) > 0$, we obtain

$$\langle \hat{\sigma u}, f\rangle = -\lambda\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-1}f(x)dx + \left(1+\frac{\lambda(\alpha+\beta+1)}{\beta}\right)f(-1) \ . \tag{2.59}$$

REMARKS. *1.* From (2.56) and (2.59), we deduce that the linear form u is the symmetrized of a Jacobi-type linear form (see [3, 11, 12, 13, 14]).
2. $Z_{2n}(x) = \frac{1}{2^n}\hat{P}_n(2x^2-1)$, $Z_{2n+1}(x) = \frac{1}{2^n}xP_n^{(\alpha,\beta)}(2x^2-1)$, $n \geq 0$.

Particular cases:

1) $\lambda = -\frac{\beta}{\alpha+\beta+1}$, $u = GG(\alpha, \beta-1)$ (see [5]).

2) $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $u = \delta_0 - \lambda Pf\frac{1}{\pi}\frac{Y(1-x^2)}{x^2\sqrt{1-x^2}}$ (see [1]), with the definition

$$\left\langle Pf\frac{Y(1-x^2)}{x^2\sqrt{1-x^2}}, f\right\rangle = \lim_{\epsilon\to 0}\left(\int_{-1}^{-\epsilon}\frac{f(x)\sqrt{1-x^2}}{x^2}dx + \int_{\epsilon}^{1}\frac{f(x)\sqrt{1-x^2}}{x^2}dx\right)$$

where Y is the characteristic function of $\mathbb{R}^+$.

3) $\alpha = \beta = \frac{1}{2}$, $\lambda = -\frac{1}{8}$, $u = \frac{1}{2}\delta_0 + 2\mathcal{U}$ where $\mathcal{U}$ is a Tchebychev form of second kind. In this case, the sequence $\{Z_n\}_{n\geq 0}$ satisfies (1.22) with

$$\gamma_{2n+1} = \frac{n+1}{4(n+2)} \ , \quad \gamma_{2n+2} = \frac{n+3}{4(n+2)} \ , \quad n \geq 0 \ .$$

In a very interesting work [10], J. Charris, G. Salas and V. Silva studied this sequence of orthogonal polynomials.

Remark. Theorem 2.3. is the main result of our paper. From it, we carry out the complete description of the symmetric semiclassical linear forms of class $s = 2$, when $\Phi(0) = 0$. Unfortunately, the case when $\Phi(0) \neq 0$ is not covered by this theorem and the description of these linear forms remains open.

Acknowledgment. We strongly believe that the referees contributions and support do make the difference in the development of this project. So, we owe a great deal to the dedicated referees whose considerable comments, helpful suggestions and valuable references have done so much to illustrate this work.

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