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Ramifications, differentials and differente on algebraic varieties of higher dimensions

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In this paper we shall study some interrelations lying among the theory of ramifications of divisors, differential forms and the differente in the covering theory of algebraic varieties of higher dimensions. In the first two paragraphs the preparatory courses will be given. In § 1 we shall recall some notions and definitions on the coverings of normal varietes and branch loci, and then in § 2 we shall prove snme elementary lemmas on uniformizing parameters which are necessary in the following investigations. After these preliminaries we shall characterize the subvarieties of codimension 1 which are ramified with respect to the covering in terms of differential forms (§ 3). In the case when the characteristic of the universal domain is zero, the ramifications of divisors can be completely described in terms of the ramification theory of discrete valuation rings and it is not necessary to use the language of differential forms. But in the modular case the matter is quite different and complicated in the case of non-tame coverings. This is one of the motivation of the present paper. In § 4 we shall study some properties of differential forms finite at a normal point. Among other things we can prove the following result. "A differential form (of highest degree) finite at a normal singular point have a zero variety containing the reference point". From this we can see why the branch loci are of purely codimension 1 at a simple point in a clearly understandable forms (§ 5). The characterization of the ramified divisors in terms of differential forms have another application. We shall define an integer $e(D^*)$, called *differential index*, for a subvariety D^* of codimension 1 of V^*.

It has the property that D^* is ramified for the convering if, and only if, $e(D^*)>0$, and this integer represent, to some extent, the order of ramifications. Since the branch loci form a bunch of varietes, the formal sum $\mathfrak{d}=\sum_{D^*} e(D^*)D^*$ represent a divisor. The famous formula of Hurwitz on canonical divisor and differente on algebraic curves can be generalized in a natural way in terms of the divisor $\mathfrak{d}$. Thus the divisor $\mathfrak{d}$ will properly be called the *differente* of the covering. At the end we shall discuss on the Jacobian set of the linear systems from the point of view of the covering theories.

§ 1. Coverings of varieties.

Let V and V^* be normal varieties defined over an algebraically closed field k, and let π be a rational map from V^* onto V defined over k. We shall say that V^* is a covering of the variety V with the *covering map* π if they satisfy the following conditions: [1]

(a) The rational map π is defined everywhere on V^*.

(b) For all points v on V, the point set $\pi^{-1}(v)$ consists of a finite number of points.

(c) The variety V^* is complete over any point of V.

(d) π is a separable map, i.e. if P^* and P are the corresponding generic points of V^* and V over the field k respectively, then $k(P^*)$ is separably algebraic over $k(P)$.

Let $K^*=k(V^*)$ and $K=k(V)$ be respectively the field of rational functions on V^* and V defined over k. Then as is well known the variety V^* is nothing other than the normalization of V in the field K^*. Conversly if the separable extension K^* of $K=k(V)$ is given, then the normalization V^* of V in K^* gives rise to the covering V^* of V and it is determined uniquely up to a biregular transformation over k. Thus there exists a 1-1 correspondence between a covering of a normal variety and a separable extension of the function field of V.

Let V^* and V be as before and let v be a point on V, and W the locus of v over k. Let $\mathfrak{o}=Q_k(W/V)$ be the quotient ring of W in V/k and let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{o}$. We shall ex-

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1) We borrowed this definition of covering from [6].

press this fact by saying "$(\mathfrak{o}, \mathfrak{m})$ *is the quotient ring of* W". Let v^* be a point on V^* corresponding to v, i.e. $\pi(v^*)=v$. Then the locus W^* of v^* over k is a subvariety of V^* corresponding to W. Let $(\mathfrak{o}^*, \mathfrak{m}^*)$ be the quotient ring of W^* in V^*/k. Then $(\mathfrak{o}^*, \mathfrak{m}^*)$ is a local ring in $K^*=k(V^*)$ lying above[2] $(\mathfrak{o}, \mathfrak{m})$. We shall say that the point v^* (or the subvariety W^*) is *unramified* over the point v (or the subvariety W) if the following conditions are satisfied ;

(1) $\mathfrak{m}^* = \mathfrak{m}\mathfrak{o}^*$

(2) $\mathfrak{o}^*/\mathfrak{m}^*$ is separably algebraic over $\mathfrak{o}/\mathfrak{m}$.

In the contrary case $v^*(W^*)$ is said to be *ramified* over $v(W)$. Let v, W and $(\mathfrak{o}, \mathfrak{m})$ be as before and let $\mathfrak{o}_i^*$ $(i=1, 2, \cdots, h)$ be the local rings of K^* lying above $\mathfrak{o}$. Then there exist h subvarieties W_i^* $(i=1, 2, \cdots, h)$ defined over k such that $\mathfrak{o}_i^*$'s are exactly the quotient rings of W_i^*'s on V^*/k. Let v_i^* be a generic point of W_i^* corresponding to v. Then we shall say that the point v is a *branch point* (or the subvariety W is a *branch subvariety*) of the covering K^*/V, if there exist at least one index i such that the point v_i^* (or the subvariety W_i^*) is ramified over v (or over W). According to Krull ([5]) the point v is not a branch point if we have

$$\sum_{i=1}^{h}[k(W_i^*):k(W)]_s = \sum_{i=1}^{h}[(\mathfrak{o}_i^*/\mathfrak{m}_i^*):(\mathfrak{o}/\mathfrak{m})]_s = [K^*:K]$$

In our case the converse is also true ([2]). Let us denote by $\Delta(K^*/V)$ the set of the branch points on V with respect to the covering K^*/V. Then it is known that there exist a finite number of branch subvarieties $\{B_j\}$ defined over k such that any branch subvariety (point) is contained in at least one of B_j's ([1]). We shall call $\Delta(K^*/V)$ the *branch loci* of the covering K^*/V.

§ 2. Uniformizing parameters.

Let V^r be a variety and let P be a simple point on V. We shall recall here the definitions of uniformizing parameters at P on V. Let $f_1, \cdots, f_r$ be functions on V, then we shall say that $f_1, \cdots, f_r$ are a set of *uniformizing parameters* at a simple point P if they satisfy the following conditions :

2) A local ring $\mathfrak{o}^*$ is said to lie above $\mathfrak{o}$, if $\mathfrak{o}^*$ is the quotient ring of the integral closure of $\mathfrak{o}$ in K^* with respect to its maxima ideal.

(1) The functions f_i's are defined and finite at P.

(2) The zero divisors of the functions $(f_i - f_i(P))$ intersect properly, locally at P, with the multiplicity 1.

The condition (2) is equivalent to the following[3)]

(2′) Let k be common field of definition for V over which P is rational, and let $(\mathfrak{o}, \mathfrak{m})$ be the quotient ring of P in V/k. Then $u_i = f_i - f_i(P)$ $(i=1, 2, \cdots, r)$ are a regular system of parameters of the local ring $\mathfrak{o}$.

DEFINITION 1. A differential form ω of degree r on V^r is said to be *finite* at a point P on V, if ω can be written in the form $\omega = a\, dt_1 \wedge \cdots \wedge dt_r$, where a and t_i's are the functions regular at P. We shall say that ω is *zero at* P, if in the above expression we can take a in such a way that $a(P)=0$.

PROPOSITION 1. *Let P be a simple point of V and let $t_1, \cdots, t_r$ be the functions on V regular at P and take the value* 0 *at P. Then $t_1, \cdots, t_r$ are a set of uniformizing parameters at P on V if, and only if, the differential $\omega = dt_1 \wedge \cdots \wedge dt_r$ is not zero at P.*

PROOF. Necessity is obvious. Let k be a field of definition of V such that P is rational over k and that the functions t_i's are all defined over k. Let $(\mathfrak{o}, \mathfrak{m})$ be the quotient ring of P in V/k, then $\mathfrak{o}$ is a regular local ring since P is a simple point on V. We shall show that t_i's are linearly independent modulo $\mathfrak{m}^2$ if $dt_1 \wedge \cdots \wedge dt_r$ is not zero at P. Assume the contrary, then there exist constants c_i's in k such that $\sum_{i=1}^{r} c_i t_i \equiv 0$ (mod $\mathfrak{m}^2$). Let $u_1, \cdots, u_r$ be a regular system of parameters of $\mathfrak{o}$. Then we have $\sum_{i=1}^{r} c_i t_i = \sum_{i,j=1}^{r} a_{ij} u_i u_j$ with a_{ij}'s in $\mathfrak{o}$. Now assume that $c_1 \neq 0$. Then multiplying both side of the equality by the differential $dt_2 \wedge \cdots \wedge dt_r$, we get

$$c_1 dt_1 \wedge \cdots \wedge dt_r = \Big(\sum_{i,j=1}^{r} (a_{ij} u_i du_j + a_{ij} u_j du_i + u_i u_j da_{ij})\Big) dt_2 \wedge \cdots \wedge dt_r.$$

Expressing the differentials da_{ij} and dt_i in terms of du_j's we see at once that the right hand side of the above equation vanishes at P, which is a contradiction.

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3) Cf. e.g. [8] and [12].

It is known that if ω is zero at a simple point P, then P is contained in the zero divisor of the differential ω ([4]). From this we can easily get the

COROLLARY 1. *Let $t_1, \cdots, t_r$ be functions on a variety V^r such that $dt_1 \wedge \cdots \wedge dt_r \neq 0$. Let Ω be the complementary set of the following bunch of subvarieties on V. 1) Multiple subvarieties of V, 2) polar varieties of the functions t_i's, 3) the zero divisor of the differential form $dt_1 \wedge \cdots \wedge dt_r$. Then the r functions t_i's are a set of uniformizing parameters at any point in Ω.*

Let W be a simple subvariety of V, then the functions $t_1, \cdots, t_r$ are said to be a set of uniformizing parameters along W on V if they are a set of uniformizing parameters at some point P on W (P is necessarily a simple point of V). The following Corollary is easy and the proof will be omitted.

COROLLARY 2. *Let $t_1, \cdots, t_r$ be a set of uniformizing parameters along a simple subvariety W on V. Then there exists a point P on W, not lying in the preassigned bunch of W, such that t_i's are a set of uniformizing parameters at P on V.*

PROPOSITION 2. *Let W be a subvariety of V, P a point on W which is simple both on V and on W, and let $t_1, \cdots, t_r$ be a set of uniformizing parameters at P on V. Let $\bar{t}_1, \cdots, \bar{t}_r$ be the trace of the functions $t_1, \cdots, t_r$ on W, then we can select a set of uniformizing parameters at P on W from among the functions $\bar{t}_1, \cdots, \bar{t}_r$.*

For the proof we shall refer to the paper [8] (Cor. 1 of Th. 1).

§ 3. Branch loci and differential forms.

PROPOSITION 3. *Let V^* be a covering of a normal variety V with the covering map π, and let k be an algebraically closed field of definition for V^*, V and π. Let W^* be a subvariety of V^* defined over k and let W be the corresponding subvariety of V. Assume that W is a simple subvariety of V and W^* is unramified over W, then any set of uniformizing parameters along W on V are still so along W^* on V^*.*

PROOF. Let $t_1, \cdots, t_r$ be a set of uniformizing parameters along W on V. Since W is not a branch variety, the set of points of W which are branch point for the covering form a proper bunch on W. The Cor. 2 of Prop. 1 asserts the existence of a point P on W rational over k such that; 1) P is a simple point of V, 2) P is not a branch point, 3) $t_1, \cdots, t_r$ are a set of uniformizing parameters at P. Let P^* be a point in $\pi^{-1}(P)$ contained in W^*, and let $\mathfrak{o}^*$ and $\mathfrak{o}$ be the quotient rings of P^* and P in V^*/k and V/k respectively. Then since $\dim \mathfrak{o}^* = \dim \mathfrak{o}$, the first condition of the unramifiedness implies that $\mathfrak{o}^*$ is a regular local ring and $(t_i - t_i(P))$'s are a regular system of parameters of $\mathfrak{o}^*$. Since P^* is k-rational, it is a simple point of V^* and t_i's are a set of uniformizing parameters at P^*, and hence along W^*, on V^*.

Let V be a normal variety and let V^* be a covering of V with the covering map π defined over an algebraically closed field k. Let P be a simple point of V rational over k and $t_1, \cdots, t_r$ a set of uniformizing parameters at P on V. We shall put $\omega = dt_1 \wedge \cdots \wedge dt_r$. Let π^* be the adjoint map of π which maps the space of differential forms on V into the space of differential forms on V^*. Since π is a separable map π^* is injective and $\pi^*\omega$ is a differential on V^* different from zero. Using these notations and assumptions we have the

PROPOSITION 4. *The component of $(\pi^*\omega)$ passing through one of the points in $\pi^{-1}(P)$ is independent of the choice of uniformizing parameters t_i's at P.*

PROOF. Let $t'_1, \cdots, t'_r$ be another set of uniformizing parameters at P and let $\omega' = dt'_1 \wedge \cdots \wedge dt'_2$. Then we have the relation $\omega' = A\omega$, where A must be a unit at P since t_i's and (t'_i)'s are uniformizing parameters at P. Hence A is, a fortiori, a unit at any one of the point in $\pi^{-1}(P)$. Thus the divisor $(\pi^*\omega)$ and the divisor $(\pi^*\omega')$ differ only in the components free from the points in $\pi^{-1}(P)$.

THEOREM 1. *Let V^* be a covering of a normal variety V with the covering map π defined over an algebraically closed field k. Let P be a simple point on V rational over k and let $t_1, \cdots, t_r$ be a set of uniformizing parameters at P defined over k. Let $\omega = dt_1 \wedge \cdots \wedge dt_r$ and assume that a subvariety D^* of V^* of codimension* 1, *containing at least one of the points in $\pi^{-1}(P)$, is a zero of the differential $\pi^*\omega$. Then D^* is ramified for the covering V^*/V. Conversely let D^* be a subvariety of condimension* 1 *which is ramified over a subvariety D of V. Let $t_1, \cdots, t_r$ be a set of uniformizing parameters along D*

on V and let $\omega = dt_1 \wedge \cdots \wedge dt_r$. Then D^* is a component of the zero of the differential $\pi^*\omega$.

PROOF. First we assume that D^* is unramified over $\pi(D^*) = D$.[4] Since D^* contains a point in $\pi^{-1}(P)$, D contains the point P. The functions $t_1, \cdots, t_r$ are a set of uniformizing parameters at P on V, hence also a set of uniformizing parameters along D. By Prop. 3 they are still so along D^* on V^*, hence D^* cannot be a component of the zero of the differential $\pi^*\omega$.

Conversly let D be a branch subvariety and D^* is ramified over D. Then, at first D^*, hence also D is defined over k. Let (R, M) and (R^*, M^*) be the quotient rings of D and D^* in V/k and V^*/k respectively. Since D and D^* are simple on V and V^*, R is a discrete valuation ring in $K = k(V)$ and R^* is its extension in $K^* = k(V^*)$. By our assumption we have either

(a) The ramification index e of R^* over R is greater than 1, i.e. $MR^* = M^{*e}$, $e > 1$.

(b) $e = 1$, and $[k(D^*) : k(D)]_i = [(R^*/M^*) : (R/M)]_i > 1$.

The proof will be divided into two cases.

The case (a). Let t and t^* be prime elements of R and R^* respectively. It is possible to find the functions $t_2, \cdots, t_r$ on V such that $t, t_2, \cdots, t_r$ are a set of uniformizing parameters along D. From Prop. 4 it is without restriction to assume that $\omega = dt \wedge dt_2 \wedge \cdots \wedge dt_r$. On the other hand we have $t = \alpha t^{*e}$ with α in R^*, $\alpha \notin M^*$. Hence we get

$$\pi^*\omega = e\alpha t^{*e-1} dt^* \wedge dt_2 \wedge \cdots \wedge dt_r + t^{*e} d\alpha \wedge dt_2 \wedge \cdots \wedge dt_r .$$

Since $e > 1$ the expression on the right hand side show that $\pi^*\omega$ contains D^* as a zero variety.

The case (b). In this case a prime element t of R is still a prime element of R^*. Let $t = t_1, t_2, \cdots, t_r$ be a set of uniformizing parameters along D on V and set $\omega = dt_1 \wedge dt_2 \wedge \cdots \wedge dt_r$. Since t is a prime element of R^* it is possible to find a set of uniformizing parameters along D^* containing t. Let $t, u_2, \cdots, u_r$ be such a set. Let us denote by $\bar{\ }$ the trace of the functions on D^* (or equivalently M^* residues). Then by Proposition 3, $\bar{u}_2, \cdots, \bar{u}_r$

4) Here $\pi(D^*)$ simply means the set theoretic image of D^* under π.

form a separating transcendence basis of $k(D^*)$ over k. From this fact we can find $(r-1)$ polynomials $G_\nu(X_2, \cdots, X_r; Y_2, \cdots, Y_r)$ $(\nu=1, \cdots, r-1)$ in $k[X; Y]$ satisfying the conditions;

(1) $G_\nu(\bar{t}_2, \cdots, \bar{t}_r; \bar{u}_2, \cdots, \bar{u}_r) = 0 \quad (\nu = 1, \cdots, r-1)$

(2) $\det|\partial G_\nu/\partial \bar{t}_i| \neq 0 \quad (\nu=1, \cdots, r-1; i=2, \cdots, r)$

In this case we must have

(3) $\det|\partial G_\nu/\partial \bar{u}_i| = 0$

since $k(\bar{u})$ is not separably algebraic over $k(\bar{t})$ by our assumption (b). From (1) and (3) we can put

(4) $G_\nu(t_2, \cdots, t_r; u_2, \cdots, u_r) = t\alpha_\nu \quad (\nu = 1, \cdots, r-1)$

(5) $(-1)^{r-1}\det|\partial G_\nu/\partial u_i| = t\beta$

where α and β are the functions contained in R^*. Taking the differentials of the both side of (4) we get

$$\sum_{i=2}^{r}(\partial G_\nu/\partial t_i)dt_i + \sum_{j=2}^{r}(\partial G_\nu/\partial u_j)du_j = \alpha_\nu dt + t d\alpha_\nu .$$

From this we get

$$\bigwedge_{\nu=1}^{r-1}\Big[\sum_{i=2}^{r}(\partial G_\nu/\partial t_i)dt_i\Big] = \bigwedge_{\nu=1}^{r-1}\Big[\sum_{j=2}^{r}(-\partial G_\nu/\partial u_j)du_j + \alpha_\nu dt + t d\alpha_\nu\Big]$$

i.e.

$$\det|\partial G_\nu/\partial t_i|\, dt_2 \wedge \cdots \wedge dt_r$$
$$(-1)^{r-1}\det|\partial G_\nu/\partial u_j|\, du_2 \wedge \cdots \wedge du_r + t\omega_1 + dt \wedge \omega_2$$

where ω_1 is a differential form of degree $r-1$ finite along D^* and ω_2 is a differential of degree $r-2$. Multiplying both sides by $dt = dt_1$ and substituting (5) we get the equation

$$\det|\partial G_\nu/\partial t_i|\, dt_1 \wedge \cdots \wedge dt_r$$
$$= t(\beta + \beta^*)\, dt_1 \wedge du_2 \wedge \cdots \wedge du_r$$

where β^* is a regular function along D^*. Since $\det|\partial G_\nu/\partial t_i|$ is regular but not zero along D^* we can see that the differential form $\pi^*\omega$ contains the variety D^* as a zero variety. Thus the proof is complete.

§4. Differential forms finite at a point.

Let V^r be a variety defined over a field k and let P be a point on V. Let us denote by $'\mathfrak{F}_P^k$ the set of r-fold differential forms defined over k finite at the point P, and by $\mathfrak{o}_P^k$ the local rings of functions defined over k regular at P. Let $\mathfrak{F}_P^k$ be the module generated over $\mathfrak{o}_P^k$ by $'\mathfrak{F}_P^k$, then as we can see easily $\mathfrak{F}_P^k$ is a finite $\mathfrak{o}_P^k$ module. Moreover if P is a simple point of V rational over k and if $t_1, \cdots, t_r$ are a set of uniformizing parameters at P, then $\mathfrak{F}_P^k$ is easily seen to be of rank 1 and $dt_1 \wedge \cdots \wedge dt_r$ is a base of $\mathfrak{F}_P^k$ over $\mathfrak{o}_P^k$. The object of this paragraph is to settle the converse problem.

PROPOSITION 5. *Let $\mathfrak{F}_P^k$ be a free $\mathfrak{o}_P^k$-module and let $\omega = dt_1 \wedge \cdots \wedge dt_r$ be a base of $\mathfrak{F}_P^k$. Then the r-derivations*[5] *$\partial/\partial t_i$ $(i=1, 2, \cdots, r)$ are regular at P, i.e. $\partial/\partial t_i$ maps $\mathfrak{o}_P^k$ into $\mathfrak{o}_P^k$. Conversely let $t_1, \cdots, t_r$ be elements of $\mathfrak{o}_P^k$ such that $dt_1 \wedge \cdots \wedge dt_r \neq 0$ and r-derivations $\partial/\partial t_i$ are regular at P, then $\mathfrak{F}_P^k$ is a free $\mathfrak{o}_P^k$-module and $dt_1 \wedge \cdots \wedge dt_r$ is its base.*

PROOF. Assume that $dt_1 \wedge \cdots \wedge dt_r$ is a $\mathfrak{o}_P^k$-base of $\mathfrak{F}_P^k$, and some of the derivations $\partial/\partial t_i$, say $\partial/\partial t_1$, is not regular at P. Then there exists an element a in $\mathfrak{o}_P^k$ such that $\partial a/\partial t_1$ is not in $\mathfrak{o}_P^k$. From this we see that $da \wedge dt_2 \wedge \cdots \wedge dt_r$ (which is in $\mathfrak{F}_P^k$ by definition)$=(\partial a/\partial t_1) dt_1 \wedge \cdots \wedge dt_r$ is not contained in $\mathfrak{o}_P^k \omega$. Contradiction! The converse is immediate.

PROPOSITION 6. *Let $\mathfrak{F}_P^k$ be a free $\mathfrak{o}_P^k$ module with the base $\omega = dt_1 \wedge \cdots \wedge dt_r$. Let V_0 be an affine representative of V containing the point P and let $x_1, \cdots, x_N$ be the coordinate functions on V_0. Then there exist indices $\alpha_1, \cdots, \alpha_r$ such that $dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_r}$ is also a base of $\mathfrak{F}_P^k$.*

PROOF. Expressing the t_i's as the quotient of polynomials in x's, we have $t_i = f_i(x)/g_i(x)$, with $g_i(P) \neq 0$. Taking the total differentials of both sides we have

$$dt_i = \sum_{j=1}^{N} \{[(\partial f_i/\partial x_j) g_i(x) - f_j(x)(\partial g_i(x)/\partial x_j)]/g_i(x)^2\}\, dx_j$$

and hence we get

$$dt_1 \wedge \cdots \wedge dt_r = \sum A_{\beta_1 \cdots \beta_r} dx_{\beta_1} \wedge \cdots \wedge dx_{\beta_r}$$

where the sum is extended over all the indices $\beta_1 < \beta_2 < \cdots < \beta_r$ taken from $1, 2, \cdots, N$, and $A_{\beta_1 \cdots \beta_r}$'s are all in $\mathfrak{o}_P^k$. Since

5) By $\partial/\partial t_i$ we mean the derivation D_i such that $D_i t_j = \delta_{ij}$ $(i, j = 1, 2, \cdots r)$.

$dx_{\beta_1}\wedge\cdots\wedge dx_{\beta_r}$ are in $\mathfrak{F}_P^k$, we have $dx_{\beta_1}\wedge\cdots\wedge dx_{\beta_r}=B_{\beta_1\cdots\beta_r}dt_1\wedge\cdots\wedge dt_r$ with $B_{\beta_1\cdots\beta_r}$'s in $\mathfrak{o}_P^k$. Substituting these relations in the preceding one we get the equality

$$1=\sum_{\beta_1<\cdots<\beta_r}A_{\beta_1\cdots\beta_r}B_{\beta_1\cdots\beta_r}$$

which proves the existence of indices $\beta_1,\cdots,\beta_r$ such that $B_{\beta_1\cdots\beta_r}$ is a unit in $\mathfrak{o}_P^k$. Hence for this set of indices, $dx_{\beta_1}\wedge\cdots\wedge dx_{\beta_r}$ is a base of $\mathfrak{F}_P^k$ over $\mathfrak{o}_P^k$.

THEOREM 2. *Let V^r be a variety defined over a field k and let P be a point rational over k. Assume that $\mathfrak{F}_P^k$ is a free $\mathfrak{o}_P^k$ module with the base $\omega=dt_1\wedge\cdots\wedge dt_r$, then P is a simple point of V if one of the following conditions holds;*

(1) *The characteristic of our universal domain is zero.*
(2) *The characteristic p is >0, and k is a perfect field, and P is normal over k.*

Moreover in this case $t_1,\cdots,t_r$ are a set of uniformizing parameters at P on V.

PROOF. On account of Prop. 5, we see that the case (1) is due to Nagata (Theorem of [3]). The case (2) is due to Zariski (Cf. the proof of Prop. 2 in [13]).

As an application of Theorem 2 we can prove the following interesting behaviour of a differential form in the neighbourhood of a normal singular point.

THEOREM 3. *Let V^r be a variety defined over a field k and let P be a normal point rational over k. Let ω be a differential form of degree r defined over k and finite at P. Then if P is a singular point and k is a perfedt field, then there exists at least one component in the zero divisor of ω such that it contains the point P.*

PROOF. By our assumption P is a singular point, hence by Th. 2 the module $\mathfrak{F}_P^k$ of differential forms finite at P is not of rank 1 over $\mathfrak{o}_P^k$. Let ω be an arbitrary differential finite at P, then by definition we can write ω in the form $\omega=a\,dt_1\wedge\cdots\wedge dt_r$ with the functions $a,t_1,\cdots,t_r$ in $\mathfrak{o}_P^k$. Since the rank of $\mathfrak{F}_P^k$ is greater than 1 some of the derivations $\partial/\partial t_i$ is not regular at P by Prop. 5, i.e. there exist an element b in $\mathfrak{o}_P^k$ and an index i such that $\partial b/\partial t_i$ is not in $\mathfrak{o}_P^k$. We assume that $b_1=\partial b/\partial t_1$ is not in $\mathfrak{o}_P^k$.

Then $(b_1)_\infty$ contains a component passing through P, because otherwise b_1 will be an element of $\mathfrak{o}_P^k$ since P is a normal point. Hence $\omega = a\,dt_1 \wedge \cdots \wedge dt_r = ab_1^{-1} db \wedge dt_2 \wedge \cdots \wedge dt_r$ and the divisor of ω is equal to $(a)+(b_1^{-1})+(db \wedge dt_2 \wedge \cdots \wedge dt_r)$. Let D be a component of $(b_1)_\infty$ which contains the point P. Then D appears in (b_1^{-1}) with a positive coefficient and in (a) and in $(db \wedge dt_1 \wedge \cdots \wedge dt_r)$ with non-negative coefficients. Thus D appears in (ω) with a positive coefficient, i.e. D is a zero variety of ω containing P.

§ 5. The purity of branch loci.

We shall apply the foregoing results to the proof of the purity of branch loci at a simple point in the case of geometric covering.

THEOREM 4. *Let V^* be a covering of a normal variety V defined over an algebraically closed field k and let W be a branch subvariety of V defined over k. Let W^* be a subvariety of V^* corresponding to W such that W^* is ramified over W. Assume that W is a simple subvariety of V, then there exists a subvariety D^* of codimension* 1 *containing W^* such that D^* is ramified over the corresponding subvariety D of V.*

PROOF. We shall first remark that it is sufficient to prove the case when W is 0-dimensional. In fact assume that the theorem is proved for the 0-dimensional case, and assume that there exists no ramified subvariety of codimension 1 containing W^*. Then there exists a k-open set Ω in W^* such that any point in Ω is not contained in any ramified subvariety of codimension 1. Thus we shall arrive at a contradiction. Let P be a simple point rational over k and let P^* be a point in $\pi^{-1}(P)$ such that P^* is ramified over P. Let $t_1, \cdots, t_r$ be a set of uniformizing parameters at P on V and let $\omega = dt_1 \wedge \cdots \wedge dt_r$. We shall divide into the two cases.

(1) The case when P^* is a simple point. Let $u_1, \cdots, u_r$ be a set of uniformizing parameters at P^* on V^* and let $\pi^*\omega = \alpha du_1 \wedge \cdots \wedge du_r$. Then the function α (regular at P) cannot be a unit at P^*, otherwise $dt_1 \wedge \cdots dt_r$ become a base of $\mathfrak{F}_{P^*}^k$ over $\mathfrak{o}_{P^*}^k$ and by Theorem 2, t_i's are a set of uniformizing parameters at P^* on V^*. Since P and P^* are rational over k this implies that P^* is unramified over P, against our assumption. Let D^* be a zero divisor of (α) containing P^*, then D^* is the zero variety of $\pi^*\omega$. By Theorem 1, D^* is ramified for the covering V^*/V.

(2) The case when P^* is a singular point. In this case we use Theorem 3. Since $\pi^*\omega$ is finite at P^* there exists a zero variety D^* of the differential $\pi^*\omega$ containing P^*. D^* is ramified for the covering on account of Theorem 1.

§ 6. Differente $\mathfrak{d}(V^*/V)$.

Let V^* be a covering of a complete normal variety V with the covering map π defined over an algebraically closed field k. Let D^* be a subvariety of codimension 1 on V^* and let D be the corresponding subvariety of V. Let $t_1, \cdots, t_r$ be a set of uniformizing parameters along D and let $\omega = dt_1 \wedge \cdots \wedge dt_r$. By Prop. 4 the multiplicity of D^* in the zero divisor of the differential $\pi^*\omega$ is a well defined integer independent of the choice of uniformizing parameters t_i's. We shall denote this integer by $e(D^*)$ and call it the *differential index* of D^*. Now we can restate the Theorem 1 in a simplified form

THEOREM 1′. *Let D^* be a subvariety of codimension* 1 *on V^*, then D^* is ramified for the covering V^*/V if, and only if, the differential index $e(D^*)$ is positive.*

With the help of the differential index we define the formal sum

$$\mathfrak{d} = \sum e(D^*)D^*$$

where the sum is extended over all subvarieties of codimension 1 of V^*. Since $\Delta(V^*/V)$ is a bunch on V, $e(D^*)=0$ except a finite number of subvarieties of V^*. Hence $\mathfrak{d}$ represents actually a V^*-divisor. We shall call $\mathfrak{d}$ the *differente* of the covering V^*/V and sometimes we shall denote it by $\mathfrak{d}(V^*/V)$. The nomenclature will be justified in the following

THEOREM 5 (*Generalization of Hurwitz's formula*). *Let V^* and V be as before and let $\mathfrak{K}^*$ and $\mathfrak{K}$ be the canonical divisors on V^* and V respectively. Then we have*

$$\mathfrak{K}^* \sim \pi^{-1}(\mathfrak{K}) + \mathfrak{d}$$

where $\mathfrak{d}$ is the differente of the covering V^*/V. More precisely let ω be a differential from of degree r on V, then we have the following equality

$$(\pi^*\omega) = \pi^{-1}((\omega)) + \mathfrak{d}\,.^{6)}$$

6) We mean by $\pi^{-1}(X)$, the algebro-geometric inverse inage of X.

For the proof the following proposition is necessary.

PROPOSITION 7. *Let V be a complete normal variety and let V^* be a covering of V with the covering map π. Let f be a function on V, then $f\circ\pi$ is a function on V^* and we have $(f\circ\pi)=\pi^{-1}((f))$.*

PROOF. When V is a non-singular variety it is proved elsewhere under weaker assumptions ([9]). Let k be a common field of definitions for the entities appeared above, and let P^* and P be the corresponding generic points of V^* and V over k. Let Λ, Γ_f and $\Gamma_{f\circ\pi}$ be the graphs of the rational map π, the numerical functions f and $f\circ\pi$ respectively and let T be the locus of $P^*\times P\times f(P)$ over k. Then by an easy calculus we get

$$(V^*\times\Gamma_f)\cdot(\Lambda\times D) = T+X\times D^{7)}$$

where X is a $V^*\times V$-divisor. Now if we denote by Θ the cycle $(0)-(\infty)$ on the projective straight line D, then

$$\pi^{-1}((f)) = \mathrm{pr}_{V^*}[\{(\Lambda\times D)\cdot(V^*\times\Gamma_f)\}\cdot(V^*\times V\times\Theta)]$$

We shall apply Th. 16 of Chap. VII in [11] to the varieties $V^*\times D$, V and the cycles $(\Lambda\times D)\cdot(V^*\times\Gamma_f)$ and $V^*\times\Theta$. Though in the cited theorem it is assumed that V is non-singular, the careful inspection of the proof allows us to use that theorem to our case. Thus we have

$$\begin{aligned}\pi^{-1}((f)) &= \mathrm{pr}_{V^*}[\mathrm{pr}_{V^*\times D}\{(\Lambda\times D)\cdot(V^*\times\Gamma_f)\}\cdot(V^*\times\Theta)]\,.\\ &= \mathrm{pr}_{V^*}[\{\Gamma_{f\circ\pi}+(\mathrm{pr}_{V^*}X)\times D\}\cdot(V^*\times\Theta)]\\ &= (f\circ\pi)\end{aligned}$$

Thus the proof is complete.

PROOF of THEOREN 5. Let D^* be a subvariety of V^* of codimension 1 and let $\pi(D^*)=D$. Let $t_1,\cdots,t_r$ be a set of uniformizing parameters along D and $\omega=w\,dt_1\wedge\cdots\wedge dt_r$. Then $v_{D^*}(\pi^*\omega)=v_{D^*}(w\circ\pi)+v_{D^*}(\pi^*(dt_1\wedge\cdots\wedge dt_r))$, and $v_{D^*}(w\circ\pi)$ is equal to the multiplicity of D^* in $\pi^{-1}((w))$ by Prop. 7. On the other hand the divisor of the function w and the differential ω differ only in the subvarieties not lying under D^*. Hence the coefficients of D^* in $\pi^{-1}((w))$ and that of D^* in $\pi^{-1}((\omega))$ are the same. On the other hand $v_{D^*}(\pi^*(dt_1\wedge\cdots\wedge dt_r))=v_{D^*}(\mathfrak{d})$ by the definition of the differente.

7) See the proof of Lemma 1 in [9].

Thus we get the relation

$$v_{D^*}(\pi^*\omega) = v_{D^*}(\pi^{-1}((\omega)))+v_{D^*}(\mathfrak{d}) .$$

Since this equality holds for any subvariety D^* of condimension 1, we get the required results.

THEOREM 6 (*Chain theorem*). *Let V^*, V' and V be normal varieties defined over an algebraically closed field k and assume that V^* is the covering of V' with the covering map λ, and V' is a covering of V with the covering map μ. Then V^* is the covering of V and we have*

$$\mathfrak{d}(V^*/V) = \mathfrak{d}(V^*/V')+\lambda^{-1}(\mathfrak{d}(V'/V)) .$$

PROOF. Let us put $\pi=\mu\circ\lambda$, and let ω be an arbitrary differential form on V. From Theorem 5 we have

$$(1) \qquad (\pi^*\omega) = \pi^{-1}((\omega))+\mathfrak{d}(V^*/V)$$

On the other hand, since $\pi^*=(\mu\circ\lambda)^*=\lambda^*\circ\mu^*$, $\pi^*\omega=\lambda^*(\mu^*\omega)$. Applying Th. 5 again to this relation we get

$$(\pi^*\omega) = \lambda^{-1}((\mu^*\omega))+\mathfrak{d}(V^*/V')$$

and

$$(\mu^*\omega) = (\mu^{-1}(\omega))+\mathfrak{d}(V'/V) .$$

Substituting the latter one in the former we get

$$(2) \qquad (\pi^*\omega) = \lambda^{-1}(\mu^{-1}(\omega))+\lambda^{-1}(\mathfrak{d}(V'/V))+\mathfrak{d}(V^*/V')$$

Combining (1) and (2) and the fact $\lambda^{-1}(\mu^{-1}((\omega)))=\pi^{-1}((\omega))$ we get the required result.

In the following we shall indicate the relation between the differential index and the ramification index of valuations under some additional conditions.

Let V^* be, as before, a covering of a normal variety defined over an algebraically closed field of definition k. Let D^* be a subvariety of codimension 1 on V^* and let D be a subvariety of V corresponding to D^*. Let R^* and R be the quotient rings of D^* and D respectively. Then R^* and R are rings of discrete valuations of $k(V^*)$ and $k(V)$ respectively over k such that $R=R^*\cap k(V)$.

PROPOSITION 8. *Let e be the ramification index of R^* over R, and assume that*

(1) $e \not\equiv 0 \pmod{p}$
(2) *$k(D^*)$ is separably algebraic over $k(D)$.*

Then the differential index $e(D^*)$ is equal to $e-1$.

PROOF. Let $t_1, \cdots, t_r$ be a set of uniformizing parameters along D on V such that t_1 is a prime element of R. Let t^* be a prime element of R^*, we shall show that $t^*, t_2, \cdots, t_r$ are a set of uniformizing parameters along D^* on V^*. We shall denote by $^-$ the trace of functions on D^* (or on D). Then by Prop. 2 $d\bar{t}_2 \wedge \cdots \wedge d\bar{t}_r \neq 0$ on D. Since $k(D^*)$ is separably algebraic over $k(D)$, $d\bar{t}_2 \wedge \cdots \wedge d\bar{t}_r \neq 0$ on D^*. By Cor. 1 of Prop. 1 there exists a point P^* on D^* which is simple on V^* and D^* such that $\bar{t}_2, \cdots, \bar{t}_r$ are a set of uniformizing parameters at P^* on D^*. This implies immediately that the r functions $t^*, t_2, \cdots, t_r$ are a set of uniformizing parameters at P^* on V^* proving the assertion. The proposition follows from this at once.

§7. Linear systems and its Jacobians.

Let V be a normal variety and let N be a linear system of divisors on V with the defining module $\mathfrak{L}$. Let $f_0, f_1, \cdots, f_s$ be a base of $\mathfrak{L}$ and let k be a common field of definition for the entities appeared above. Let P be a generic point of V over k, then the point $Q=(f_0(P), f_1(P), \cdots, f_s(P))$ in a projective s-space L^s has a locus V' over k. Then as is well known the variable part of the linear system N can be represented in a unique way by the intersection product

$$\mathrm{pr}_V[\Lambda \cdot (V \times H)]$$

where Λ is the graph of the rational map α such that $\alpha(P)=Q$, and H is a hyperplane in L. The rational map α defined above will be called the *rational map associated with the linear system N.* Let $K'=k(V')$ and let U[8] be the normalization of V' (in K'). Then there exists a rational map π from V onto U which is induced in a natural way by α. We shall call this map a *normal map associated with the linear system N.* First we shall seek under what conditions the normal map π associated with the linear system N gives rise to the covering map from V onto U.

8) Since V' is a projective variety, U, as a normalization of V', is also a projective variety (Cf. [3]).

PROPOSITION 9. *Assume that the linear system N satisfies the following conditions;*

(B) *N has no base point,*

(F) *For any point P on V and a subvariety W of V containing P, there exists a member D of the linear system N such that D contains P but does not contain W.*

Then the rational map α associated with N has the following properties:

(1) *α is regular everywhere on V.*

(2) *α maps V onto a variety V' of the same dimension.*

(3) *V does not contain any fundamental subvariety with respect to α, i.e. for any point v' on V', $\alpha^{-1}(v')$ consists of a finite number of points of V.*

PROOF. (1) is immediate. To prove (2), assume that α sends V onto a variety V' of lower dimension than V. Let P and P' be the corresponding generic points of V and V' over k respectively. Then a component W of $\alpha^{-1}(P')$ containing P is of dimension $\geqq 1$. It is easy to see that the subvariety W and the point P do not satisfy the condition (F). The proof of (3) is quite similar to the proof of (2) and will be omitted.

PROPOSITION 10. *The rational map α is separably algebraic if, and only if, the linear system satisfies the condition,*

(S) *There exists a simple point P different from the base points, and the divisors $D_1, \cdots, D_r$ in the linear system N such that*

(1) *P is a simple point of D_i's.*[9]

(2) *D_i's are transversal to each other at P. In other word the intersection product $(D_1 \cdots D_r)$ is defined locally at P and contains P with the multiplicity* 1.

PROOF. Since P is not a base point of the linear system, there exists a divisor D_0 in N such that D_0 does not contain P. Let t_i be the function on V such that $(t_i)=D_i-D_0$. Then the condition (2) implies that $t_1, \cdots, t_r$ are a set of uniformizing parameters at P on V. In particular $k(V)$ is separably algebraic

9) P is said to be a simple point of a divisor X if, X contains only one component D containing P with multiplicity 1, and P is a simple point of D.

over $k(t)$. Since as the defining module of N we can take the one which contains $1, t_1, \cdots, t_r$ the rational map α must be separable. The converse can be proved in a similar way as above using Prop. 1.

Combining the above propositions we get the

THEOREM 7. *Let V be a normal variety and N a linear system on V. Let π be a normal map associated with the linear system N which maps V onto a projective variety U.*[10] *Then V is a covering of U with the covering map π if, and only if, the linear system N satisfies the conditions* (B), (F) *and* (S).

Let N be a linear system of dimension r on a normal variety V^r and assume that N satisfies the conditions (B), (F) and (S). Then the rational map π associated with N is, automatically, a normal map from V onto a projective r-space. Thus we have a representation of V as a covering variety of a projective space with the covering map π. We shall call such a linear system an *r-system* in the following.

DEFINITION 2. *Let N be an r-system on a normal variety V^r and let π be the covering map associated with N onto a projective r-space L. The differente for the above covering V/L is called the Jacobian set of the linear system N and it will be denoted by N_j.*

The following Proposition shows the characteristic properties of the Jacobian set.

PROPOSITION 11. *Let V^r be a normal variety and let N be an r-system on V without fixed component. Then the Jacobian set N_j is composed of the points of the following character:*

(1) *Multiple points of V*

(2) *Multiple points of the linear system N, i.e. the point which is a multiple point for a suitable member D of N containing the point P.*

Before the proof, it is convenient to prove the following auxiliary results.

PROPOSITION 12. *Let V^* be a covering of a normal variety V*

10) Cf. the foot note (8).

with the covering map π defined over a field k. Let P^* be a simple point of V^* and let P be a point of V corresponding to P^*. Assume that P is also a simple point of V, then P^* is unramified over P if, and only if, the point $P^* \times P$ is contained in the intersection product $\Lambda(V^* \times P)$ with the multiplicity 1, *where Λ is the graph of the rational map π.*

Proof follows from the criterion of multiplicity 1 given in [10], P. 79.

LEMMA. *Let S_1 and S_2 be affine spaces of dimension r and T^r be a linear variety of dimension r in $S_1 \times S_2$. Let P be a point in S_2 and let H be a hyperplane of S_2 containing P. Then if $T \cap S_1 \times P$ is not empty and* 0*-dimensional, then $T \cap S_1 \times H$ is $(r-1)$-dimensional. If $T \cap S_1 \times P$ is of dimension $\geqq 1$, then there exists a hyperplane H such that T is contained in $S_1 \times H$.*

The proof is easy.

PROOF of the PROPOSITION 11. We shall denote by $\sup(N_j)$ the point set attached to N_j. Let P be a point not contained in $\sup(N_j)$. Then P is unramified for the covering V/L, and hence P must be a simple point, proving that the multiple point are contained in $\sup(N_j)$. Moreover by Prop. 12, the intersection of Λ and $V \times P_0$ is transversal to each other at $P \times P_0$, where Λ is the graph of the covering map associated with N and P_0 is a point lying under P. Let H be any hyperplane in L, then Λ and $V \times H$ is transversal to each other at $P \times P_0$ by the preceding Lemma. Hence if we put $D \doteq \mathrm{Pr}_V[\Lambda \cdot (V \times H)]$, P must be a simple point of D by Prop. 21 of Chap. V in [11]. Conversely assume that P is a point in $\sup(N_j)$ and it is a simple point of V. Let P_0 be the point on L lying under P. Then P is ramified over P_0 and we see that Λ and $V \times P_0$ is not transversal to each other at $P \times P_0$ by Prop. 11. Hence the preceding Lemma tells us the existence of a hyperplane H_0 containing P_0 such that Λ and $V \times H_0$ are not transversal at $P \times P_0$. Then P must be a multiple point of the divider $\mathrm{Pr}_V[\Lambda \cdot (V \times H_0)]$.

The properties of the Jacobian set can be derived from the properties of differente in a natural way. If we remark that the canonical class on a projective r-space is given by the class of

$-(r+1)H$, where H is a hyperplane in L^r, we get the following theorem as a special case of Theorem 5.

THEOREM 8. *Let V^r be a normal variety and let N be an r-system on V. Let $\mathfrak{K}$ be the canonical divisor and let N_j be the Jacobian set of N. Then we have*

$$N_j-(r+1)N\sim\mathfrak{K}.$$

COROLLARY 1. *Let N and M be two r-systems on a normal variety V and let N_j and M_j be their Jacobians, then we have*

$$N_j-(r+1)N\sim M_j-(r+1)M$$

Hence if $|N_j-(r+1)N|$ exists for an r-system, then $|M_j-(r+1)M|$ exists for any r-system on V and they belong to the one and the same linear class.[11]

COROLLARY 2. *Let N be an r-system of hypersurfaces of order n in a projective r-space. Then the Jacobian set N_j is the hypersurface of order $(r+1)(n-1)$.*

Let $|C|$ be a complete linear system of dimension $\geqq r$ on a normal variety V^r satisfying the conditions (B), (F) and (S). Then we can extract various r-systems N from $|C|$. Let N_j be the Jacobian of the extraceted system. Then we see from Theorem 8, that N_j is contained in a fixed complete linear system. We shall call thus defined linear system the *Jacobian system* attached to the linear system $|C|$, and we shall denote it by $|C_j|$.

THEOREM 9. *Let $|C|$ and $|C'|$ be two linear systems satisfying the conditions* (B), (F) *and* (S). *Then the sum $|C+C'|$ also satisfies these conditions and we have the following relation on the Jacobian systems,*

$$|C+C'|_j=|C_j+(r+1)C'|=|C'_j+(r+1)C|$$

PROOF. The first half of the Theorem is seen by the straight forward verifications of the definitions. By Th. 8 we have $(C+C')_j \sim\mathfrak{K}+(r+1)(C+C')$, $C_j\sim\mathfrak{K}+(r+1)C$, $C'_j\sim\mathfrak{K}+(r+1)C'$. Hence

$$|C_j+(r+1)C'|=|C'_j+(r+1)C|=|\mathfrak{K}+(r+1)(C+C')|=|(C+C')_j|.$$

At the end we shall give a theorem on the existence of an r-system on a given variety.

11) It was one of the classical method to define the canonical class on a surface.

THEOREM 10. *Let V be a normal variety defined over an algebraically closed field k. Then V admits an r-system if, and only if, V is a projective variety.*

PROOF. Assume that V admits an r-system. Then V is a covering of a projective space, in other words, it is a normalization of a projective variety. Hence V must be projective automatically. Conversely assume that V is projective and let L^N be its ambiant space. Let L_1 be a hyperplane not containing V and let P be a point contained neither in V nor in L_1. It is always possible to take P and L_1 such that they are rational over k. Projection of V from the point P into L_1 defines a rational map π_1. If we choose P carefully we can see easily that the rational map π_1 satisfies the conditions (a)-(d) of §1. Applying the similar construction for $V_1=\pi_1(V)$ we get the projective variety V_2 and the rational map π_2 having the similar properties. Thus we have a series of a projective varieties of the same dimension r, $V, V_1, \cdots, V_i, \cdots$ and the rational maps $\pi_1, \pi_2, \cdots, \pi_i, \cdots$ satisfying the conditions (a)-(d) of §1, until $i=N-r$ (V_{N-r} is a r-dimensional linear variety). Then the rational map $\pi=\pi_{N-r} \cdots \pi_2 \cdot \pi_1$ is the desired rational map giving the r-system on V.

q. e. d.

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