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On Riemann's period relations on open Riemann surfaces

By

Yukio KUSUNOKI

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Introduction.

The theory of Abelian differentials of the first kind on abstract open Riemann surfaces was first developped in 1940 by R. Nevanlinna [13]. This theory was established for parabolic Riemann surfaces by considering the complete orthogonal system of such differentials, and was completed by Virtanen [23] in 1950 for general Riemann surfaces. On the other hand, in view of period relations Ahlfors [1], Virtanen [22] treated this theory for parabolic Riemann surfaces, where Riemann's bilinear relation plays a fundamental role. For the case of hyperelliptic surfaces of infinite genus this relation was investigated in detail by Hornich [6], P. J. Myrberg [12] and recently Pfluger [17].

In the present paper also the problem on the periods of Abelian integrals on an abstract open Riemann surface will be treated.[1] Ahlfors [1] proved the existence of an exhaustion and corresponding canonical homology basis of $R \in O_G$ (class of parabolic Riemann surfaces) such that for any two harmonic differentials du_1, du_2 with finite Dirichlet integrals the mixed Dirichlet integral $D_R(u_1, u_2)$ is equal to

$$D_R(u_1, u_2) = \lim_{n\to\infty} \sum_{i=1}^{k_n} \left(\int_{A_i} d\bar{u}_1 \int_{B_i} d\bar{u}_2^* - \int_{A_i} d\bar{u}_2^* \int_{B_i} d\bar{u}_1 \right),$$

where $d\bar{u}_1$ and $d\bar{u}_2^*$ are the modified quantities of du_1 resp. du_2^* (conjugate harmonic differential of du_2) which depend on the exhaustion.

To obtain the corresponding formula which is expressed by

1) The principal results in this paper have been announced and partly proved in my notes [8], [9].

the periods of du_1 and du_2^* I have to impose further some conditions, that is, for certain restricted class O' of O_G, Riemann's first and second (bilinear) relations with an infinite number of their periods will be obtained (§ 2, 3).

In connexion with O' we shall consider another subclass O'' of O'. These classes are defined by the extremal length. In § 1 we shall study their properties, above all, those concern with the problem of limit (at ideal boundary) of bounded harmonic functions.

Finally I shall extend Riemann's second relation to the ultimate form when du_1 (or du_2^*) has only a finite number of non vanishing periods, that is, it will be established for Riemann surfaces R of class O_{HD} (on which no harmonic function with finite Dirichlet integral exists). On the other hand, it will be also obtained when du_1 and du_2^* have only a finite number of non vanishing A-periods, if we impose some conditions on the structure of $R \in O_{HD}$. (§ 3).

§ 1. Two classes of Riemann surfaces.

1. *Extremal length.*——To define the subclasses of class O_G we shall start with preliminaries on the extremal length on Riemann surfaces (cf. Ahlfors, Beurling [3], Hersch [5], Ohtsuka [16]). Now let R be an arbitrary Riemann surface and G be a domain on R. We consider a system of curves $\{c\} \neq \phi$ (ϕ: empty set) on G each curve of which consists of a finite or a countable number of curves on G. Let (P) be the set of non-negative covariants ρ defined on G, i.e. $\rho(z)|dz|$ is non-negative invariant metric under the transformations of local parameter z at p, such that

$$L(\rho, \{c\}) = \inf_{c \in \{c\}} \underline{\int_c} \rho(z)\,|dz|$$

(1. 1)

$$A(\rho) = \overline{\iint_G} \rho^2 dxdy, \quad z = x + iy$$

are not simultaneously 0 or ∞.[2] Then the extremal length with

2) $\underline{\int}$ is the lower integral, $\overline{\iint}$ is the upper integral in Darboux's sense.

respect to $\{c\}$ is defined by

$$\lambda_{(P)}\{c\} = \sup_{\rho\in(P)} \frac{L(\rho, \{c\})^2}{A(\rho)} \geqq 0. \tag{1. 2}$$

We take $\lambda_{(P)}\{c\}=0$ if (P) is empty.
Next we consider another class (Q) of non-negative covariants ρ such that, for any curve $c\in\{c\}$,

$$\int_c \rho(z)\,|dz| \geqq 1. \tag{1. 3}$$

Then another simple definition of the extremal length is

$$\frac{1}{\lambda_{(Q)}\{c\}} = \inf_{\rho\in(Q)} A(\rho). \tag{1. 4}$$

We say that ρ is *admissible* for $\{c\}$ when (1. 3) is satisfied. If there is no admissible covariant, i.e. $(Q)=\phi$, $\lambda_{(Q)}\{c\}=0$.

PROPOSITION 1.——

$$\lambda_{(P)}\{c\} = \lambda_{(Q)}\{c\}. \tag{1. 5}$$

In the following therefore we denote this common value by $\lambda\{c\}$.

Proof. First we suppose $(P)\neq\phi$.

(I) The case where $L(\rho, \{c\}) < \infty$ for all $\rho\in(P)$.

(i) When there exists at least one $\rho\in(P)$ such that $0<L(\rho, \{c\}) <\infty$, we choose a constant k such that $L(\rho', \{c\})=1$, $\rho'=k\rho\in(Q)$. Then it follows easily

$$0 \leqq \lambda_{(P)}\{c\} = \sup_{\rho'} 1/A(\rho') \leqq \lambda_{(Q)}\{c\}. \tag{1. 6}$$

Now in general $(Q)=P_1\cup P_2\cup P_3$ where $P_1=\{\rho_1;\ 1\leqq L(\rho_1, \{c\})<\infty\}$, $P_2=\{\rho_2;\ L(\rho_2, \{c\})=\infty,\ A(\rho_2)<\infty\}$ and $P_3=\{\rho_3;\ L(\rho_3, \{c\})=A(\rho_3)=\infty\}$. $P_1\subset(P)$, $P_2\subset(P)$, but since $P_2=\phi$ in the present case, we have conversely

$$\lambda_{(P)}\{c\} \geqq \sup_{\rho\in P_1} \frac{L(\rho; \{c\})^2}{A(\rho)} \geqq \sup_{\rho\in P_1} \frac{1}{A(\rho)} = \sup_{\rho\in P_1\cup P_3} \frac{1}{A(\rho)} = \lambda_{(Q)}\{c\}. \tag{1. 7}$$

(ii) If $L(\rho, \{c\})=0$ for all $\rho\in(P)$, then $(Q)=\phi$ or for any $\rho\in(Q)$ $L(\rho)=A(\rho)=\infty$, hence $\lambda_{(P)}\{c\}=\lambda_{(Q)}\{c\}=0$ by definition.

(II) The case where there exists at least one $\rho_0\in(P)$ for which $L(\rho_0, \{c\})=\infty$.

Then $A(\rho_0)<\infty$ and $\lambda_{(P)}\{c\}=\infty$. Since the covariants $\rho_n=\rho_0/n$ $(n=1, 2, \ldots.)$ are admissible for (Q),

$$0 \leqq \lambda_{(Q)}\{c\}^{-1} = \inf_{\rho\in(Q)} A(\rho) \leqq \lim_{n\to\infty} A(\rho_n)/n^2 = 0.$$

Therefore $$\lambda_{(P)}\{c\} = \lambda_{(Q)}\{c\} = \infty.$$

Now if $(P) = \phi$, then $\lambda_{(P)}\{c\} = 0$ and for any non-negative covariant ρ $L(\rho, \{c\})$and $A(\rho)$ are simultaneously zero or ∞. Hence it is proved that $\lambda_{(P)}\{c\} = \lambda_{(Q)}\{c\} = 0$, q.e.d.

We shall use the following properties of the extremal length.

PROPOSITION 2.——(i) *If* $\{c_1\} \subset \{c_2\}$, *then* $\lambda\{c_1\} \geqq \lambda\{c_2\}$.

(ii) $$\lambda\{\{c_1\} \cup \{c_2\}\}^{-1} \leqq \lambda\{c_1\}^{-1} + \lambda\{c_2\}^{-1}.$$

(iii) *If* $\{c_1\} \subset G_1 \subset G$, $\{c_2\} \subset G_2 \subset G$ *and* $G_1 \cap G_2 = \phi$, *then*

$$\lambda\{\{c_1\} \cup \{c_2\}\}^{-1} = \lambda\{c_1\}^{-1} + \lambda\{c_2\}^{-1}.$$

(iv) *If every curve* $c \in \{c\}$ *contains at least one* $c_1 \in \{c_1\}$ *and one* $c_2 \in \{c_2\}$ *where* $\{c_1\} \subset G_1 \subset G$, $\{c_2\} \subset G_2 \subset G$ *and* $G_1 \cap G_2 = \phi$, *then*

$$\lambda\{c_1\} + \lambda\{c_2\} \leqq \lambda\{c\}.$$

2. In the following we take $G = R$ without loss of generality. Let B be a union of a finite number of disjoint ring domains B_i on R each of which has boundaries α_i and β_i which consist of a finite number of disjoint analytic Jordan closed curves respectively. Let $\{c\}$ be the set of closed curves c on B such that $c = \sum_i c_i$, c_i is homologous to α_i, i.e. $c_i \sim \alpha_i \sim \beta_i$. Now let $\{c^*\}$ be a subset of $\{c\}$ which consist of *analytic* Jordan closed curves. Suppose $\{\tilde{c}\}$ and $\{\tilde{c}^*\}$ denote the corresponding sets for the union of curves in B_i which connect α_i to β_i. By Prop. 2 (i) $\lambda\{c\} \leqq \lambda\{c^*\}$, $\lambda\{\tilde{c}\} \leqq \lambda\{\tilde{c}^*\}$, but we have

PROPOSITION 3.——

$$\lambda\{c\} = \lambda\{c^*\} = D_B(\omega) = \frac{1}{\lambda\{\tilde{c}\}} = \frac{1}{\lambda\{\tilde{c}^*\}}. \tag{1. 8}$$

where $D_B(\omega)$ *stands for the Dirichlet integral over B of the harmonic measure* ω *(in B) with resp. to* β, *i.e.* $\omega = \omega_i$ *in* B_i, ω_i *is harmonic measure of* B_i *with resp. to* β_i.

Proof. Let L_λ be the level curve $\omega = \lambda$, $0 \leqq \lambda \leqq 1$ except a finite number of λ for which L_λ contain the points where grad $\omega = 0$. Obviously $L_\lambda \in \{c^*\} \subset \{c\}$. Suppose ρ is admissible for $\{c^*\}$. Since $\Omega = \omega + i\omega^*$ is considered as a uniformizer at B with a finite number of suitable slits Γ: $\omega^* = $const.,

$$1 \leqq \int_{L_\lambda} \rho d\omega^*, \quad 0 \leqq \lambda \leqq 1.$$

Since $\int_{L_\lambda} d\omega^* = \int_{L_1} d\omega^* = D_B(\omega)$, by using Schwarz's inequality and integrating we have

$$1 \leqq D_B(\omega) \overline{\iint_B} \rho^2 d\omega d\omega^*.$$

Therefore

$$D_B(\omega)^{-1} \leqq \inf_\rho \overline{\iint_B} \rho^2 d\omega d\omega^* \leqq \inf_\rho A(\rho) = \lambda\{c^*\}^{-1} \leqq \lambda\{c\}^{-1}. \tag{1. 9}$$

On the other hand,

$$\tilde{\rho}(p(\zeta)) = \begin{cases} D_B^{-1}(\omega)\,|d\Omega/d\zeta|, & p(\zeta) \in B', \ B' = B - \Gamma \\ 0, & p(\zeta) \in R - B' \end{cases} \tag{1. 10}$$

is admissible for $\{c\} \supset \{c^*\}$. Hence we have conversely

$$\lambda\{c^*\}^{-1} \leqq \lambda\{c\}^{-1} \leqq A(\tilde{\rho}) = D_B(\omega)^{-1}. \tag{1. 11}$$

We can prove analogously that $D_B(\omega) = \lambda\{\tilde{c}\}^{-1} = \lambda\{\tilde{c}^*\}^{-1}$.

3. In the following we say that K is a *compact domain with analytic boundaries* when K is a compact domain and its boundary ∂K consists of a finite number of disjoint analytic Jordan closed curves. Now we consider the system of curves $\{C\} \subset R - R_0$ (R_0 is the image of a parameter disc), such that $C \in \{C\}$ consists of a finite number of disjoint *analytic* Jordan closed curves which is homologous to ∂R_0, i.e. $C \sim \partial R_0$. Let $\{\Gamma\}$ be a system of analytic curves in $R - R_0$ each of which extends from ∂R_0 to the ideal boundary $\mathfrak{J}$ of R. This means as follows. The ideal boundary $\mathfrak{J}$ of R is the set of ideal boundary elements α which is defined as follows (Stoilow [21]) : Let $\{\Omega_n\}$ be a sequence of domains on R such that

(1) The relative boundary C_n of Ω_n consists of an analytic Jordan closed curve on R.

(2) $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \supset \cdots$, $\bar{\Omega}_1 \neq R$[3)]

(3) $\bigcap_{n=1}^{\infty} \Omega_n = \phi$.

Then we say $\{\Omega_n\}$ defines an ideal boundary element α. We find that Ω_n are non-compact by (3) and C_n divides R into two disjoint

3) The barred letter stands for the closure of set.

parts Ω_n and $R-\bar{\Omega}_n$ (we shall frequently write such a curve or cycle as ~ 0 (mod $\mathfrak{J}$) i.e. when it is the boundary of an infinite 2-dimensional chain). Two such sequences $\{\Omega_n\}$ and $\{\Omega_n'\}$ are called *equivalent* each other if for given i, j there exist k, l such that

$$\Omega_i' \supset \Omega_k\,, \quad \Omega_j \supset \Omega_l'.$$

We understand that two equivalent sequences determine the same boundary element. Next we say that a sequence of points $\{P_n\}$ or curves $\{\gamma_n\}$ on R *tend* to α (or $\mathfrak{J}$) resp. to $\mathfrak{J}$ according as all P_n or γ_n except a finite number of points or curves belong to every Ω_m (or $R-\bar{R}_m$) resp. to $R-\bar{R}_m$ and that a curve Γ extends to $\mathfrak{J}$ if there exists a sequence of points $\{P_n\}$ on Γ tending to $\mathfrak{J}$.

PROPOSITION 4 (Ohtsuka [16])——*R is of parabolic type (Nullrand) if and only if*

$$\lambda\{C\}=\lim_{n\to\infty}\lambda\{C^n\}=1/\lambda\{\Gamma\}=\lim_{n\to\infty}1/\lambda\{\Gamma^n\}$$

is equal to zero, where $\{C^n\}$ is the subset of $\{C\}$ lying inside of $G_n=R_n-R_0$, here $\{R_n\}$ is the usual exhaustion of R, and $\{\Gamma^n\}$ is the set of analytic curves in G_n connecting ∂R_0 to ∂R_n.

Proof. Since $\{C^n\}\subset\{C^{n+1}\}\subset\{C\}$, by Prop. 2 (i), 3 we have

$$\lim_{n\to\infty}\lambda\{C^n\}=\lim_{n\to\infty}d_n\equiv d\geqq\lambda\{C\}\geqq 0 \tag{1. 12}$$

where $d_n=D_{G_n}(\omega_n)$ and ω_n is the harmonic measure of G_n. Hence if $d=0$, $d=\lambda\{C\}=0$. If $d>0$, since ω_n converge uniformly to a non-constant harmonic function ω on every compact set in $R-\bar{R}_0$, hence we have for any $C\in\{C\}$

$$\int_C d\omega^*=\lim_{n\to\infty}\int_C d\omega_n^*=\lim_{n\to\infty}d_n=d\,.$$

Therefore the covariant

$$\bar{\rho}(p(\zeta))=\begin{cases} d^{-1}|d\Omega/d\zeta|\,, & p(\zeta)\in R-R_0-\Gamma, \quad \Omega=\omega+i\omega^* \\ 0\,, & p(\zeta)\in R_0+\Gamma \end{cases}$$

where Γ denotes a countable number of suitable slits $\omega^*=$const. through the points where $\Omega'=0$, is admissible with respect to $\{C\}$ and

$$\lambda\{C\}^{-1}\leqq\iint_R\bar{\rho}^2dx\,dy=d^{-2}\iint_{R-R_0}d\omega\, d\omega^*\leqq d^{-1}$$

therefore together with (1.12) we obtain

$$d=\lambda\{C\}=\lim_{n\to\infty}\lambda\{C^n\}. \tag{1. 13}$$

We can also prove easily that $\lambda\{\Gamma\}=d^{-1}$. On the other hand, according to R. Nevanlinna's theorem [13], R is of parabolic type if and only if $d=0$, q.e.d.

4. Now we shall consider a subset $\{\gamma\}$ of $\{C\}$ which contains an infinite number of curves $\in\{C\}$ tending to $\mathfrak{J}$. Let $\{\gamma^*\}$ be the complementary set of $\{\gamma\}$ with respect to $\{C\}$. Since

$$\lambda\{C\}^{-1}\leqq\lambda\{\gamma\}^{-1}+\lambda\{\gamma^*\}^{-1},$$
$$0\leqq\lambda\{C\}\leqq\lambda\{\gamma\},\quad 0\leqq\lambda\{C\}\leqq\lambda\{\gamma^*\},$$

we have

PROPOSITION 5.——*R is of parabolic type if and only if at least $\lambda\{\gamma\}=0$ or $\lambda\{\gamma^*\}=0$ holds.*

PROPOSITION 6.——*Let $K\supset R_0$ be a compact domain with analytic boundaries. Let $\{\gamma_K\}=\{\gamma_K;\gamma_K\in\{\gamma\},\gamma_K\cap K=\phi\}$*[4] *and $\{\gamma_K^*\}$ be the complementary set of $\{\gamma_K\}$ with respect to $\{\gamma\}$. In order that $\lambda\{\gamma\}=0$ it is necessary and sufficient that $\lambda\{\gamma_K\}=0$.*

Proof. Since $0\leqq\lambda\{\gamma\}\leqq\lambda\{\gamma_K\}$, it is sufficient. Therefore it is enough to prove $\lambda\{\gamma_K^*\}>0$, since $\lambda\{\gamma\}^{-1}\leqq\lambda\{\gamma_K\}^{-1}+\lambda\{\gamma_K^*\}^{-1}$. Let K_0 be a compact domain with analytic boundaries containing K completely and $K_0-R_0=K_0^*$. Let $\omega(p)$ be the harmonic measure of K_0^*. Put

$$0<\max_{p\in\partial K}\omega(p)=m<1\,,\quad \int_{\partial R_0}d\omega^*=d>0\,. \tag{1. 14}$$

The covariant

$$\tilde{\rho}(p(\zeta))=\begin{cases}d'|d\Omega/d\zeta|,\ d'=1/\min(d,\,2(1-m)),\ p(\zeta)\in K_0^*-\Gamma\\ 0,\quad p(\zeta)\in R-K_0^*+\Gamma\end{cases}$$

where $\Omega=\omega+i\omega^*$ and Γ denotes slits (cf. (1.10)), is admissible for $\{\gamma_K^*\}$ because

$$\int_{-\gamma_K^*}\tilde{\rho}|dz|=d'\int_{\gamma_K^*\cap K_0^*}|d\Omega|\begin{cases}=d'\displaystyle\int_{\gamma_K^*}|d\Omega|\geqq d'|\int_{\gamma_K^*}d\omega^*|=d'd\geqq1,\ \text{ if } \gamma_K^*\subset K_0^*.\\ \geqq d'\displaystyle\int_{\gamma_K^*\cap K_0^*}|d\omega|\geqq d'2(1-m)\geqq1,\ \text{ if } \gamma_K^*\ \ K_0^*.\end{cases}$$

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4) In the following we shall use also such a notation.

Therefore we have

$$\lambda\{\gamma_K^*\}^{-1}\leqq\iint_R\tilde{\rho}^2dxdy=d''^2d<\infty\,,\quad \text{q.e.d.}$$

PROPOSITION 7.——*Suppose that φ_1 and φ_2 are any two non-negative covariants sequare integrable over $R-K$ (K is a compact domain with analytic boundaries). If $\lambda\{\gamma\}=0$, then there exists a sequence of curves $\gamma_n\in\{\gamma\}$ $(\gamma_n\cap K=\phi)$ tending to the ideal boundary $\mathfrak{J}$ of R such that*

$$\int_{\gamma_n}\varphi_1|dz|\int_{\gamma_n}\varphi_2|dz|\to 0\quad \textit{for } n\to\infty.$$

Proof. Now we assume that for any $\gamma_K\in\{\gamma_K\}$

$$\int_{\gamma_K}\varphi_1|dz|\int_{\gamma_K}\varphi_2|dz|\geqq\eta>0\,. \tag{1. 15}$$

Since φ_1 and $\varphi_2\geqq 0$, we have $\int_{\gamma_K}\varphi_1|dz|\geqq\sqrt{\eta}$ or $\int_{\gamma_K}\varphi_2|dz|\geqq\sqrt{\eta}$. Let $\{\gamma_K^i\}=\{\gamma_K^i;\int_{\gamma_K^i}\varphi_i|dz|\geqq\sqrt{\eta},\ \gamma_K^i\in\{\gamma_K\}\}$ $(i=1, 2)$, then $\{\gamma_K\}=\{\gamma_K^1\}\cup\{\gamma_K^2\}$, hence $\lambda\{\gamma_K\}^{-1}\leqq\lambda\{\gamma_K^1\}^{-1}+\lambda\{\gamma_K^2\}^{-1}$. Since $\lambda\{\gamma_K\}=0$ (Prop. 6), it follows that $\lambda\{\gamma_K^1\}=0$ or $\lambda\{\gamma_K^2\}=0$, e.g. $\lambda\{\gamma_K^1\}=0$. Then the covariant $\psi=\varphi_1/\sqrt{\eta}$ for $p\in R-K$ and $=0$ for $p\in K$ is admissible for $\{\gamma_K^1\}$ and

$$\lambda\{\gamma_K^1\}^{-1}\leqq\iint_R\psi^2dxdy=1/\eta\iint_{R-K}\varphi_1^2dxdy<\infty$$

which is absurd. That is, for any given $\eta>0$ there exists a curve $\gamma_K\in\{\gamma_K\}$ such that the inverse inequality of (1. 15) holds. Therefore we can prove this proposition at once, q.e.d.

COROLLARY——*If $R\in O_G$ and df is an Abelian differential on R with finite Dirichlet integral taken over $R-K$. Then for any cycle $C\sim 0$ (mod $\mathfrak{J}$), $C\subset R-K$*

$$\int_C df=0\,.$$

Proof. Since $R\in O_G$, $\lambda\{C\}=0$ (Prop. 4), hence by Prop. 7 there exists a sequence of curves $C_n\in\{C\}$ tending to $\mathfrak{J}$ such that $\int_{C_n}|df|\to 0$ $(n\to\infty)$. Since C is homologous to a cycle C_n' on C_n with bounded ($\leqq K$) coefficients and df has no pole on $R-K$

$$\left|\int_C df\right| = \left|\int_{C'_n} df\right| \leq K \int_{C_n} |df| \to 0, \quad \text{q.e.d.}$$

5. Now we shall consider two subsets $\{\Gamma\}$, $\{L\}_E$ of $\{C\}$.

(I) $\{\Gamma\}$: $\{\Gamma\}$ is the set of $\Gamma \in \{C\}$ such that in the decomposition of Γ into components, i.e. $\Gamma = \sum_i \Gamma_i$, $\Gamma_i \cap \Gamma_j = \phi$ $(i \neq j)$, each curve Γ_i divides R into two disjoint parts.

(II) $\{L\}_E$: This is the system of curves of $\{\Gamma\}$ depending on an exhaustion $E = \{R_n\}$ such that $L_n \equiv \partial R_n \in \{\Gamma\}$. That is, $\{L\}_E = \bigcup_{n=1}^{\infty} \{L_n\}$ where $\{L_n\}$ is the set of curves of $\{\Gamma\}$ contained in annuli including L_n.[5)]

First of all we note $\{\Gamma\}$ and $\{L\}_E$ contain an infinite number of curves tending to $\mathfrak{J}$ (cf. Sario [18], p. 466).

DEFINITION.——*We shall denote by O' or O'' the classes of Riemann surfaces for which $\lambda\{\Gamma\} = 0$ resp. $\lambda\{L\}_E = 0$ for certain exhaustion E.*

Since $\{L\}_E \subset \{\Gamma\} \subset \{C\}$ and $\lambda\{C\} = 0$ is equivalent to $R \in O_G$, we have $O'' \subset O' \subset O_G$. In the following of this paragraph we study on the properties of classes O' and O''.

6. *The single-valued harmonic function outside of a compact set.*——Let K be a compact domain on $R \in O_G$ with analytic boundaries and u be a single-valued harmonic function defined on $R - K$,[6)] and bounded to one side (e.g. bounded or positive harmonic function). Then it holds for instance the following properties.

PROPOSITION 8.——(a) *If u is bounded, then the maximum and minimum principle hold.*

(a′) *Maximum principle also holds even if* $\varliminf_{n \to \infty} \max_{p \in \partial R_n} u(p)/\lambda\{\Gamma^n\} = 0$. *That is, if u is unbounded, then* $\max_{p \in \partial R_n} |u(p)| \geq \eta \lambda\{\Gamma^n\}$ $(\eta > 0)$ (Kusunoki [7]).

(b) *u is bounded if and only if u has a finite Dirichlet integral. Moreover then* $\int_{\partial K} du^* = 0$ (R. Nevanlinna [13], [14]).

(b′) *u is bounded if and only if* $\int_{\partial K} du^* = 0$.

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5) By annulus including $l \in \{\Gamma\}$ we mean the union of doubly connected ring domains each of which includes a component of l.

6) This is not necessarily connected, but we mean hereby $R - K$ a component of it.

Here we prove only the sufficient condition of (b′). We assume e.g. $u > -M$. Let v_n be a harmonic function such that $v_n = u$ on ∂K and $= -M$ on ∂R_n. Then a suitable subsequence, say $\{v_n\}$ tend to a bounded harmonic function v which is equal to u on ∂K. Suppose u is unbounded, then $U = u - v = \lim_{n\to\infty}(u - v_n) \geqq 0$ is non-constant and $=0$ on ∂K. Since $\int_{\partial K} dv^* = 0$ by (b), we have $\int_{\partial K} dU^* = \int_{\partial K}(\partial U/\partial\nu)ds = 0$. It follows $U^* = \text{const.}$ on ∂K, because $\partial U/\partial\nu = 0$ on ∂K. Now since the curve ∂K is analytic, the function $U + iU^*$ is also analytic on ∂K by the principle of reflection, therefore $U \equiv \text{const.} = 0$ which is absurd, q.e.d.

Now the problem of limits (at ideal boundary) of bounded harmonic function is more complicated. Let f be a real or complex valued continuous function defined on $R - K$ and S_α^f be the set of limit values at $\alpha \in \mathfrak{J}$, i.e. $S_\alpha^f = \{\beta ;\ \lim_{n\to\infty} f(p_n) = \beta,\ p_n \to \alpha\}$, then for any two equivalent sequences $\{\Omega_n\}$ and $\{\Omega_m'\}$ determining α

$$S_\alpha^f = \bigcap_{n=1}^{\infty} \overline{f(\Omega_n)} = \bigcap_{m=1}^{\infty} \overline{f(\Omega_m')}. \tag{1. 16}$$

This is a closed set. Now if f is a bounded analytic function, it has always the limit, i.e. for any $\alpha \in \mathfrak{J}$, S_α^f consists of a single point. (Heins [4], A. Mori [10]). But in case of a bounded harmonic function there exists an example of Riemann surface $\in O_G$ of infinite genus for which it does not have a limit (Heins [4]). For this problem we have the following

THEOREM 1.——*Suppose $R \in O'$ and $u(p)$ be a single-valued bounded harmonic function on $R - K$. Then $u(p)$ has always a limit when p tends to any ideal boundary element α.*

Proof. Since $R \in O' \subset O_G$, by Prop. 8 (b) u has a finite Dirichlet integral over $R - K$. Therefore by Prop. 7 there exists a sequence of curves $\Gamma_n \in \{\Gamma\}$, $n = 1, 2, \cdots$ tending to $\mathfrak{J}$ such that

$$\int_{\Gamma_n} |w'|\,|dz| = \int_{\Gamma_n} |dw| \to 0 \quad \text{for } n \to \infty\ , \tag{1. 17}$$

where $w = u + iu^*$ in $R - K$, $=0$ unless u is defined. Now let $\{\Omega_n\}$ be a determining sequence of α. Since $\Gamma_n \sim \partial K$ tends to $\mathfrak{J}$, there exists a number m_1 such that $\Omega_{n_1} (\equiv \Omega_1) \cup \Gamma_{m_1} \neq \phi$, then there exists also a number n_2 such that $\Gamma_{m_1} \cap \Omega_{n_2} = \phi$, since $\Omega_n \to \mathfrak{J}$. Now let

$\Gamma_{m_1}^{i_1} \sim 0$ (mod $\mathfrak{J}$) be a component of Γ_{m_1} which divides C_{n_1} from C_{n_2}. We write by Ω_1' the non-compact domain which is bounded by the relative boundary $\Gamma_{m_1}^{i_1}$ and contains the domain Ω_{n_2}. By the same way we determine the Ω_2' such that $\Omega_{n_2} \supset \Omega_2' \supset \Omega_{n_3}$. And so on. Thus we have a sequence of domains $\{\Omega_m'\}$ which is equivalent to $\{\Omega_n\}$. Because for given i, j there exist p, q such that $\Omega_i \supset \Omega_{n_p} \supset \Omega_p'$, $\Omega_j' \supset \Omega_{n_{j+1}} \equiv \Omega_q$. Therefore by (1.16) $S_\alpha^u = \bigcap_{n=1}^{\infty} \overline{u(\Omega_n')}$. Now by Prop. 8 (a) $u(p)$ attains to $\sup_{p\in R-K} u(p)$ and $\inf_{p\in R-K} u(p)$ at the relative boundary ∂K. If S_α^u contains two different values a and b, then $a, b \in \overline{u(\Omega_n')}$ for all n and

$$|a-b| \leqq \max_{p\in L_n} u(p) - \min_{p\in L_n} u(p)$$

where $L_n \equiv \Gamma_{m_n}^{i_n}$ denotes the relative boundary of Ω_n'. Since $L_n \sim 0$ (mod $\mathfrak{J}$) and consists of a single component, it follows

$$\max_{p\in L_n} u(p) - \min_{p\in L_n} u(p) \leqq \int_{L_n} |du| \leqq \int_{\Gamma_{m_n}} |dw| ,$$

hence $0 < |a-b| \leqq \int_{\Gamma_{m_n}} |dw|$ for all n, which contradicts with (1.17), q.e.d.

7. Now we shall prove a sufficient condition for which R should belong to class O'' therefore to O'. Let D_n, $n=1, 2, \cdots$ be a sequence of annuli which are *disjoint* each other and include the curves L_n of $\{\Gamma\}$ and let $\{c_n\}$ be the set of curves of $\{\Gamma\}$ lying in D_n, then we have by Prop. 3

$$\lambda\{c_n\} = 2\pi/\log \mu_n \tag{1.18}$$

where μ_n denotes the Sario-Pfluger's ring modul of D_n. Since $D_m \cap D_n = \phi\,(n \neq m)$ by Prop. 2 (iii) we have

$$\lambda\{\bigcup_{n=1}^{N}\{c_n\}\}^{-1} = \sum_{n=1}^{N} \lambda\{c_n\}^{-1}$$

and by Prop. 2 (i)

$$\lambda\{\Gamma\} \leqq \lambda\{L\}_E \leqq \lambda\{\bigcup_{n=1}^{N}\{c_n\}\} \quad \text{for any } N.$$

Hence

$$\frac{1}{\lambda\{\Gamma\}} \geqq \frac{1}{\lambda\{L\}_E} \geqq \frac{1}{2\pi} \log \prod_{n=1}^{N} \mu_n . \tag{1.19}$$

Therefore we have

THEOREM 2.——*Let D_n, $n=1, 2, \cdots$ be a sequence of annuli which are disjoint each other and include the curves of $\{\Gamma\}$ and let μ_n be the Sario-Pfluger's modul of D_n. If*

$$\prod_{n=1}^{\infty} \mu_n = \infty ,$$

then $R \in O'' \subset O'$.

COROLLARY (Heins [4]).——*Let R be a parabolic Riemann surface which has only one ideal boundary element α. Let $\{D_n\}$ be a sequence of doubly connected domains with analytic Jordan boundaries which are disjoint each other and D_n separate D_{n-1} from α. If the product of modul of D_n diverges, then every single-valued bounded harmonic function on an end (i.e. $R-K$) has a limit at α.*

8. Let $R \in O_G$ be of finite genus. Then we can take a compact domain K so large that each component B_i of $R-K$ is of planar character (schlichtartig). Therefore we can construct the (Evans) potential U_i on B_i such that $U_i(p) \to \infty$ for $p \to \mathfrak{J}$, $p \in B_i$ and $U=0$ on ∂K, $\int_{\partial K} dU^* = 2\pi$, where $U=U_i$ in B_i. Let

$$\lambda_1 < \lambda_2 < \cdots\cdots$$

be all values of U for which $U=\lambda_i$ contain the points where grad $U=0$, then each component of D_n:

$$D_n = \{p\,;\ \lambda_{n-1} < U(p) < \lambda_n\}$$

is a doubly connected ring domain (cf. Ahlfors [1], p. 16), moreover e.g. the level curve L_n: $U=\lambda_n'$ $(\lambda_{n-1} < \lambda_n' < \lambda_n)$ belong to $\{\Gamma\}$, because each B_i is of planar character. Since $\mu_n = \exp.\ (\lambda_n - \lambda_{n-1})$, it follows $\prod_{n=1}^{N} \mu_n \to \infty$ $(N \to \infty)$,[7] therefore by Theorem 2 we have $R \in O'' \subset O'$, i.e. $O_G = O'' = O'$. But if R is of infinite genus, there exists a Riemann surface which belongs to O_G, but not O' (cf. sec. 6).

THEOREM 3.——*If R is of finite genus, then $O_G = O' = O''$. If R is of infinite genus, we have $O'' \subset O' \subsetneqq O_G$.*

THEOREM 4.——*Let R, R' be two Riemann surfaces, and K, K' be compact domains with analytic boundaries on R, R'. Suppose that there exists a one to one conformal transformation $p \longleftrightarrow p'$*

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7) Take $\lambda_{N+1} = \infty$ if the number of λ's is finite N.

between the (not necessarily connected) complements $R-K$ and $R'-K'$. Then $R\in O'$ or O'' if and only if $R'\in O'$ or O'' respectively.

This shows that O' or O''-property of Riemann surface depends only on its ideal boundary.

Proof. Under our assumption any curve $\Gamma_K\in\{\Gamma\}$, $L_K\in\{L\}_E$ on $R-K$ are transformed to $\Gamma'_{K'}\in\{\Gamma''\}$, resp. $L'_{K'}\in\{L'\}_{E'}$ on $R'-K'$. Now if $R\in O'$ (or O''), then $\lambda\{\Gamma_K\}=0$ ($\lambda\{L_K\}_E=0$) (Prop. 6). Since the extremal length is invariant under the conformal transformation, we have $\lambda\{\Gamma'_{K'}\}=0$ ($\lambda\{L'_{K'}\}_{E'}=0$). Therefore $R'\in O'$ (or O'') by Prop. 6, q.e.d.

§ 2. Riemann's first period relation

1. *Canonical homology basis.*——Let R be an arbitrary Riemann surface and Σ, Σ' denotes respectively the cell-division of R and its dual subdivision. There exists a canonical homology basis A_1, $B_1, \cdots, A_n, B_n, \cdots$ on R where A_n belong to Σ, B_n to Σ' and satisfy the following condition; (Ahlfors [1]).

(1) Any cycle C on R is expressed as

$$C\sim\sum_{n=1}^{N}(p_n A_n+q_n B_n) \qquad (\text{mod } \mathfrak{F}) .$$

(2) The intersection numbers N between them are characterized by

$$N(A_m, A_n)=N(B_m, B_n)=0 . \quad N(A_m, B_n)=\delta_n^m \quad \text{(Kronecker)}.$$

Now let $\{R_n\}$ be an exhaustion of R. Then there exists a canonical homology basis satisfying moreover the following condition: (Ahlfors [1]).

(3) $A_1, B_1, \cdots, A_{k_n}, B_{k_n}$ are the relative homology basis of R_n mod ∂R_n, i.e. any cycle $C\subset R_n$ is expressed as

$$C\sim\sum_{i=1}^{k_n}(p_i A_i+q_i B_i) \quad (\text{mod } \partial R_n).$$

We shall call such basis a canonical homology basis of *$\mathfrak{A}$-type* with respect to $\{R_n\}$. For instance let S be a two sheeted Riemann surface of hyperelliptic type whose branch points lie on the real positive axis g and accumulate only at ∞. Then Fig. 1 shows a canonical basis of *$\mathfrak{A}$-type* on S with respect to $E=\{R_n\}$ such that ∂R_n (e.g. $|z|=r_n$) pass through I_n. Let $\{L\}$ be the set of analytic

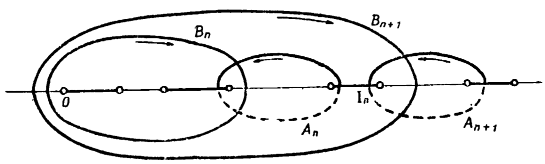

Fig. 1

Jordan closed curves on z-plane which separate the circle $|z|=r_0$ from ∞ and meet once with g at slits I_n. Then it is easily seen that $\lambda\{L\}=0$ implies $\lambda\{L\}_E=0$. Now when $\lambda\{L\}_E=0$, the canonical basis of $\mathfrak{A}$-*type* is useful. (cf. Pfluger [17] and Th. 5, 6).

2. *Riemann's first period relation.*——In this section we shall always consider Abelian differentials each of which has finite Dirichlet integral taken over R except the neighbourhoods of a finite number of singularities. Now let $R\in O'$, then there exists a sequence of curves $l_\nu\in\{\Gamma'\}$ tending to $\mathfrak{F}$ such that they are disjoint each other and for two Abelian differentials df_1, df_2

$$\int_{l_\nu}|df_1|\int_{l_\nu}|df_2|\to 0\,,\quad \nu\to\infty\quad \text{(Prop. 7)}. \tag{2. 1}$$

If we choose $A_1, B_1, \cdots, A_n, B_n, \cdots$ as a canonical homology basis of $\mathfrak{A}$-*type* with respect to this exhaustion $\{R^\nu\}$ $(\partial R^\nu=l_\nu)$, then we can prove the following Riemann's relation. While, if $R\in O''$, i.e. $\lambda\{L\}_E=0$, $\{L\}_E=\bigcup\limits_{n=1}^{\infty}\{L_n\}$ for certain exhaustion $E=\{R_n\}(\partial R_n=L_n)$, then under the canonical basis of $\mathfrak{A}$-*type* with respect to E Riemann's relation holds for any two differentials. Here we shall prove for this case. The former case is proved analogously, rather more simply.

Let $A_1, B_1, \cdots, A_n, B_n, \cdots$ be the canonical basis of $\mathfrak{A}$-*type* with respect to E. Now we consider two arbitrary Abelian differentials df_1 (1st or 2nd kind) and df_2 which have a finite number of singularities P_μ, where they have locally the expansions

$$\begin{aligned} df_1&=\left[-\left(\frac{pa_{-p}}{z^{p+1}}+\cdots+\frac{a_{-1}}{z^2}\right)+a_1+2a_2z+\cdots\right]dz \\ df_2&=\left[-\left(\frac{qb_{-q}}{z^{q+1}}+\cdots+\frac{b_0}{z}\right)+b_1+2b_2z+\cdots\right]dz\,. \end{aligned} \tag{2. 2}$$

Then we can find a sequence of curves $l_\nu\in\{L_{n_\nu}\}$ tending to $\mathfrak{F}$ such that they are disjoint each other and relation (2. 1) holds. Let

$A_1, B_1, \cdots, A_{k^\nu}, B_{k^\nu}$ $(k^\nu \equiv k_{n_\nu})$ be its relative basis on R_{n_ν} (mod L_{n_ν}), then we note that it is possible to replace A_i, B_i by A_i', B_i' which are homologous to A_i, B_i respectively, moreover A_i', B_i' $(i=1, 2, \cdots, k^\nu)$ are contained in the compact domain R^ν bounded by $l_\nu (\in \{L_{n_\nu}\} \subset \{\Gamma\})$. For instance in an annulus including L_{n_ν} and l_ν we replace the parts of A_i, B_i lying outside of R^ν by its homologous counterparts in R^ν. Now we may assume that the canonical basis are realized on the 1-dimensional elements and $\sum_{j=1}^{\lambda_\nu} a_j^2 = R^\nu$ where a_j^i denotes i-dimensional element, moreover that a_j^1 are all analytic curves not containing any pole of df_1 and df_2. We take R^ν so large that $R^\nu \supset R_{n_0}$, where R_{n_0} contains all singularities P_μ of df_1 and df_2. Now consider the sum of line integrals

$$(2.\ 3) \qquad I = \sum_{j=1}^{\lambda_\nu} \int_{\partial a_j^2} f_1\, df_2$$

where the branch of f_1 is defined as follows (cf. Ahlfors [1]), i.e. for a chain (curve) $L(\not\ni P_\mu)$ connecting a fixed point $b_0 (\neq P_\mu)$ on R_{n_0} to a point $b_j (\neq P_\mu)$ in a_j^2 we define by

$$(2.\ 4) \quad f_1(b_j) = \int_L df_1 + \sum_{i=1}^{k^\nu} [-N(A_i', L) \int_{B_i'} df_1 + N(B_i', L) \int_{A_i'} df_1].$$

Thus defined value is independent of the choice of L, because the difference of two such chains forms a cycle C and the difference of corresponding values of f_1 is equal to the period of df_1 along the cycle $C + \sum_i [-N(A_i', C) B_i' + N(B_i', C) A_i']$, which vanishes since df_1 is of first or second kind and $C \sim \sum [N(A_i', C) B_i' - N(B_i', C) A_i']$ (mod l_ν) (Cor. of Prop. 7). Now each a_j^1 inside of R^ν appears twice in these integrals (2. 3) and the corresponding difference of f_1 is equal to

$$\sum_i [-N(A_i', b_j^1) \int_{B_i'} df_1 + N(B_i', b_j^1) \int_{A_i'} df_1]$$

where b_j^1 is the dual element of a_j^1. On the other hand

$$\sum_j N(A_i', b_j^1) \int_{a_j^1} df_2 = \int_{A_i'} df_2, \quad \sum_j N(B_i', b_j^1) \int_{a_j^1} df_2 = \int_{B_i'} df_2 .$$

Since $A_i' = A_i, B_i' = B_i (i=1, \cdots, k_{n_0})$ and df_2 has no pole outside of R_{n_0},

$$\int_{A_i'} df_j = \int_{A_i} df_j,\ \int_{B_i'} df_j = \int_{B_i} df_j,\ i=1,\cdots,k^\nu,\ j=1,2,$$

hence we obtain

$$I=\sum_{i=1}^{k^\nu}\Big(\int_{A_i} df_1\int_{B_i} df_2-\int_{A_i} df_2\int_{B_i} df_1\Big)+\int_{l_\nu} f_1\,df_2.$$

Now $l_\nu\in\{\Gamma\}$, $l_\nu=\sum_i l_\nu^i$, $l_\nu^i\sim 0$ (mod $\mathfrak{F}$), therefore $\int_{l_\nu^i} df_2=0$ (Cor. of Prop. 7). Hence for fixed points $p_i\in l_\nu^i$

$$\Big|\int_{l_\nu^i} f_1\,df_2\Big|=\Big|\int_{l_\nu^i}(f_1(p)-f_1(p_i))df_2\Big|\leqq\int_{l_\nu^i}|df_1|\cdot\int_{l_\nu^i}|df_2|,$$

therefore

$$\Big|\sum_i\int_{l_\nu^i} f_1\,df_2\Big|\leqq\int_{l_\nu}|df_1|\int_{l_\nu}|df_2|\to 0\quad\text{for } \nu\to\infty.$$

While, $I=2\pi i\sum$ (residues of $f_1\,df_2$ on R^ν), hence we have

THEOREM 5.——*For each Riemann surface $R\in O'$ there exists an exhaustion and corresponding canonical basis of $\mathfrak{A}$-type such that for two Abelian differentials df_1 (1st or 2nd kind) and df_2 with finite Dirichlet integrals over R except the neighbourhoods of a finite number of singularities P_μ where they have locally the expansions* $(2\cdot 2)$, *we have*

$$(2.\ 5)\qquad \lim_{\nu\to\infty}\sum_{i=1}^{k^\nu}\Big(\int_{A_i} df_1\int_{B_i} df_2-\int_{A_i} df_2\int_{B_i} df_1\Big)$$
$$=2\pi i\sum_{P_\mu}\Big(\sum_{n=1}^{p} na_{-n}b_n-a_0b_0-\sum_{n=1}^{q} na_nb_{-n}\Big)\equiv I,$$

where a_0 is the constant term of f_1 at P_μ defined by $(2\cdot 4)$. *We have $I=0$ if both df_1 and df_2 are of first kind. If $R\in O''$, i.e. $\lambda\{L\}_I=0$, then for the canonical basis of $\mathfrak{A}$-type with respect to E* $(2\cdot 5)$ *holds always for any two such differentials.*[8] *If R is of finite genus, it is valid for any canonical basis on $R\in O_G$.*

Remark. 1°) On $R\in O_G$ there exist Abelian differentials dw_i (1st kind), dt_q^r (2nd kind, $r\geqq 2$) with finite Dirichlet integrals

8) If necessary, we should remove A_i, B_i a little so as to avoid P_μ.

except a neighbourhood of singularity q where $dt_q^r = (-r/z^{r+1} + \text{regular term})\, dz$, such that $\int_{A_j} dw_i = \delta_i^j$, $\int_{A_i} dt_q^r = 0$ (Virtanen [22]). We write here $A_1, B_1, \cdots$, as $K_1, K_2, \cdots$. If we take in Theorem 5 e.g. $df_1 = dw_\mu$, $df_2 = dw_\nu$; $df_1 = dt_q^r$, $df_2 = dw_\nu$, then

$$\int_{K_\nu} dw_\mu = \int_{K_\mu} dw_\nu\,, \quad \int_{K_{2\mu}} dt_q^r = \frac{-2\pi i}{(r-1)!} \frac{d^r w_\mu(q)}{dq^r}\,, \quad \mu, \nu = 1, 2, \cdots.$$

In connexion with elementary integrals on general surface (e.g. Sario [19]) we have corresponding formulas. (cf. Schiffer-Spencer [20] p. 74–76).

2°) Riemann's relation (2.5) holds also for certain restricted class of Abelian differentials having an infinite number of periods and singularities. For instance let $\{U_n\}$ be a sequence of disjoint compact domains on R and $U = \bigcup_{n=1}^{\infty} U_n$. We consider a class of Abelian differentials which are of first or second kind and have finite Dirichlet integrals over $R-U$. Now if the extremal length vanishes for subsets of $\{\Gamma\}$ or $\{L\}_E$ lying on $R-U$, then by modifying Prop. 7 and its Cor. we can obtain the corresponding formula (2.5) for differentials of this class. Although it gives a relation between an infinite number of periods and singularities, there is no gurantee for the convergency of the infinite series in (2.5), but it will converge for instance under further restriction such as (3.13).

§ 3. Riemann's second (bilinear) relation

1. By the same way as the proof of Theorem 5 we have

THEOREM 6.——*For each Riemann surface $R \in O'$ there exists an exhaustion and corresponding canonical homology basis of $\mathfrak{A}$-type on R such that for two Abelian differentials $df_j = du_j + idv_j$ $(j=1, 2)$ of the first kind with finite Dirichlet integrals we have*

$$(3.1) \qquad D_R(u_1, u_2) = \lim_{\nu\to\infty} \sum_{i=1}^{k_\nu} \Big(\int_{A_i} du_1 \int_{B_i} dv_2 - \int_{B_i} du_1 \int_{A_i} dv_2 \Big).$$

If $R \in O''$, i.e. $\lambda\{L\}_E = 0$, then for the canonical basis of $\mathfrak{A}$-type with respect to E (3.1) holds for any two such differentials.

2. For another extension of Riemann's second relation we

shall use the Ahlfors' theory of Schottky differentials under the same notations as Ahlfors [2].

THEOREM 7.——*Let $R \in O_{HD}$ and $A_1, B_1, \cdots, A_n, B_n, \cdots$ be an arbitrary canonical homology basis on R and let $df_j = du_j + i dv_j$ ($j = 1, 2$) be any two Abelian differentials of the first kind with finite Dirichlet integrals. If u_1 (or v_2) has only a finite number of non-vanishing periods, then we have*

$$D_R(u_1, u_2) = \sum_{i=1}^{N} \Big(\int_{A_i} du_1 \int_{B_i} dv_2 - \int_{B_i} du_1 \int_{A_i} dv_2 \Big). \tag{3.2}$$

Moreover, this theorem does not hold for $R \notin O_{HD}$.

Proof. Since $\Omega_i \equiv du_i$ ($i=1, 2$) are harmonic differentials with finite (Dirichlet) norm, for any cycle $C \sim 0$ (mod $\mathfrak{F}$)

$$\int_C \Omega_i = 0 \quad (i=1, 2). \tag{3.3}$$

Suppose e.g.

$$\int_{A_n} \Omega_1 = \int_{B_n} \Omega_1 = 0, \quad n \geqq N+1. \tag{3.4}$$

Now let $\{R_n\}$ be an arbitrary exhaustion of R. We take R_{n_0} so large that it contains completely the cycles $A_1, B_1, \cdots, A_N, B_N$. We define the branch in $R_n (n > n_0)$ of u_1 as before (2.4), i.e.

$$u_1(b_j) = \int_L du_1 + \sum_{i=1}^{N} [-N(A_i, L) \int_{B_i} du_1 + N(B_i, L) \int_{A_i} du_1].$$

Under (3.3), (3.4) we find that this value is independent of the choice of L connecting b_0 to b_j as before. Now let $\tau_n^* \in S_0^*(R_n)$ be the Schottky harmonic differential on $R_n (n > n_0)$ with the same periods as $(\Omega_2^*)^* = -\Omega_2$ on R_n, then $-\Omega_2 - \tau_n^*$ becomes an exact differentials, say dV_n.

$$\Omega_2^* = \tau_n - dV_n^*. \quad \tau_n = 0 \text{ on } \partial R_n. \tag{3.5}$$

$$D_{R_n}(u_1, u_2) = \iint_{R_n} \Omega_1 \Omega_2^* = \iint_{R_n} \Omega_1 \tau_n - \iint_{R_n} \Omega_1 dV_n^* \equiv I_1^n + I_2^n. \tag{3.6}$$

By Schwarz's inequality

$$|I_2^n|^2 \leqq \|\Omega_1\|^2_{R_n} \|dV_n^*\|^2_{R_n}.$$

Since $R \in O_{HD}$, we have

$$(3.\ 7)\qquad \|dV_n^*\|^2_{R_n} = \|\Omega_2^*\|^2_{R_n} - \|\tau_n\|^2_{R_n} \to 0 \quad (n \to \infty),$$

therefore under the condition $\|\Omega_1\|_R < \infty$

$$(3.\ 8)\qquad I_2^n \to 0 \quad (n \to \infty).$$

On the other hand, by Green's formula and integration by parts we have

$$(3.\ 9)\qquad I_1^n = \sum_j \int_{\partial a_j^2} u_1 \tau_n .$$

Now each a_j^1 inside of R_n appears twice in these line integrals and the corresponding difference of u_1 is equal to

$$(3.\ 10)\qquad \sum_{i=1}^{N} [-N(A_i, b_j^1) \int_{B_i} du_1 + N(B_i, b_j^1) \int_{A_i} du_1].$$

Since $R_n \supset R_{n0} \supset A_i, B_i (i=1, 2, \cdots, N)$, for $i=1, \cdots, N$

$$(3.\ 11)\qquad \sum_j N(A_i, b_j^1) \int_{a_j^1} \tau_n = \int_{A_i} \tau_n , \quad \sum_j N(B_i, b_j^1) \int_{a_j^1} \tau_n = \int_{B_i} \tau_n .$$

Since $\tau_n = 0$ on ∂R_n and τ_n converge uniformly to $\Omega_2^* = dv_2$ on every compact set on R, we obtain the desired result (3. 2) by (3. 6) $\sim$ (3. 11) when $n \to \infty$. Especially if we take $f_1 = f_2 = f$, then

$$(3.\ 12)\qquad D_R(u) = \sum_{i=1}^{N} (\int_{A_i} du \int_{B_i} dv - \int_{B_i} du \int_{A_i} dv).$$

Now if this theorem holds for $R \notin O_{HD}$, then for any single-valued harmonic function u with finite Dirichlet integral we have $D_R(u) = 0$ by (3. 12), i.e. $u \equiv$ const., therefore $R \in O_{HD}$, which is absurd, q.e.d.

Remark.——This theorem includes the Virtanen's result for $R \in O_G$ (Virtanen [22]). By Theorem 7 we can immediately extend the Virtanen's theory on A-periods to Riemann surface $\in O_{HD}$: The necessary and sufficient condition in order that there exists an Abelian integral of the first kind with finite norm and having the given A-periods. The existence of Abelian integral of the second kind with finite norm (except the neighbourhood of singularities) without any A-periods etc.

3. Let $A_1, B_1, \cdots, A_n, B_n, \cdots$ be a canonical homology basis of $\mathfrak{A}$-type with respect to an exhaustion $\{R_n\}$ of R. We denote

$\{A_i^{(n)}\}$, $\{B_i^{(n)}\}$ $(1\leqq i\leqq k_n)$ the set of cycles which are homologous to A_i resp. B_i and are contained in R_n. Then $\lambda\{A_i^{(n)}\}$, $\lambda\{B_i^{(n)}\}\geqq 0$[9] are monotone decreasing for fixed i.

THEOREM 8.——*Let $R\in O_{HD}$ and $A_1, B_1, \cdots, A_n, B_n, \cdots$ be a canonical homology basis of $\mathfrak{A}$-type with respect to an exhaustion $\{R_n\}$ of R. If for any n*

$$\sum_{i=1}^{k_n}\sqrt{\lambda\{A_i^{(n)}\}\lambda\{B_i^{(n)}\}}<M<\infty \tag{3. 13}$$

where k_n denotes the genus of R_n, then we have for any differentials df_1, df_2 with finite normes

$$D_R(u_1, u_2)=\lim_{n\to\infty}\sum_{i=1}^{k_n}\Big(\int_{A_i}du_1\int_{B_i}dv_2-\int_{A_i}dv_2\int_{B_i}du_1\Big).$$

Hence if du_1 and dv_2 have only a finite number of non-vanishing A-periods, (3. 2) *holds.*

Proof. Suppose $\lambda\{A_i^{(n)}\}$ and $\lambda\{B_i^{(n)}\}>0$, then by (1. 2) there exist the cycles A_i^n, B_i^n in R_n which are homologous to A_i, B_i respectively and for given $0<\varepsilon_n<\min(\sqrt{\lambda\{A_i^{(n)}\}}\|dV_n\|_{R_n}, \sqrt{\lambda\{B_i^{(n)}\}}\|df_1\|_{R_n})$

$$\begin{aligned}&\Big|\int_{B_i}du_1\Big|=\Big|\int_{B_i^n}du_1\Big|\leqq\int_{B_i^n}|df_1|\leqq\sqrt{\lambda\{B_i^{(n)}\}}\|df_1\|_R\\ &+\varepsilon_n<2\sqrt{\lambda\{B_i^{(n)}\}}\|df_1\|_R,\\ &\Big|\int_{A_i}dV_n^*\Big|=\Big|\int_{A_i^n}dV_n^*\Big|\leqq\int_{A_i^n}|dw_n|\leqq\sqrt{\lambda\{A_i^{(n)}\}}\|dV_n\|_{R_n}\\ &+\varepsilon_n<2\sqrt{\lambda\{A_i^{(n)}\}}\|dV_n\|_{R_n},\end{aligned} \tag{3. 14}$$

where $dw_n=dV_n+idV_n^*$ in R_n and $=0$ in $R-R_n$. These inequalities still hold even if $\lambda\{A_i^{(n)}\}$ (or $\lambda\{B_i^{(n)}\})=0$, because if $\big|\int_{A_i}dw_n\big|=\eta>0$, then $\varphi=|w_n'|/\eta$ is admissible for $\{A_i^{(n)}\}$ and

$$\lambda\{A_i^{(n)}\}^{-1}\leqq\iint_R\varphi^2dxdy=1/\eta^2\iint_{R_n}|w_n'|^2dxdy<\infty$$

9) The extremal length with respect to the set of cycles is analogously defined by taking the line integral along the cycle. (cf. Hersch [5]). Or, from the beginning we consider the canonical basis of $\mathfrak{A}$-type such that $A_1, B_1, \cdots$ are all Jordan closed curves. (cf. R. Nevanlinna [15]).

i.e. $\lambda\{A_i^{(n)}\}>0$ which is absurd, hence $\int_{A_i} dV_n^* = \int_{A_i} dw_n = 0$, etc. Now by (3. 5) we have analogously

$$D_{R_n}(u_1, u_2) = \sum_{i=1}^{k_n} \left(\int_{A_i} du_1 \int_{B_i} dv_2 - \int_{B_i} du_1 \int_{A_i} dv_2 \right)$$

$$+ \sum_{i=1}^{k_n} \left(\int_{A_i} du_1 \int_{B_i} dV_n^* - \int_{A_i} dV_n^* \int_{B_i} du_1 \right).$$

Hence by (3. 13), (3. 14), (3. 7)

$$\left| \sum_{i=1}^{k_n} \int_{B_i} du_1 \int_{A_i} dV_n^* \right| < 4 \|df_1\|_R \|dV_n^*\|_{R_n} \sum_{i=1}^{k_n} \sqrt{\lambda\{A_i^{(n)}\}\lambda\{B_i^{(n)}\}} \to 0 \quad (n \to \infty)$$

and analogously

$$\left| \sum_{i=1}^{k_n} \int_{B_i} dV_n^* \int_{A_i} du_1 \right| \to 0 \quad (n \to \infty).$$

Therefore we have the conclusion for $n \to \infty$, q.e.d.

On such a Riemann surface every Abelian integral of the first kind with finite norm is therefore determined except a constant by its A-periods. (cf. Ahlfors [1], L. Myrberg [11]).

Finally we note that the conditions in Theorems 6 and 8 are both concerned with the extremal length with respect to the set of cycles; the one is the set of cycles dividing Riemann surface, and the other the set of non dividing cycles.

Mathematical Institute
Kyoto University
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