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Note on the boundedness and the ultimate boundedness of solutions of $x' = F(t, x)$

By

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In order to obtain existence theorems of periodic solutions of the non-linear differential equation of the second order, Reuter [5]* and various authors have discussed the boundedness of solutions. The present author has also researched conditions for the boundedness or the ultimate boundedness of solutions for the purpose of using Massera's theorem (Theorem 2 in [4]) in the discussion of the existence of a periodic solution ([6], [7] and [10]). And utilizing the properties of solutions, the author has obtained necessary and sufficient conditions for the boundedness of solutions ([8] and [10]). A certain function which appears in these conditions resembles that of Liapounoff's research [2] for the stability of the solution.

The author thinks that the stability and the asymptotic stability correspond to the boundedness and the ultimate boundedness respectively and *in certain sense* they are of the same concepts respectively. Of course, with regard to the independent variable, both are of the problems "*in the large*", but with regard to unknown functions, the former is of the problem "*in the small*", while the other is of the problem "*in the large*". Now we will discuss the boundedness and the ultimate boundedness. Massera [3] and several authors have discussed problems reciprocal to Liapounoff's condition for the stability. We will see that we obtain some results analogous to those of their researches. The author has also obtained necessary and sufficient conditions for the stability ([9]) and of course we can obtain results for boundednesses analogous to them.

Now we consider a system of differential equations,

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* Numbers is [ ] refer to the bibliography at the end of the paper.

$$\frac{dx}{dt}=F(t, x), \tag{1}$$

where x denotes an n-dimensional vector and $F(t, x)$ is a given vector field which is defined and continuous in the domain

$$\Delta:\ 0\leqq t<\infty,\quad |x|<\infty.$$

$|x|$ represents the sum of the squares of its components. And let

$$x=x(t; x_0, t_0)$$

be a solution of (1) through the initial point (t_0, x_0).

Here we state the following definitions for the boundedness.

a) The solution $x=x(t; x_0, t_0)$ issuing from (t_0, x_0) to the right is said to be *bounded*, if there exists a positive number B such that $|x(t; x_0, t_0)|<B$ for $t\geqq t_0$. This B may be determined for each solution.

b) The solutions issuing from $t=t_0$ to the right are said to be *equi-bounded*, if for $|x_0|\leqq\alpha$, B in a) is determined depending only on α but independent of the particular solution considered.

c) The solutions are said to be *uniformly bounded*, if for every t_0, B is determined depending only on α and independent of t_0.

In the foregoing paper [8] we have obtained a necessary and sufficient condition for the boundedness. Now we will modify it so as to correspond to the considered conditions for the ultimate boundedness. To simplify the statement, we assume that every solution of (1) is unique for Cauchy-problem (cf. Theorem 3 and Remark 1 in [8]), then we have

in order that every solution of (1) *is bounded, it is necessary and sufficient that there exists a function* $\varphi(t, x)$ *satisfying the following conditions in* Δ; *namely*

1° $\varphi(t, x)>0$,

2° $\varphi(t, x)$ *tends to infinity uniformly for* t, *when* $|x|\to\infty$,

3° *for any solution of* (1), $x=x(t)$, *the function* $\varphi(t, x(t))$ *is a non-increasing function of* t.

For the condition of the *equi-boundedness* we require moreover

4° *there exists a positive constant* κ *such that*

$$\varphi(t_0, x)\leqq\kappa(K), \tag{2}$$

provided $|x|\leqq K$ (K: *arbitrary*).

Moreover for the *uniform boundedness,* $\varphi(t, x)$ *may be defined in* $|x| \geqq R_0 (R_0 > 0$ *may be sufficiently great) in* $\varDelta$ *and* κ *in* (2) *depends only on* K *and is independent of* t_0. Of course, we can assume that $\varphi(t, x)$ exists in $\varDelta$.

When $\varphi(t, x)$ is continuous with its partial derivatives of the first order, the condition 3° becomes

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} \cdot F(t, x) \leqq 0,$$

where the dot represents a scalar product. In this case there is κ for each t such as for $|x| \leqq K$, we have $\varphi(t, x) \leqq \kappa$ by the continuity of $\varphi(t, x)$ and hence the solutions are equi-bounded.

The following example shows that the boundedness and equi-boundedness are actually different concepts.

Example 1. Consider the following system in polar coordinates,

$$r' = r \frac{g'(t, \theta)}{g(t, \theta)}, \qquad \theta' = 0,$$

where $g'(t, \theta)$ is the derivative by t and $g(t, \theta)$ is the function

$$g(t, \theta) = \frac{(1+t) \sin^4 \theta}{\sin^4 \theta + (1 - t \sin^2 \theta)^2} + \frac{1}{1 + \sin^4 \theta} \cdot \frac{1}{1+t^2}.$$

The general solution of this equation is then

$$r = r_0 g(t, \theta_0), \qquad \theta = \theta_0.$$

If $\theta_0 = m\pi$ (m: integer), the solution is

$$r = \frac{r_0}{1+t^2}, \qquad \theta = m\pi,$$

and if $\theta_0 \neq m\pi$, the solution may be written, if we put $\tau = \frac{1}{\sin^2 \theta_0}$,

$$r = r_0 \left(\frac{1+t}{1 + (t-\tau)^2} + \frac{\tau^2}{1+\tau^2} \cdot \frac{1}{1+t^2} \right), \qquad \theta = \theta_0.$$

It is clear that every solution is bounded and if θ_0 is very near $m\pi$, the solution will have a great value r for $t = \tau$ which is as large as we please, so that the solutions are not equi-bounded.

Clearly if $n = 1$, the boundedness and the equi-boundedness are equivalent. Moreover if the system (1) is linear and the solutions are bounded, they are equi-bounded. In fact, if we make the equation into a homogeneous linear equation by the transformation $x = X + x(t)$, $x(t)$ being a solution issuing from $t = 0$ to the right, we have

$$X(t\,;X_0, 0)=A(t)\cdot X_0,$$

where $A(t)$ is a matrix and the dot represents matrix multiplication with the column vector X_0. By this boundedness, every element $A_{ij}(t)$ of $A(t)$ is bounded and hence there exists a positive number B^* such as $|A_{ij}(t)|<B^*$. From this the equi-boundedness is proved.

Thus when the system (1) is of the first order ($n=1$) or linear, if the solutions are bounded, they are equi-bounded, but they are not necessarily uniformly bounded. Massera's Example 1 ([3]) shows this fact.

Example 2. Let

$$g(t)=\sum_{m=1}^{\infty}\frac{1}{1+m^4(t-m)^2}.$$

Consider the first order linear differential equation

$$x'=\frac{g'(t)}{g(t)}x. \tag{3}$$

The solution of (3) satisfying the initial condition $x=x_0$ when $t=t_0$ is

$$x=\frac{x_0}{g(t_0)}g(t)$$

and if we make t_0 large, $g(t_0)$ has a value which is as near zero as we please. Therefore the solutions are not uniformly bounded.

Theorem 1. *We suppose that $F(t, x)$ in the system (1) is periodic of t. If its solutions issuing from $t=0$ are equi-bounded and the solutions issuing from $t>0$ are simply bounded, then they are uniformly bounded.*

Proof. Without the loss of the generality, we can assume that the period of $F(t,x)$ is 1. Consider the solutions issuing from $t=t_0$ to the right, where $0\leqq t_0<1$. For any given positive number α, all the solutions such as $|x_0|\leqq\alpha$ arrive at $t=1$, since they are bounded. Hence there exists a positive number β independent of t_0 for which we have

$$|x(1\,;x_0, t_0)|<\beta$$

(Theorem 3 in [1]). Thus it is sufficient to consider the solutions such as $|x_0|\leqq\beta$ at $t=1$ which are the same as the solutions such as $|x_0|\leqq\beta$ at $t=0$, because $F(t,x)$ is periodic of period 1. Since at $t=0$, the solutions are equi-bounded, there exists a positive number γ for β such as if $|x_0|\leqq\beta$, then $|x(t;x_0,1)|<\gamma$. Therefore for the solutions $x=x(t;\ x_0,t_0)$ such as $0\leqq t_0<1$, $|x_0|\leqq\alpha$, we have $|x(t;x_0,t_0)|<\gamma$. On the other hand, the conditions of solutions for $t_0\geqq1$ are the same as those for $0\leqq t_0<1$ by the periodicity of $F(t,x)$. Hence we have $|x(t;x_0,t_0)|<\gamma$ so long as we have $|x_0|\leqq\alpha$. Namely the solutions are uniformly bounded.

To simplify the statements, we give here *some promises*. When a continuous function $\varphi(t,x)$ below is defined in the domain

$$0\leqq t<\infty,\qquad |x|\geqq R_0\ \ (R_0 \text{ may be sufficiently great}),$$

$\varphi(t,x)$ is said to be defined in Δ^*. $\varphi(t,x)$ is always positive in its definition domain and satisfies locally the Lipschitz condition with regard to x. We represent

$$\overline{\lim_{h\to0}}\frac{1}{h}\{\varphi(t+h,x+hF(t,x))-\varphi(t,x)\}$$

by $D_F\varphi$. We will say briefly $\varphi(t,x)$ has the property A when there exists a κ (κ may depend on K) such that $\varphi(t,x)\leqq\kappa(K)$, provided $|x|\leqq K$, where K is an arbitrary positive number. Moreover we will say $D_F\varphi$ has the property B when there is a λ such as $D_F\varphi\leqq-\lambda(K)<0$, provided $|x|\leqq K$.

Theorem 2. *If $\varphi(t,x)$ is defined in Δ, $\varphi(t,x)$ tends to infinity uniformly as $|x|\to\infty$ and in the interior of Δ we have $D_F\varphi\leqq0$, the solutions of* (1) *are equi-bounded.*

Theorem 3. *If $\varphi(t,x)$ is defined in Δ^*, $\varphi(t,x)\to\infty$ uniformly as $|x|\to\infty$ and has the property A and in the interior of Δ^* we have $D_F\varphi\leqq0$, the solutions of* (1) *are uniformly bounded (cf. Theorem* 1 *in* [7]).

Two theorems above are sufficient conditions for the boundedness. Here we state the following definitions for the ultimate boundedness.

a) For the solution $x=x(t;x_0,t_0)$ issuing from (t_0,x_0) to the right, if there exist positive numbers B and T such that $|x(t;x_0,t_0)|<B$ for $t\geqq t_0+T$, the solutions of (1) are said to be *ultimately bounded,* where B is independent of the particular solution while T may depend on the solution. Here t_0 is arbitrary.

b) The solutions issuing from $t=t_0$ to the right are said to be *equi-ultimately bounded,* if for $|x_0|\leqq\alpha$, T in a) is determined depending only on α and t_0.

c) The solutions are said to be *uniformly ultimately bounded*, if for every t_0, T is determined depending only on α and independent of t_0.

For the ultimate boundedness we have

Theorem 4. *If there exists a function $\varphi(t, x)$ defined in Δ^* which tends to infinity uniformly for t as $|x|\to\infty$ and has the property A and if $D_F\varphi$ has the property B, the solutions of* (1) *are uniformly ultimately bounded.*

Proof. Since $\varphi(t, x)\to\infty$ uniformly as $|x|\to\infty$, there exists a positive number R such that if $|x|\geqq R(\geqq R_0)$, for a given $G>0$ we have $G<\varphi(t, x)$. Now we consider the domain Δ^{**} such as

$$0\leqq t<\infty, \quad |x|\geqq R.$$

Then there is a positive number B such that if $|x_0|\leqq R$,

$$|x(t; x_0, t_0)|<B,$$

since the solutions of (1) are uniformly bounded by Theorem 3. Now we consider $x=x(t; x_0, t_0)$ such as $|x_0|\leqq\alpha$, where α is an arbitrary positive number and $\alpha>R$. Then there exists a positive number β depending only on α and we have $|x(t; x_0, t_0)|<\beta$ for $t_0\leqq t<\infty$. Considering $\varphi(t, x)$ in the domain Δ^{***} such as

$$0\leqq t<\infty, \quad R\leqq|x|\leqq\beta,$$

there exists a positive number λ depending on β such as $D_F\varphi\leqq-\lambda(\beta)$, because $D_F\varphi$ has the property B. Now we take a function $\varphi(t, x)e^{Nt}$, where N is a suitable positive constant. Then this function is positive and tends to infinity uniformly as $t\to\infty$, since $\varphi(t, x)e^{Nt}>Ge^{Nt}$. Moreover we have

$$D_F\varphi e^{Nt}=e^{Nt}(D_F\varphi+N\varphi)\leqq e^{Nt}(-\lambda+N\varphi).$$

As $\varphi(t, x)$ has the property A, there is κ such as $\varphi(t, x)\leqq\kappa(\beta)$ in Δ^{***}. Hence determining N such as

$$-\lambda+N\kappa\leqq 0,$$

we have

$$D_F\varphi e^{Nt}\leqq 0.$$

Choosing T such that

$$\kappa e^{Nt_0}<Ge^{N(t_0+T)},$$

we can see that for any solution satisfying $R<|x_0|\leqq\alpha$ we have $|x|=R$ at some $t'(t_0\leqq t'\leqq t_0+T)$. Namely we can prove that $|x(t;x_0,t_0)|<B$ for $t\geqq t_0+T$, where T is determined depending only on α, since it is sufficient that we have

$$\kappa<Ge^{NT}.$$

Therefore the solutions of (1) are uniformly ultimately bounded.

Corollary. *When the solutions of* (1) *are uniformly bounded, if there exists a function* $\varphi(t,x)$ *in* Δ^* *such that it has the property A and* $D_F\varphi$ *has the property B, the solutions are uniformly ultimately bounded.*

We assume in Theorem 4 that $\varphi(t,x)$ has the property A, but weakening this property, we have

Theorem 5. *If* $\varphi(t,x)$ *tends to infinity uniformly as* $|x|\to\infty$ *and* $D_F\varphi$ *has the property B and if there exist some* R $(\geqq R_0)$ *and* κ *such that* $\varphi(t,x)\leqq\kappa$ *for* $|x|=R$, *the solutions of* (1) *are equi-ultimately bounded.*

Proof. If $t=t_0$ (t_0 : arbitrary) and $|x_0|\leqq R$, there exists a positive number B such as

$$|x(t;x_0,t_0)|<B$$

for $t\geqq t_0$. For any given $\alpha>R$, we consider solutions issuing from $t=t_0$, $|x_0|\leqq\alpha$. If $|x_0|\leqq R$, of course, we have $|x(t;x_0,t_0)|<B$ for $t\geqq t_0$, and hence we assume that $R<|x_0|\leqq\alpha$. Since there exists

$$\max_{R\leqq|x_0|\leqq\alpha}\varphi(t_0,x_0)>0,$$

we represent that value by M. And we take a positive number β such as

$$M<\inf_{|x|=\beta}\varphi(t,x).$$

Then we can see that if the solution $x=x(t;x_0,t_0)$ issuing from $t=t_0$, $R<|x_0|\leqq\alpha$ to the right lies always in $|x|>R$ for $t\geqq t_0$, we have $|x(t;x_0,t_0)|<\beta$. Consider the domain $t_0\leqq t$, $R<|x|\leqq\beta$. If this solution lies always in this domain, there arises a contradiction, because we should have by the property B

$$\varphi(t,x(t))-\varphi(t_0,x(t_0))\leqq-\lambda(t-t_0).$$

Therefore for the solution $x=x(t;x_0,t_0)$ satisfying $|x_0|\leqq\alpha$ we must have at some t'

$$|x(t';x_0,t_0)|\leqq R.$$

Concluding, we see that if $|x_0|\leqq\alpha$, there exists T depending only on t_0 and α for which we have

$$|x(t;x_0,t_0)|<B, \quad \text{if } t\geqq T.$$

Thus the solutions of (1) are equi-ultimately bounded.

For instance, as a simple example,

Example 3. We consider the first order linear differential equation

$$x'=-\frac{x}{t+1}. \tag{4}$$

For this equation we put

$$\varphi(t,x)=(t+1)(x^2-R)+1,$$

where R is a positive constant. Then as this $\varphi(t,x)$ satisfies the conditions in Theorem 5, the solutions of (4) issuing from every $t=t_0$ are equi-ultimately bounded. The general solution of this equation is $x=x_0(t_0+1)/t+1$ and hence, for a positive number R, the solutions are clearly equi-ultimately bounded, though they are not uniformly ultimately bounded; for the solution such as $|x_0|\leqq\alpha$ satisfies $|x|<R$ when

$$t>t_0+(t_0+1)\left(\sqrt{\frac{\alpha}{R}}-1\right).$$

Therefore such $\varphi(t,x)$ as in Theorem 4 does not exist. The property A is not implied by the ultimate boundedness and the equi-ultimate boundedness.

The solutions of $x''=-x$ are clearly uniformly bounded, but they are not ultimately bounded. In this case, we have to consider the system $x'=y$, $y'=-x$ and we may define $\varphi(t,x,y)=x^2+y^2$. For this φ, $D_F\varphi$ has not the property B. Hence it seems that the property B is required by the ultimate boundedness. On the other hand, when the uniform boundedness of solutions is known already and $D_F\varphi$ has the property B, even if $\varphi(t,x)$ has not the property A, the solutions are ultimately bounded, since there arises a contradiction by the relation

$$\varphi(t,x(t))-\varphi(0,x(0))\leqq-\lambda t\to-\infty\ (t\to\infty)$$

and hence we have $|x|\leqq R_0$ at some t. Therefore it seems that the property A is a required condition for the uniform boundedness or the uniform ultimate boundedness. Example 2 shows that the assumption that $\varphi(t, x)$ has the property A cannot be dropped in Theorem 4. For the equation (3), there exists a function $\varphi(t, x)$ which satisfies all the conditions in Theorem 4 except the property A, but the solutions of (3) are not ultimately bounded; of course they are not uniformly bounded as stated before.

Weakening the condition that $D_F\varphi$ has the property B, we have the following theorem.

Theorem 6. *Suppose that the solutions of the system* (1) *are uniformly bounded and that* $\varphi(t,x)$ *is defined in* Δ^* *and has the property* A. *Moreover we assume that there exists a positive function* $\varphi^*(t, x)$ *defined in* Δ^* *satisfying the following conditions; namely* $-\varphi^*(t, x)$ *has the property* B *and*

$$\lim_{t\to\infty} (D_F\varphi+\varphi^*) = 0 \tag{5}$$

uniformly in any domain $R_0 \leqq |x| \leqq K$.

Then the solutions of (1) *are uniformly ultimately bounded.*

Proof. By the uniform boundedness of the solutions, there exists a positive number β depending only on α such that if $|x_0| \leqq \alpha$, then $|x(t; x_0, t_0)| < \beta$. For this β, a positive number γ is so determined depending only on β that if $|x_0| \leqq \beta$, then $|x(t; x_0, t_0)| < \gamma$. Namely γ is determined depending only on α. Now considering in $R_0 \leqq |x| \leqq \gamma$, there is λ such as $\varphi^*(t, x) \geqq \lambda (>0)$, for $-\varphi^*(t, x)$ has the property B. By (5), there exists T such as

$$D_F\varphi+\varphi^* \leqq \frac{\lambda}{2} \qquad (t \geqq T)$$

and hence $D_F\varphi \leqq -\frac{\lambda}{2}$ for $t \geqq T$. While for the solutions issuing from $|x_0| \leqq \beta$, $t_0 \geqq T$ to the right, we can verify the uniform ultimate boundedness in the same way as in the proof of Theorem 4. Therefore for the solutions issuing from $|x_0| \leqq \alpha$, $t = t_0$ (t_0: arbitrary), we can verify the uniform ultimate boundedness.

If $D_F\varphi$ has the property B, there exists a $\varphi^*(t, x)$ stated in this theorem, since it is sufficient to put $\varphi^* = -D_F\varphi$, but the existence of $\varphi^*(t, x)$ does not imply that $D_F\varphi \leqq 0$. Hence this condition is weaker than the condition that $D_F\varphi$ has the property B. Example 3 shows that the assumption that $\varphi(t, x)$ has the property

A cannot be dropped in this theorem. In fact, since the solution of (4) such as $x=x_0$ when $t=t_0$ is $x=x_0(t_0+1)/t+1$, the solutions are uniformly bounded and $\varphi^*=-D_F\varphi$.

Theorem 7. *Suppose that $\varphi(t,x)$ is defined in Δ^* and $\varphi(t,x)\to\infty$ uniformly as $|x|\to\infty$ and that there exist R and κ such as $\varphi(t,x)\leqq\kappa$ when $|x|=R$. Moreover suppose that $\varphi^*(t,x)$ is positive and $-\varphi^*(t,x)$ has the property B (φ^* may be defined in Δ^{**}) and that*

$$\lim_{t\to\infty}(D_F\varphi+\varphi^*)=0$$

uniformly in any domain $R\leqq|x|\leqq K$.

Then if the solutions of (1) *are bounded, they are ultimately bounded, and if they are equi-bounded, they are equi-ultimately bounded.*

Proof is omitted.

Remark. In this theorem, even if the solutions are uniformly bounded, they are not necessarily uniformly ultimately bounded. It is clear from Example 3. The condition " $\varphi(t,x)\to\infty$ uniformly as $|x|\to\infty$" is not essential and can be replaced by other conditions, because it is utilized only to show that the solutions such as $|x_0|\leqq R$ are uniformly bounded.

Theorem 6 may be modified as follows; if the solutions of (1) are bounded and $\varphi^*(t,x)\geqq\kappa(>0)$ when $R_0\leqq|x|$ and (5) is true uniformly for $R_0\leqq|x|$ and if the condition " $\varphi\to\infty$ uniformly as $|x|\to\infty$" is added to the properties of $\varphi(t,x)$, the solutions of (1) are uniformly ultimately bounded.

As we have seen in Example 1, the ultimate boundedness and the equi-ultimate boundedness are different concepts, while we have the following theorems.

Theorem 8. *If the system* (1) *is of the first order and the solutions are ultimately bounded, they are also equi-ultimately bounded.*

Theorem 9. *If the system* (1) *is linear and the solutions are ultimately bounded, they are also equi-ultimately bounded and besides they are equiasymptotically stable (cf. p.* 706 *in* [3]).

Proof. Since we can transform the equation into a homogeneous linear one by the transformation $x=X+x(t)$, $x(t)$ being a solution starting from $t=0$ to the right, we may consider the equation to be homogeneous from the beginning. Then the general solution is of the form

$$x(t; x_0, 0) = A(t) \cdot x_0,$$

where $A(t)$ is a matrix and the dot represents matrix multiplication with the column vector x_0. If the solutions are ultimately bounded for the bound B, all the elements of $A(t)$ must tend to zero when $t \to \infty$: otherwise, for a certain element $A_{ij}(t)$ and a positive number a, there exists a divergent sequence $\{t_m\}$ such as $|A_{ij}(t_m)| \geqq a$. For a solution satisfying the initial condition $x_{0,1} = x_{0,2} = \cdots = x_{0,j-1} = x_{0,j+1} = \cdots = x_{0,n} = 0$, $|x_{0,j}| > \dfrac{B}{a}$, we have

$$|x_i(t_m)| > a \cdot \frac{B}{a} = B,$$

since $|x_i| = |A_{ij}(t)|\ |x_{0,j}|$, and hence it is not ultimately bounded for the bound B. Therefore all the elements must tend to zero as $t \to \infty$. Namely $x = 0$ is equiasymptotically stable. In this case, the equi-ultimate boundedness and the equiasymptotic stability are same concepts in the small.

Example 3 shows that in Theorems 8 and 9, from the ultimate boundedness we may derive the equi-ultimate boundedness, but not the uniform ultimate boundedness.

Theorem 10. *We suppose that $F(t, x)$ in the system* (1) *is periodic of t and that every solution of* (1) *is unique for Cauchy-problem. Moreover we assume that there exists a positive number κ such as if $|x_0| \leqq B$, $|x(t; x_0, 0)| < \kappa$. Then if the solutions are ultimately bounded for the bound B, they are uniformly ultimately bounded.*

Proof. Without the loss of generality, we can assume that the period of $F(t, x)$ is 1. As the solutions are ultimately bounded for the bound B, of course, they are so for the bound κ. Now we will show that they are uniformly ultimately bounded for the bound κ. Considering the solutions issuing from $t = 0$, $|x| \leqq \alpha$ to the right, we suppose that they are not equi-ultimately bounded for the bound κ. Then for any given k (integer) there exist x_k^0 and t_k such that $|x_k^0| \leqq \alpha$, $t_k \geqq k$ and

$$|x(t_k; x_k^0, 0)| \geqq \kappa. \tag{6}$$

The points x_k^0 have a point of accumulation x_0 and we have $|x_0| \leqq \alpha$. By the assumption, there exists an integer $N > 0$ such as

$$|x(N; x_0, 0)| < B.$$

Now at $t=N$, take a sufficiently small neighborhood U of $x(N; x_0, 0)$ and then all the solutions starting from a suitable neighborhood of x_0 at $t=0$ go into U by the uniqueness of the solution. Therefore there are indexes k as large as we please, for which

$$|x_{k,N}| = |x(N; x_k^0, 0)| < B.$$

On the other hand since $x(t; x_{k,N}, N) = x(t-N; x_{k,N}, 0)$ by the periodicity of the system, we have

$$|x(t; x_k^0, 0)| = |x(t; x_{k,N}, N)| < \kappa \quad \text{if } t \geqq N$$

which contradicts (6). Hence the solutions are equi-ultimately bounded.

Then since all the solutions such as $|x_0| \leqq \alpha$, $0 \leqq t_0 \leqq 1$ arrive at $t=1$, there exists a positive number β such as $|x(1; x_0, t_0)| < \beta$. Thus it is sufficient to consider the solutions such as $|x_0| \leqq \beta$ at $t=1$ which are the same as those such as $|x_0| \leqq \beta$ at $t=0$ and hence we have $|x(t; x_0, t_0)| < \kappa\,(0 \leqq t_0 \leqq 1)$ if $t \geqq T+1$, where T is one for the solutions such as $|x_0| \leqq \beta$, $t_0 = 0$. Therefore if $1+T+t_0 \leqq t$, we have $|x(t; x_0, t_0)| < \kappa\,(|x_0| \leqq \alpha)$. Namely the solutions are uniformly ultimately bounded.

When this theorem holds good, considering in $0 \leqq t \leqq T+1$, for the solutions such that $|x_0| \leqq \alpha$, $0 \leqq t_0 \leqq 1$, there exists a positive number β such as $|x(t; x_0, t_0)| < \beta$ for $0 \leqq t \leqq T+1$, since all the solutions are continuable. If $t \geqq T+1$, we have $|x(t; x_0, t_0)| < \kappa$. After all there is $\bar{\kappa}$ such as $|x(t; x_0, t_0)| < \bar{\kappa}$ for $t \geqq 0$. Hence by the periodicity, there exists a positive number γ for which we have $|x(t; x_0, t_0)| < \gamma$, provided $|x_0| \leqq \alpha$ (t_0: arbitrary). Namely the solutions are uniformly bounded.

In the stability, the definition of the asymptotic stability contains the condition that the solution is stable (cf. p. 706 in [3]). In Example 1, solutions tend to $r=0$ as $t \to \infty$, but $r=0$ is not stable and hence it is not said to be asymptotically stable. If we will make the ultimate boundedness correspond to the asymptotic stability, we may add the equi-boundedness of solutions to the definition of the ultimate boundedness. If so, Massera's Example 4 ([3]) shows that the ultimate boundedness and the equi-ultimate boundedness are different concepts. When the definition of the ultimate boundedness is modified as this, Theorem 10 may be stated simply " if the solutions are ultimately bounded, they are uniformly ultimately bounded".

As a corollary of Theorem 10, we have

Corollary. *Suppose that $F(t, x)$ in the system* (1) *is periodic of t and every solution is unique for Cauchy-problem. If the solutions issuing from $t=0$ are equi-bounded and all the solutions are ultimately bounded, they are uniformly ultimately bounded.*

Theorem 11. *Suppose that $F(t, x)$ in the system* (1) *is bounded when $|x|$ is bounded and that there exists $\varphi(t, x) \geqq 0$ defined in Δ satisfying the following conditions;*

1° $\varphi(t, x) \to \infty$ *uniformly as* $|x| \to \infty$,

2° $D_F \varphi \leqq 0$,

3° *for a positive number R which may be sufficiently great, $D_F\varphi$ has the property B, if $|x| \geqq R$.*

Then the solutions of (1) *are ultimately bounded.*

The proof is omitted, since it is similar as one of Theorem 4 in [3]. Remark that if we make $|x|$ large suitably, we have $\varphi > 0$ for $|x| \geqq R^*$, since $\varphi \to \infty$, and then we have to use $\bar{R} = \max(R, R^*)$. The solutions are ultimately bounded for the bound $2\bar{R}$. Example 2 shows that the boundedness of $F(t, x)$ cannot be dropped in this theorem.

Now in order to consider the reciprocal problem of Theorem 4 or Theorem 5, we assume that $F(t, x)$ in the system (1) has *continuous partial derivatives of the first order with respect to x.* Hence, of course, every solution is unique for Cauchy-problem. Then we have the following theorem.

Theorem 12. *If $F(t, x)$ is periodic of t and the solutions issuing from $t=0$ are equi-bounded and the solutions are ultimately bounded, then there exists a positive function $\varphi(t, x)$ defined in Δ^* (R_0 be sufficiently great) which is continuous with its partial derivatives of the first order and tends to infinity uniformly as $|x| \to \infty$ and satisfies the following conditions; namely*

1° $\varphi(t, x)$ *has the property A,*

2° $\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} \cdot F(t, x)$ *has the property B, where the dot represents the scalar product.*

Proof. Now we suppose that the solutions are ultimately bounded. Then by Theorem 10 they are uniformly ultimately bounded. Hence we assume that the solutions are uniformly ultimately bounded for the bound B. Moreover by Theorem 1, the solutions of (1) are uniformly bounded. In order to construct a function $G(\eta)$, at first we put

$$\sup_{\substack{0\leqq t_0<\infty \\ |x_0|\leqq\eta \\ t_0\leqq\tau<\infty}} |x(\tau\,;x_0,t_0)|=f(\eta).$$

$f(\eta)$ is uniquely determined, since the solutions are uniformly bounded. $f(\eta)$ is non-decreasing with respect to η. Then if we put

$$\sup_{\substack{0\leqq t<\infty \\ |x_0|\leqq f(\eta)}} |F(t,x_0)|=g(\eta),$$

then $g(\eta)$ is R-integrable, because $g(\eta)$ is non-decreasing. Let $h(\eta)$ be

$$\int_{\eta}^{\eta+1} g(s)\,ds=h(\eta),$$

then $h(\eta)$ is continuous and clearly we have $g(\eta)\leqq h(\eta)$. Moreover if we put

$$\int_{\eta}^{\eta+1} h(s)\,ds=k(\eta),$$

then $k(\eta)$ is continuous and differentiable with respect to η and we have $h(\eta)\leqq k(\eta)$. Now we consider a function of η for $\eta\geqq 0$ as follows; namely

$$G(\eta)=\begin{cases}[k(4\eta)-k(4B)]^2+(\eta-B)^2 & (\eta\geqq B)\\ 0 & (0\leqq\eta<B).\end{cases}$$

Then $G(\eta)$ is continuous for $\eta\geqq 0$ and for $0\leqq\eta\leqq B$ we have $G(\eta)=0$ and for $\eta>B$ it is positive increasing and it has the continuous derivative which is positive for $\eta>B$ and vanishes for $0\leqq\eta\leqq B$. And $G(\eta)$ tends to infinity as $\eta\to\infty$.

Now we put

$$\varphi(t,x)=\int_t^\infty G(|x(\tau\,;x,t)|)\,d\tau. \tag{7}$$

This $\varphi(t,x)$ exists indeed, since by the uniform ultimate boundedness of x for the bound B, there is T depending only on α such as if $t\geqq t_0+T$, then $|x(t\,;x_0,t_0)|<B$ holds, where $|x_0|\leqq\alpha$, and hence we have

$$\varphi(t, x) = \int_t^{t+T} G(|x(\tau; x, t)|)\, d\tau.$$

Thus $\varphi(t, x)$ is continuous and clearly it is positive when $|x| > B$.

Since $F(t, x)$ is assumed to have continuous partial derivatives of the first order with respect to x,

$$\frac{\partial |x(t; x_0, t_0)|}{\partial t_0} \quad \text{and} \quad \frac{\partial |x(t; x_0, t_0)|}{\partial x_0}$$

exist and are continuous and hence $\varphi(t, x)$ has continuous partial derivatives of the first order.

Since there exists β independent of t_0 such that if $|x_0| \leqq \alpha$, then $|x(t; x_0, t_0)| < \beta$ by the uniform boundedness of the solutions, we have

$$\varphi(t, x) \leqq \int_t^{t+T} G(\beta)\, d\tau = T \cdot G(\beta).$$

Therefore $\varphi(t, x)$ has the property A.

If the point (t, x) moves along a fixed solution, say the solution through (t_0, x_0), we have

$$\varphi(t, x(t)) = \int_t^{\infty} G(|x(\tau; x_0, t_0)|)\, d\tau$$

so that

$$\frac{d}{dt}\varphi(t, x(t)) = -G(|x(t; x_0, t_0)|) = -G(|x|)$$

and hence $\dfrac{d\varphi}{dt}$ has the property B, provided we consider $\varphi(t, x)$ in $|x| \geqq R_0 (> B)$.

Finally, we will see that $\varphi(t, x) \to \infty$ uniformly as $|x| \to \infty$. Clearly

$$\varphi(t, x) = \int_0^{\infty} G(|x(t+\tau; x, t)|)\, d\tau.$$

As we may assume that $g(|x|)$ is positive, for ρ such as

$$\rho = \frac{\sqrt{|x|}}{2g(|x|)\sqrt{n}},$$

we have

$$\varphi(t, x) \geqq \int_0^{\rho} G(|x(t+\tau; x, t)|)\, d\tau.$$

On the other hand, we have

$$|x_i(t+\tau\,;\,x,t)-x_i|\leqq g(|x|)\,\tau$$

and hence for $0\leqq\tau\leqq\rho$

$$\sum_{i=1}^{n}\{x_i(t+\tau\,;\,x,t)-x_i\}^2\leqq n\{g(|x|)\}^2\rho^2=\frac{|x|}{4}\,.$$

From this we have

$$|x(t+\tau\,;\,x,t)|\geqq\frac{|x|}{4}\,.$$

Therefore

$$\varphi(t,x)\geqq\int_0^\rho G\Big(\frac{|x|}{4}\Big)\,d\tau=\frac{\sqrt{|x|}}{2g(|x|)\sqrt{n}}\,G\Big(\frac{|x|}{4}\Big)\,.$$

As we may assume that $|x|$ is sufficiently great, we have

$$\varphi(t,x)\geqq\frac{\sqrt{|x|}}{2k(|x|)\sqrt{n}}\Big\{[k(|x|)-k(4B)]^2+\Big(\frac{|x|}{4}-B\Big)^2\Big\},$$

since $g(|x|)\leqq k(|x|)$. The right-hand side of this inequality tends to infinity as $|x|\to\infty$ and hence $\varphi(t,x)$ tends to infinity uniformly as $|x|\to\infty$.

Remark. Of course, the assumptions in this theorem may be those in Theorem 10.

When the system (1) is linear, as we have stated before, if the solutions are ultimately bounded, they are equiasymptotically stable, while the necessary condition for it has been discussed by Massera (Theorem 9 in [3]).

When in Theorem 11, $F(t,x)$ is periodic of t, then Theorem 12 becomes its converse. Moreover observing the construction of $\varphi(t,x)$ in Theorem 12, clearly we know that there exists a similar function $\varphi(t,x)$, if $F(t,x)$ is bounded for $|x|$ bounded and moreover if the solutions are uniformly bounded and are uniformly ultimately bounded. Conversely if there exists such a function $\varphi(t,x)$, the solutions are uniformly bounded and are uniformly ultimately bounded.

In case of $F(t,x)$ not differentiable with respect to x, if we assume only the continuity of $F(t,x)$ and the uniqueness of solutions for Cauchy-problem, the statement that $\varphi(t,x)$ is differentiable is not necessary in Theorem 12, while 2° is replaced by

$$\frac{d}{dt}\varphi(t,x(t))\leqq-\lambda(K)<0,\quad\text{provided}\quad R_0\leqq|x(t)|\leqq K.$$

This is the same expression as those in [8] and [9]. Of course, its converse is true.

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