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Some remark on rational points

By

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§ **1.** It seems to us that, in spite of their importance, little is known about properties of rational points on algebraic varieties. In this short note we shall prove[1)]

THEOREM. *Let U^r, V be abstract varieties, V be complete, π be a rational function defined on U with values in V, and k be a common field of definition for U, V and π. If U has a rational point P over k which is simple on U, then V has also a rational point over k.*

We begin by two lemmas which are proved in elementary way.

LEMMA 1. *Let U^r be an algebraic variety in S^n, $P=(x)$ be a simple point of U and k be a field over which U and (x) are rational. Then there is a subvariety W of dimension $r-1$ with the following properties:*

(i) *P is contained in W as a simple point,*

(ii) *W is defined over a purely transcendental extention $k(u)$ of k.*

(iii) *every rational function π defined on U is regular along W.*

PROOF. Let H be the hyperplane of S^n defined by the equation

$$\sum_{i=1}^{n} u_i (X_i - x_i) = 0,$$

where (u) is a set of independent variables over k.

Then, as P is simple on U and H is transversal to U at P, it

1) This problem is proposed to us by Y. Nakai.

2) It is noted that, instead of using a component of hyperplane section of U, we may take as W the most general hypersurface section of sufficiently high degree containing P, which is itself absolutely irreducible subvariety. See M. Nishi and Y. Nakai, " On the hypersurface sections of algebraic varieties embedded in a projective space." Mem. Coll. Sci. Univ. of Kyoto, vol. XXIX, 1955.

is well-known that there is one and only one component W^{r-1} of $U \cap H$ containing P, and this W has the properties (i), (iii) mentioned in lemma 1.

As to (ii), in the first place W is algebraic over $k(u)$. If σ be any automorphism of $\overline{k(u)}$ over $k(u)$, then P belongs also to the conjugate W^{σ} of W, which implies $W^{\sigma}=W$.

On the other hand, the order of inseperability of W over $k(u)$ is equal to 1, since the intersection multiplicity $j(U.H, W)=1$.

Therefore W is defined over $k(u)$. Thus we have proved lemma 1.

LEMMA 2. *Let V be a complete abstract variety defined over k. If V has a rational point Q over $k(u)$ where $(u)=(u_1, \cdots, u_n)$ is a set of independent variables over k, then V has also a rational point Q' over k.*

PROOF. It is sufficient to prove in the case where $n=1$. Let E_1 be a numerical straight line with reference to k, which is locus of point (u) over k.

Then, since Q is rational over $k(u)$, there is a rational function θ: $\theta(u)=Q$, defined on E_1 with values in V with reference to k. As E_1 is a nonsingular curve, and V is complete, θ is defined at every point, and so particularly at a rational point (a) of E_1 over k. Then $Q'=\theta(a)$ is a rational point of V over k. q. e. d.

PROOF OF THE THEOREM. Now we shall prove the theorem, using induction on the dimension r of U. When $r=1$, the assertion follows from the fact that π is defined at every simple point, particularly at P.

Let U be a representative of $\boldsymbol{U}$ in which $\boldsymbol{P}$ has a representative P. Since U, V, P and k satisfy the conditions in our theorem, without loss of generality, we may assume that $\boldsymbol{U}$ is a variety embedded in an affine space S^n.

Then, as has been verified in lemma 1, there is a subvariety W^{r-1} of U defined over $k(u)$, containing P as a simple and rational point over $k(u)$. It is clear that W, V, P and restriction π_W of π to W satisfy the conditions in our theorem for dimension $r-1$, with reference to $k(u)$.

Therefore, by the induction assumption, we conclude that, with reference to $k(u)$, V has a rational point Q. Hence lemma 2 is applicable, and with reference to k V has also a rational point Q'. q. e. d.

§ **2.** In the above theorem, the assumption that P is simple is essential. The following example[3] shows that even if P is a normal point, V has not always a rational point.

First we shall prove a lemma probably well-known.

LEMMA 3. *Let $x_1, \cdots, x_n$ be n independent variables over a field k of characteristic $p \neq 2$, and let $z = \sqrt{f(x)}$, where $f(x)$ is a polynomial with no multiple prime factor in $k[x]$. Then the ring $\mathfrak{o} = k[x, z]$ is integrally closed in its quotient field.*

PROOF. Let $w = r_1(x) + r_2(x)z \in k(x, z)$ be any integral element over $\mathfrak{o}$, where $r_i(x) \in k(x)$; $i=1, 2$.

Then the conjugate w^σ of w over $k(x)$, hence $w + w^\sigma$, $w \cdot w^\sigma$, and therefore, $r_1(x)$, $r_2(x)^2 z^2$ are integral over $\mathfrak{o}$. This implies $r_1(x) \in k[x]$, $r_2(x)^2 z^2 = r_2(x)^2 f(x) \in k[x]$.

Let an expression of r_2 be $r_2(x) = h(x)/g(x)$, where $g(x)$, $h(x)$ are relatively prime polynomials in $k[x]$, then $g(x)^2$ divides $f(x)$, which implies $g(x)$ is an unit in $k[x]$, by the assumption of our lemma for $f(x)$.

Hence $r_2(x) \in k[x]$, therefor $w \in k[x, z] = \mathfrak{o}$ q. e. d.

The example (for any value of characteristic $p \neq 2$) is the following:

U^2 is the surface in S^3 defined by the equation

$$x^4 + ux^2 + vy^2 + wz^2 = 0\,,$$

containing a rational point $P = (0, 0, 0)$,

V^2 is the surface in projective space L^3 defined by the equation

$$x^2 + ut^2 + vy^2 + wz^2 = 0\,,$$

in homogeneous co-ordinates (x, y, z, t); where u, v, w are independent variables over a prime field κ.

It is readily seen U^2 and V^2 are birationally equivalent over $k = \kappa(u, v, w)$.

Since $f(x, y) = -\dfrac{1}{w}(x^4 + ux^2 + vy^2)$ is irreducible in $k\,[x, y]$, lemma 3 is applicable and U^2 is everywhere normal.

We shall now show that V^2 has no rational points over k. In fact, let, say in affine representative $z=1$, there be a rational point $(\alpha/\delta, \beta/\delta, 1, \gamma/\delta)$ in V^2 over $k = \kappa(u, v, w)$, where $\alpha, \beta, \gamma, \delta \in \kappa[u, u, w]$.

We have

3) I owe this example to M. Nagata.

$$\alpha^2+u\gamma^2+v\beta^2+w\delta^2=0.$$

After removing, if necessary, common factors from both sides of the above relation, specialize $(u, v, w)\to(u, v, o)$ over κ and we have

$$\alpha'^2+u\gamma'^2+v\beta'^2=0,$$

where $(\alpha', \beta', \gamma')\neq(0, 0, 0)$, for otherwise α, β, γ, δ would have a common factor w.

Next, specialize $(u, v)\to(u, 0)$ over κ, and we have

$$\alpha''^2+u\gamma''^2=0,$$

where, on the same reason as above, we may assume $(\alpha'', \gamma'')\neq(0, 0)$, which against our assumption that u is a variable over κ.