Hyperbolic span and pseudoconvexity

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To Professor Yukio Kusunoki on the occasion of his 90th birthday

Abstract A planar open Riemann surface R admits the Schiffer span $s(R,\zeta)$ to a point $\zeta \in R$. M. Shiba showed that an open Riemann surface R of genus one admits the hyperbolic span $\sigma_H(R)$. We establish the variation formulas of $\sigma_H(t) := \sigma_H(R(t))$ for the deforming open Riemann surface $R(t)$ of genus one with complex parameter t in a disk Δ of center 0, and we show that if the total space $\mathcal{R} = \bigcup_{t\in\Delta}(t, R(t))$ is a two-dimensional Stein manifold, then $\sigma_H(t)$ is subharmonic on Δ. In particular, $\sigma_H(t)$ is harmonic on Δ if and only if $\mathcal{R}$ is biholomorphic to the product $\Delta \times R(0)$.

1. Introduction

Let R_0 be an open Riemann surface of genus one, and let $\chi_0 = \{A_0, B_0\}$ be a fixed canonical homology basis of R_0 modulo dividing cycles. Consider a triplet (R, χ, i) consisting of a (closed) torus R, a canonical homology basis $\chi = \{A, B\}$ of R, and a conformal embedding i of R_0 into R such that $i(A_0)$ (resp., $i(B_0)$) is homologous to the cycle A (resp., B). We say that two such triplets (R, χ, i) and (R', χ', i') are *equivalent* if there is a conformal mapping f of R onto R' with $f \circ i = i'$ on R_0. Each equivalence class is denoted by $[R, \chi, i]$ and is called a *closing* of (R_0, χ_0).

As is well known, the closing $[R, \chi, i]$ carries a unique holomorphic differential ϕ^R with $\int_A \phi^R = 1$. It will be called the *normal differential* for (R, χ). We put

$$\tau[R, \chi, i] = \int_B \phi^R,$$

which is referred to as the modulus of $[R, \chi, i]$. We denote by $\mathcal{C}(R_0, \chi_0)$ the set of closings of (R_0, χ_0) and put

$$\mathfrak{M}(R_0, \chi_0) = \bigl\{\tau \in \mathbb{C} \bigm| \tau = \tau[R, \chi, i], [R, \chi, i] \in \mathcal{C}(R_0, \chi_0)\bigr\}.$$

The set $\mathfrak{M}(R_0, \chi_0)$ obviously lies in the upper half-plane $\mathbb{H}$.

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THEOREM 1.1 (M. SHIBA, [10, P. 306], [11, P. 123])

(1) $\mathfrak{M}(R_0,\chi_0)$ *is a closed disk (which may degenerate to a singleton); there exists* $\tau^*\in\mathbb{H}$ *and* $\rho\in\mathbb{R}$ *such that* $0\le\rho<\Im\tau^*$ *and*

$$\mathfrak{M}(R_0,\chi_0)=\bigl\{\tau\in\mathbb{H}\bigm| |\tau-\tau^*|\le\rho\bigr\}.$$

(2) *The hyperbolic diameter* $\sigma_H(R_0)$ *of* $\mathfrak{M}(R_0,\chi_0)$ *in* $\mathbb{H}$ *is determined solely by the surface* R_0*; it is invariant under any change of canonical homology basis of* R_0.

We call $\mathfrak{M}(R_0,\chi_0)$ the *moduli disk* for (R_0,χ_0) and call $\sigma_H(R_0)$ the *hyperbolic span* for the open torus R_0. We shall study how $\sigma_H(R_0(t))$ varies when $R_0(t)$ deforms with complex parameter t from the several complex variables point of view.

Let $(\widetilde{\mathcal{R}},\widetilde{\pi},\Delta)$ be a holomorphic family such that $\widetilde{\mathcal{R}}$ is a two-dimensional complex manifold, $\Delta=\{t\in\mathbb{C}_t \mid |t|<r\}$ is a disk, and $\widetilde{\pi}$ is a holomorphic projection from $\widetilde{\mathcal{R}}$ onto Δ. We assume that the fiber $\widetilde{R}(t):=\widetilde{\pi}^{-1}(t), t\in\Delta$, is noncompact, irreducible, and nonsingular in $\widetilde{\mathcal{R}}$, so that $\widetilde{R}(t)$ is an open Riemann surface. Let $(\mathcal{R},\pi,\Delta)$ be a holomorphic subfamily of $(\widetilde{\mathcal{R}},\widetilde{\pi},\Delta)$ such that $\mathcal{R}\subset\widetilde{\mathcal{R}}, \partial\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is a C^ω smooth real three-dimensional (open) surface, $R(t):=\pi^{-1}(t)\Subset\widetilde{R}(t)$, $t\in\Delta$, and $R(t)$ is a bordered Riemann surface of genus one with C^ω smooth boundary $\partial R(t)$ in $\widetilde{R}(t)$. We set

$$\mathcal{R}=\bigcup_{t\in\Delta}\bigl(t,R(t)\bigr)\subset\widetilde{\mathcal{R}},\qquad \partial\mathcal{R}=\bigcup_{t\in\Delta}\bigl(t,\partial R(t)\bigr)\subset\widetilde{\mathcal{R}}.$$

We identify $\mathcal{R}$ with the deformation of the open torus $R(t)$,

$$\mathcal{R}: t\in\Delta\to R(t)\Subset\widetilde{R}(t).$$

Each $R(t)$, $t\in\Delta$, admits the hyperbolic span $\sigma_H(t):=\sigma_H(R(t))$. Then we have the following main theorem.

THEOREM 1.2

If $\mathcal{R}$ *is a pseudoconvex domain in* $\widetilde{\mathcal{R}}$*, then*

(1) *the hyperbolic span* $\sigma_H(t)$ *is subharmonic on* Δ*,*

(2) $\sigma_H(t)$ *is harmonic on* Δ *if and only if* $(\mathcal{R},\pi,\Delta)$ *is a trivial holomorphic family;* $(\mathcal{R},\pi,\Delta)\approx\Delta\times R(0)$.

In the appendix we prove the following corollary as a generalization of this theorem. Let $(\mathcal{R},\pi,\Delta)$ be a holomorphic family such that $\mathcal{R}$ is an $(n+1)$-dimensional complex manifold, Δ is a domain in $\mathbb{C}_t^n$, and $R(t)=\pi^{-1}(t)$, $t\in\Delta$, is irreducible and nonsingular in $\mathcal{R}$ such that $R(t)$ is an open torus with finite ν (independent of $t\in\Delta$) ideal boundary components. We denote by $s_H(t)$ the hyperbolic span for $R(t)$.

COROLLARY 1.3

(1) *Assume that, for each $t_0 \in \Delta$, there exists a small ball δ of center t_0 in Δ such that $\mathcal{R}|_\delta$ is a Stein manifold. Then $s_H(t)$ is plurisubharmonic on Δ.*

(2) *Assume that*

(i) *$\mathcal{R}$ is a Stein manifold and Δ is a pseudoconvex domain in $\mathbb{C}^n_t$,*

(ii) *$\mathcal{R}$ is a topologically trivial family modulo dividing cycles,*

(iii) *the ideal boundary component of $R(t)$, $t \in \Delta$, is nonparabolic.*

Then $s_H(t)$ is pluriharmonic on Δ if and only if $(\mathcal{R}, \pi, \Delta)$ is a trivial holomorphic family.

2. Variation formulas of the second order for $\Im\tau_1(t)$ and $\Im\tau_0(t)$

Let R be a bordered Riemann surface of genus one in a Riemann surface $\widetilde{R}$ such that $R \Subset \widetilde{R}$ and ∂R consists of C^ω smooth contours, $\partial R = C_1 + \cdots + C_\nu$. Let ϕ be a holomorphic differential on $\overline{R} = R \cup \partial R$, precisely, on a neighborhood of $\overline{R}$ in $\widetilde{R}$. If ϕ is semiexact on R and $\Im\phi = 0$ on C_j, $j = 1, \dots, \nu$, then ϕ is called a *canonical differential* on R in the sense of Kusunoki; in other words, on a thin tubular neighborhood V_j of C_j, the branch on V_j of the abelian integral $\Phi(z) = \int_{\zeta_0}^z \phi$ is a single-valued holomorphic function on V_j such that $\Im\Phi(z) = \text{const}$ on C_j (see [7, p. 241], [1, Chapter III]).

Let $\chi = \{A, B\}$ be a canonical homology basis of R modulo dividing cycles such that the orientation of A and B are equal to the x- and y-axis in $\mathbb{C}_z$. It is simply written $A \times B = 1$. For s, $-1 < s \leq 1$, there uniquely exists a holomorphic differential ϕ_s on R such that

(i) $e^{-\frac{\pi i}{2}s}\phi_s$ is a canonical differential on R,

(ii) $\int_A \phi_s = 1$.

We set $\tau_s = \int_B \phi_s$. Then there uniquely exists a closing $[R_s, \{A_s, B_s\}, i_s]$ of (R, χ) such that the transplant of ϕ_s by i_s^{-1} extends to the normal differential ϕ^{R_s} for $(R_s, \{A_s, B_s\})$, so that $\tau[R_s, \{A_s, B_s\}, i_s] = \tau_s$. In the special case $s = 1$ (resp., 0) we simply call ϕ_1 (resp., ϕ_0) the L_1- (resp., L_0-) *differential* for (R, A), so that $\Re\phi_1 = 0$ (resp., $\Im\phi_0 = 0$) on C_j, and $\tau_1 = \int_B \phi_1$ (resp., $\tau_0 = \int_B \phi_0$).

THEOREM 2.1 ([10, P. 306], [11, P. 123])

(1) $\Re\tau_1 = \Re\tau_0$.

(2) $\partial\mathfrak{M}(R, \chi) = \{\tau \in \mathbb{H} \mid |\tau - \tau^*| = \rho\}$, *where*

$$\tau^* = \frac{1}{2}(\tau_1 + \tau_0), \qquad \rho = \frac{1}{2i}(\tau_1 - \tau_0) > 0.$$

(3) $\tau_s = \tau^* + \rho e^{(s-\frac{1}{2})\pi i}$, $-1 < s \leq 1$.

(4) *The hyperbolic span $\sigma_H(R)$ is written into $\sigma_H(R) = \log \frac{\Im\tau_1}{\Im\tau_0}$.*

Now let $(\widetilde{\mathcal{R}}, \widetilde{\pi}, \Delta)$ be the holomorphic family stated in Section 1 for Theorem 1.2. For $t \in \Delta$, we fix a canonical homology basis $\chi(t) = \{A(t), B(t)\}$ of $R(t)$ modulo dividing cycles such that $A(t)$ and $B(t)$ move continuously in $\mathcal{R}$ with $t \in \Delta$. Then we have the L_1- (resp., L_0-) differential $\phi_1(t,z)$ (resp., $\phi_0(t,z)$) for $(R(t),A(t))$, so that $\int_{A(t)}\phi_1(t,z)=\int_{A(t)}\phi_0(t,z)=1$. We set

$$\tau_1(t)=\int_{B(t)}\phi_1(t,z),\qquad \tau_0(t)=\int_{B(t)}\phi_0(t,z).$$

As usual we write $\phi_i(t,z)=f_i(t,z)\,dz$, $i=1,0$, by use of the local parameter of $R(t)$. Then we have the following variation formulas for $\Im\tau_1(t)$ and $\Im\tau_0(t)$.

LEMMA 2.2

For $t\in\Delta$,

$$(1)\quad \frac{\partial^2\Im\tau_1(t)}{\partial t\partial\bar t}=\frac{1}{2}\int_{\partial R(t)}k_2(t,z)\big|f_1(t,z)\big|^2|dz|+\Big\|\frac{\partial\phi_1(t,z)}{\partial\bar t}\Big\|^2_{R(t)},$$

$$(2)\quad \frac{\partial^2\Im\tau_0(t)}{\partial t\partial\bar t}=-\Big(\frac{1}{2}\int_{\partial R(t)}k_2(t,z)\big|f_0(t,z)\big|^2|dz|+\Big\|\frac{\partial\phi_0(t,z)}{\partial\bar t}\Big\|^2_{R(t)}\Big),$$

where

$$k_2(t,z)=\mathcal{L}\varphi(t,z)/\Big|\frac{\partial\varphi}{\partial z}\Big|^3\quad \textit{on } \partial\mathcal{R},$$

$$\mathcal{L}\varphi(t,z)=\frac{\partial^2\varphi}{\partial t\partial\bar t}\Big|\frac{\partial\varphi}{\partial z}\Big|^2-2\Re\Big\{\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar z}\frac{\partial^2\varphi}{\partial\bar t\partial z}\Big\}+\Big|\frac{\partial\varphi}{\partial t}\Big|^2\frac{\partial^2\varphi}{\partial z\partial\bar z}\quad \textit{on } \partial\mathcal{R},$$

$$\varphi(t,z)\quad \textit{is the } C^2 \textit{ smooth defining function for } \partial\mathcal{R} \textit{ in } \widetilde{\mathcal{R}}.$$

The Levi form $\mathcal{L}\varphi(t,z)$ on $\partial\mathcal{R}$ depends on the choice of the defining function $\varphi(t,z)$ of $\partial\mathcal{R}$, but $k_2(t,z)$ does not depend on it. Further, $k_2(t,z)/|dz|$ is a form on $\partial\mathcal{R}$ with respect to the holomorphic family $(\mathcal{R},\pi,\Delta)$, so that $k_2(t,z)|f_i(t,z)|$, $i=1,0$, is a real-valued function on $\partial\mathcal{R}$ (see [8, (1.2)]).

3. Proof of Lemma 2.2

For Lemma 2.2, it suffices to prove it at $t=0$. By Gunning and Narasimhan [2], $\widetilde{R}(0)$ is conformally equivalent to a sheeted Riemann surface $\mathbf{D}$ over $\mathbb{C}_z$ without branch points. Thus, if necessary, take a smaller disk Δ (of center 0) in $\mathbb{C}_t$. We may assume the following.

(a) $\mathcal{R}$ is an unramified domain over $\Delta\times\mathbb{C}_z$ (i.e., $\mathcal{R}\subset\Delta\times\mathbf{D}$) so that each $R(t),t\in\Delta$, is a relatively compact domain of genus one in $\mathbf{D}$ such that $\partial R(t)$ consists of C^ω smooth contours $C_j(t)$, $j=1,\dots,\nu$. We write $\overline{R(t)}=R(t)\cup\partial R(t)=R(t)\cup(\bigcup_{j=1}^{\nu}C_j(t))\Subset\mathbf{D}$.

(b) $A(t)=A(0)$ and $B(t)=B(0)$ for $t\in\Delta$ such that $A(0)$ and $B(0)$ are smooth Jordan curves and $A(0)\cap B(0)$ consists of a single point ζ_0 with $A(0)\times B(0)=1$. We write $A(0)=A$ and $B(0)=B$.

For $t\in\Delta$, we have the L_1- (resp., L_0-) differential $\phi_1(t,z)$ (resp., $\phi_0(t,z)$) for $(R(t),A)$. If necessary, take a smaller disk Δ. Since each $C_j(t)$ is of class C^ω in $\widetilde{R}(0)$, we have a tubular neighborhood V_j of $C_j(0)$ in $\mathbf{D}$ such that $\phi_1(t,z)$ (resp.,

$\phi_0(t,z))$ is holomorphically extended to $R(0)\cup V_j$. We set

$$\mathbf{V}=\bigcup_{j=1}^{\nu} V_j \qquad \text{and} \qquad \mathbf{R}=R(0)\cup\mathbf{V}(\Subset\mathbf{D}). \tag{3.1}$$

Then $\phi_1(t,z)$ (resp., $\phi_0(t,z)$) is defined in the product domain $\Delta\times\mathbf{R}$ and is holomorphic for $z\in\mathbf{R}$, *but not for* $t\in\Delta$ *in general.*

Let us prove Lemma 2.2(1). Each $\phi_1(t,z)$, $t\in\Delta$, is written in the form

$$\phi_1(t,z)=f_1(t,z)\,dz \quad \text{on } \mathbf{R},$$

where $f_1(t,z)$ is a single-valued holomorphic function for z on $\mathbf{R}$. Fix $t\in\Delta$, and consider the abelian integral $\Phi_1(t,z)=\int_{\zeta_0}^{z}\phi_1(t,\cdot)$ on $\mathbf{R}$. The branch $\Phi_1(t,z)$ with $\Phi_1(t,\zeta_0)=0$ is a single-valued holomorphic function for z on $\mathbf{R}\setminus(A\cup B)$. We have $\partial(R(0)\setminus(A\cup B))=\partial R(0)+[A^+B^+A^-B^-]$, where $[A^+B^+A^-B^-]$ is a *simple* closed curve in $R(0)$. From Cauchy's theorem we have

$$\int_{\partial R(0)+[A^+B^+A^-B^-]}\Phi_1(t,z)f_1(0,z)\,dz=0.$$

Since $\phi_1(t,z)$ is a semiexact holomorphic differential on $R(0)$, we have by the bilinear relation that

$$\begin{aligned}
&\int_{A^+A^-}\Phi_1(t,z)f_1(0,z)\,dz=\Big(-\int_B\phi_1(t,z)\Big)\Big(\int_A\phi_1(0,z)\Big)=-\tau_1(t),\\
&\int_{B^+B^-}\Phi_1(t,z)f_1(0,z)\,dz=\Big(\int_A\phi_1(t,z)\Big)\Big(\int_B\phi_1(0,z)\Big)=\tau_1(0),\\
&\qquad\therefore \tau_1(t)-\tau_1(0)=\int_{\partial R(0)}\Phi_1(t,z)\,d\Phi_1(0,z),\quad t\in\Delta.
\end{aligned} \tag{3.2}$$

We set

$$\Phi_1(t,z)=U_1(t,z)+iU_1^*(t,z) \quad \text{on } \mathbf{R}\setminus(A\cup B),$$

where $U_1(t,z)$ and $U_1^*(t,z)$, $t\in\Delta$, are single-valued harmonic functions on $\mathbf{R}\setminus(A\cup B)$. Since $\phi_1(0,z)$ is the L_1-differential on $R(0)$, we have

$$\int_{C_j(0)}dU_1^*(0,z)=0 \qquad \text{and}$$

$$U_1(0,z)=\mathrm{const}\,a_j(0) \quad \text{on } C_j(0),$$

so that

$$\Im\tau_1(t)-\Im\tau_1(0)=\sum_{j=1}^{\nu}\int_{C_j(0)}U_1(t,z)\,dU_1^*(0,z),\quad t\in\Delta,$$

$$\therefore \frac{\partial^2\Im\tau_1(t)}{\partial t\partial\bar t}=\sum_{j=1}^{\nu}\int_{C_j(0)}\frac{\partial^2U_1(t,z)}{\partial t\partial\bar t}\,dU_1^*(0,z),\quad t\in\Delta.$$

Since $U_1(t,z) = \mathrm{const}\, a_j(t)$ on $C_j(t)$, we apply Hamano's formula (see [3, (1.2)]) to obtain

$$\begin{aligned}\frac{\partial^2 U_1}{\partial t \partial \bar{t}}\, dU_1^* &= 2k_2(t,z)\Big|\frac{\partial U_1}{\partial z}\Big|^2 |dz| + 4\Im\Big\{\frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz\Big\} \\ &\quad + \frac{\partial^2 a_j}{\partial t \partial \bar{t}}\, dU_1^* - 4\Im\Big\{\frac{\partial a_j}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz\Big\} \quad \text{along } C_j(t).\end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned}\frac{\partial^2 \Im\tau_1}{\partial t \partial \bar{t}}(0) &= 2\int_{\partial R(0)} k_2(0,z)\Big|\frac{\partial U_1}{\partial z}\Big|^2 |dz| + 4\Im\Big\{\int_{\partial R(0)} \frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz\Big\} \\ &\quad + \sum_{j=1}^{\nu} \frac{\partial^2 a_j}{\partial t \partial \bar{t}} \int_{C_j(0)} dU_1^* - 4\Im\Big\{\sum_{j=1}^{\nu} \frac{\partial a_j}{\partial t}\int_{C_j(0)} \frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz\Big\} \\ &\equiv I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where each integrand is evaluated at $t = 0$ and $z \in \partial R(0)$ or $C_j(0)$. From $\frac{\partial U_1}{\partial z} = \frac{1}{2} f_1$ on $\Delta \times \mathbf{V}$ we have $I_1 = \frac{1}{2}\int_{\partial R(0)} k_2(0,z)|f_1(0,z)|^2 |dz|$. Since $U_1^*(t,z)$, $t \in \Delta$, is single-valued in V_j, we have $\int_{C_j(0)} dU_1^*(t,z) = 0$, and $I_3 = 0$. Since $U_1(t,z)$ is of class C^ω for $(t,z) \in \Delta \times V_j$, we have $\int_{C_j(0)} \frac{\partial^2 U_1(t,z)}{\partial \bar{t}\partial z}\, dz = \frac{1}{2}\frac{\partial}{\partial \bar{t}}\int_{C_j(0)} d\Phi_1(t,z) = 0$, and $I_4 = 0$. It remains to calculate I_2. Since $\frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}$, $t \in \Delta$, is a single-valued function of class C^ω on $\mathbf{R} \setminus (A \cup B)$, we have by Green's formula that

$$\begin{aligned}&\int_{\partial R(0)+[A^+B^+A^-B^-]} \frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz \\ &\quad = \iint_{R(0)\setminus(A\cup B)} d\Big(\frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz\Big), \quad t \in \Delta.\end{aligned} \tag{3.4}$$

Since $d_z(\frac{\partial U_1}{\partial t}) = \frac{\partial}{\partial t}\Re\{\phi_1(t,z)\}$ is a semiexact harmonic differential on $R(t)$, we have

$$\begin{aligned}\int_{A^+A^-} \frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz &= -\Big(\frac{\partial}{\partial t}\Re\Big\{\int_B \phi_1(t,z)\Big\}\Big)\Big(\frac{1}{2}\frac{\partial}{\partial \bar{t}}\int_A \phi_1(t,z)\Big) \\ &= -\frac{1}{2}\Big(\frac{\partial}{\partial t}\Re\tau_1(t)\Big)\Big(\frac{\partial}{\partial \bar{t}}1\Big) = 0.\end{aligned}$$

Similarly,

$$\int_{B^+B^-} \frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz = \Big(\frac{\partial}{\partial t}\Re\Big\{\int_A \phi_1(t,z)\Big\}\Big)\Big(\frac{1}{2}\frac{\partial}{\partial \bar{t}}\int_B \phi_1(t,z)\Big) = 0.$$

Since $U_1(t,z)$ is harmonic for $z \in R(0)$, we have $\frac{\partial U_1}{\partial z}\, dz = \frac{1}{2}\phi_1$ and

$$\begin{aligned}\iint_{R(0)\setminus(A\cup B)} d\Big(\frac{\partial U_1}{\partial t}\frac{\partial^2 U_1}{\partial \bar{t}\partial z}\, dz\Big) &= \iint_{R(0)}\Big(\Big|\frac{\partial^2 U_1}{\partial t \partial \bar{z}}\Big|^2 + \frac{\partial U_1}{\partial t}\frac{\partial^3 U_1}{\partial \bar{t}\partial z \partial \bar{z}}\Big)\, d\bar{z} \wedge dz \\ &= \frac{i}{4}\Big\|\frac{\partial \phi_1}{\partial \bar{t}}\Big\|^2_{R(0)}.\end{aligned}$$

By (3.4) we have $I_2 = \|\frac{\partial \phi_1}{\partial \bar t}(0,z)\|^2_{R(0)}$, and

$$\frac{\partial^2 \Im \tau_1}{\partial t \partial \bar t}(0) = \frac{1}{2}\int_{\partial R(0)} k_2(0,z)\big|f_1(0,z)\big|^2 |dz| + \Big\|\frac{\partial \phi_1}{\partial \bar t}(0,z)\Big\|^2_{R(0)},$$

which is formula (1) at $t=0$.

Let us prove Lemma 2.2(2) by a method similar to that for (1). We write

$$\phi_0(t,z) = f_0(t,z)\,dz \quad \text{on } \mathbf{R},$$

where $f_0(t,z)$, $t\in\Delta$, is a single-valued holomorphic function for $z\in\mathbf{R}$. We consider the abelian integral $\Phi_0(t,z)$ of $\phi_0(t,z)$ on $\mathbf{R}$ with $\Phi_0(t,\zeta_0)=0$, which is a single-valued holomorphic function on $\mathbf{R}\setminus(A\cup B)$. By the same way we obtained (3.2), we have

$$\tau_0(t)-\tau_0(0) = \int_{\partial R(0)} \Phi_0(t,z) f_0(0,z)\,dz, \quad t\in\Delta.$$

We set, for $t\in\Delta$,

$$\Phi_0(t,z) = U_0(t,z) + iU_0^*(t,z) \quad \text{on } \mathbf{R}\setminus(A\cup B).$$

Since $\phi_0(0,z)$ is the L_0-differential for $(R(0),A)$, $\Phi_0(t,z)$ is a single-valued holomorphic function on $\mathbf{R}\setminus(A\cup B)$ with $U_0^*(0,z)=\mathrm{const}\, b_j(0)$ on $C_j(0)$. We have

$$\begin{aligned}\Im\tau_0(t)-\Im\tau_0(0) &= \sum_{j=1}^{\nu}\int_{C_j(0)} U_0^*(t,z)\,dU_0(0,z)\\ &= -\sum_{j=1}^{\nu}\int_{C_j(0)} U_0^*(t,z)\,d\big(U_0^*(0,z)\big)^*, \quad t\in\Delta,\\ \therefore \frac{\partial^2\Im\tau_0(t)}{\partial t\partial\bar t} &= -\sum_{j=1}^{\nu}\int_{C_j(0)} \frac{\partial^2 U_0^*(t,z)}{\partial t\partial \bar t}\,d\big(U_0^*(0,z)\big)^*, \quad t\in\Delta.\end{aligned}$$

Since $U_0^*(t,z)=\mathrm{const}\, b_j(t)$ on $C_j(t)$, similarly to (3.3) we have

$$\begin{aligned}\frac{\partial^2\Im\tau_0}{\partial t\partial\bar t}(0) =& -\Big(2\int_{\partial R(0)} k_2(0,z)\Big|\frac{\partial U_0^*}{\partial z}\Big|^2|dz| + 4\Im\Big\{\int_{\partial R(0)}\frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial\bar t\partial z}\,dz\Big\}\\ &+\sum_{j=1}^{\nu}\frac{\partial^2 b_j}{\partial t\partial\bar t}\int_{C_j(0)} d(U_0^*)^* - 4\Im\Big\{\sum_{j=1}^{\nu}\frac{\partial b_j}{\partial t}\int_{C_j(0)}\frac{\partial^2 U_0^*}{\partial \bar t\partial z}\,dz\Big\}\Big)\\ \equiv& -(J_1+J_2+J_3+J_4),\end{aligned}$$

where each integrand is evaluated at $t=0$ and $z\in\partial R(0)$ or $C_j(0)$. Since $\frac{\partial U_0^*}{\partial z} = \frac{1}{2i}f_0$, we have $J_1 = \frac{1}{2}\int_{\partial R(0)} k_2(0,z)|f_0(0,z)|^2|dz|$. By the same reasons that $I_3 = I_4 = 0$ we have $J_3=J_4=0$. For J_2 we have by Green's formula

$$\int_{\partial R(0)+[A^+B^+A^-B^-]}\frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial\bar t\partial z}\,dz = \iint_{R(0)\setminus(A\cup B)} d\Big(\frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial\bar t\partial z}\,dz\Big), \quad t\in\Delta.$$

Since $\int_A \phi_0(t,z) = 1$ for $t \in \Delta$, we have

$$\int_{A^+ A^-} \frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial \bar{t}\partial z}\,dz = -\Big(\frac{\partial}{\partial t}\Im\Big\{\int_B \phi_0(t,z)\Big\}\Big)\frac{1}{2i}\Big(\frac{\partial}{\partial \bar{t}}\int_A \phi_0(t,z)\Big) = 0,$$

$$\int_{B^+ B^-} \frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial \bar{t}\partial z}\,dz = \Big(\frac{\partial}{\partial t}\Im\Big\{\int_A \phi_0(t,z)\Big\}\Big)\frac{1}{2i}\Big(\frac{\partial}{\partial \bar{t}}\int_B \phi_0(t,z)\Big) = 0.$$

It follows that

$$\begin{aligned}\int_{\partial R(0)} \frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial \bar{t}\partial z}\,dz &= \iint_{R(0)\setminus\{A\cup B\}} d\Big(\frac{\partial U_0^*}{\partial t}\frac{\partial^2 U_0^*}{\partial \bar{t}\partial z}\,dz\Big)\\ &= \frac{i}{4}\Big\|\frac{\partial \phi_0(t,z)}{\partial \bar{t}}\Big\|^2_{R(0)}, \quad t \in \Delta,\end{aligned}$$

and $J_2 = \|\frac{\partial \phi_0}{\partial \bar{t}}(0,z)\|^2_{R(0)}$. We thus have (2) at $t = 0$.

COROLLARY 3.1

If the total space $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then

(1) *$\Im\tau_1(t)$ is subharmonic on Δ,*
(2) *$\Im\tau_0(t)$ is superharmonic on Δ,*
(3) *the Euclidean radius $\rho(t)$ of the moduli disk $\mathfrak{M}(R(t),\chi(t))$ is subharmonic on Δ.*

Proof

Since $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, we have $\mathcal{L}\varphi(t,z) \geq 0$ on $\partial\mathcal{R}$. It follows from Lemma 2.2 that, for $t \in \Delta$,

$$\frac{\partial^2 \Im\tau_1(t)}{\partial t\partial \bar{t}} \geq \Big\|\frac{\partial \phi_1(t,z)}{\partial \bar{t}}\Big\|^2_{R(t)} \geq 0, \qquad \frac{\partial^2 \Im\tau_0(t)}{\partial t\partial \bar{t}} \leq -\Big\|\frac{\partial \phi_0(t,z)}{\partial \bar{t}}\Big\|^2_{R(t)} \leq 0, \tag{3.5}$$

which proves (1) and (2). These inequalities with Theorem 2.1(2) yield (3). □

REMARK 1

In the case of deforming planar open Riemann surfaces, we showed the variation formulas of type (1) and (2) of Lemma 2.2, for the Schiffer span (see [9] for the definition) in [4], and for the harmonic span in [6]. In [5] we showed the relation between both spans. We showed further in [3] and [6] that, for the deformation of an open Riemann surface of positive genus, a formula of type (1) holds but a formula of type (2) does not hold. Formulas (1) and (2) in Lemma 2.2 with the remarkable contrast are the first example in the case of the deforming nonplanar open Riemann surface.

4. Proof of the main theorem

Let R be a bordered Riemann surface of genus one with C^ω smooth boundary in a larger $\widetilde{R}$, $R \Subset \widetilde{R}$, and let $\{A, B\}$ be a canonical homology basis of R modulo dividing cycles. We denote by ϕ_1 (resp., ϕ_0) the L_1- (resp., L_0-) differential for

(R, A). From Theorem 2.1(1) and (2) we have

$$\int_B \phi_1 = \tau_1 := \xi + i\eta_1, \qquad \int_B \phi_0 = \tau_0 := \xi + i\eta_0,$$

where $i = \sqrt{-1}$ and ξ, η_1, η_0 are real numbers with $\eta_1 > \eta_0 > 0$. Consider any canonical homology basis $\{A', B'\}$ of R:

$$\begin{cases} A' = mA + nB \\ B' = m'A + n'B \end{cases} \quad \text{modulo dividing cycles,}$$

where $m, n, m', n' \in \mathbb{Z}$ with $mn' - nm' = 1$. Then we have the L_1- and L_0-differentials ψ_1 and ψ_0 for (R, A'): $\int_{A'} \psi_1 = \int_{A'} \psi_0 = 1$, and

$$\int_{B'} \psi_1 = \tau_1' := \alpha + i\beta_1, \qquad \int_{B'} \psi_0 = \tau_0' := \alpha + i\beta_0,$$

respectively, where α, β_1, β_0 are real numbers with $\beta_1 > \beta_0 > 0$. Then we have the following result.

LEMMA 4.1

We have

$$\begin{cases} \psi_1 = \frac{1}{X}((m + n\xi)\phi_1 - in\eta_1\phi_0) & \textit{on } R, \\ \psi_0 = \frac{1}{X}((m + n\xi)\phi_0 - in\eta_0\phi_1) & \textit{on } R, \end{cases} \tag{4.1}$$

$$\begin{cases} \alpha = \frac{1}{X}((m + n\xi)(m' + n'\xi) + nn'\eta_1\eta_0), \\ \beta_1 = \frac{1}{X}\eta_1, \\ \beta_0 = \frac{1}{X}\eta_0, \end{cases} \tag{4.2}$$

where

$$X = (m + n\xi)^2 + n^2\eta_1\eta_0 > 0.$$

Proof

From the uniqueness of the L_1-differential for (R, A'), ψ_1 must be written in the form

$$\begin{cases} \psi_1 = a\phi_1 + ib\phi_0 \quad \text{on } R \text{ for } \textit{some } a, b \in \mathbb{R}, \\ \int_{A'} \psi_1 = 1. \end{cases}$$

We have

$$\begin{aligned} \int_{A'} \psi_1 &= \int_{mA+nB} a\phi_1 + ib\phi_0 \\ &= a(m + n\tau_1) + ib(m + n\tau_0) \\ &= a(m + n\xi) - bn\eta_0 + i\big(an\eta_1 + b(m + n\xi)\big), \end{aligned}$$

so that

$$\begin{cases} a(m + n\xi) - bn\eta_0 = 1, \\ an\eta_1 + b(m + n\xi) = 0. \end{cases}$$

Since $X>0$ from $mn'-nm'=1$ and $\eta_1,\eta_0>0$, it follows that

$$a=\frac{1}{X}(m+n\xi), \qquad b=\frac{1}{X}(-n\eta_1),$$

which yield the expression of ψ_1 in (4.1). Hence,

$$\begin{aligned}\tau_1' &= \frac{m+n\xi}{X}\int_{m'A+n'B}\phi_1 - i\frac{n\eta_1}{X}\int_{m'A+n'B}\phi_0 \\ &= \frac{1}{X}\{(m+n\xi)(m'+n'\tau_1)-in\eta_1(m'+n'\tau_0)\} \\ &= \frac{1}{X}\{\big((m+n\xi)(m'+n'\xi)+nn'\eta_1\eta_0\big)+i\eta_1\} \qquad \text{by } mn'-nm'=1,\end{aligned}$$

which yields the expressions of α and β_1 in (4.2). Since ψ_0 must be written in the form

$$\begin{cases}\psi_0 = i\widetilde{a}\phi_1+\widetilde{b}\phi_0 & \text{on } R \text{ for } \textit{some } \widetilde{a},\widetilde{b}\in\mathbb{R}, \\ \int_{A'}\psi_0=1, \end{cases}$$

in the same way as we obtained ψ_1, we have the expressions of ψ_0 in (4.1) and β_0 in (4.2). $\square$

By (4.2) we see that ψ_1, ψ_0, β_1, and β_0 do not depend on the choice of m',n' with $mn'-nm'=1$, that is, of B', and that $\frac{\Im\tau_1}{\Im\tau_0}=\frac{\Im\tau_1'}{\Im\tau_0'}$, which shows Theorem 2.1(4).

Proof of Theorem 1.2(1)

We do not lose generality in assuming (a) and (b) stated in Section 3. By Theorem 2.1(4), it suffices to show the following: if $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then

$$\frac{\partial^2}{\partial t\partial\bar{t}}\log\frac{\Im\tau_1(t)}{\Im\tau_0(t)}\geq 0, \quad t\in\Delta. \tag{4.3}$$

In (b), for $t\in\Delta$ we defined the canonical homology basis $\{A,B\}$ of the Riemann surface $R(t)$ of genus one over $\mathbb{C}_z$, and we considered the L_1- (resp., L_0-) differential $\phi_1(t,z)$ (resp., $\phi_0(t,z)$) for $(R(t),A)$, so that $\int_A\phi_1(t,z)=\int_A\phi_0(t,z)=1$. We put

$$\int_B\phi_1(t,z)=\tau_1(t):=\xi(t)+i\eta_1(t), \qquad \int_B\phi_0(t,z)=\tau_0(t):=\xi(t)+i\eta_0(t), \tag{4.4}$$

where $\xi(t),\eta_1(t),\eta_0(t)$ are real numbers with $\eta_1(t)>\eta_0(t)>0$. Then, for *arbitrary* $m,n\in\mathbb{Z}$ with $(m,n)=\pm1$ it holds that

$$\begin{aligned}&\frac{\eta_1(t)}{(m+n\xi(t))^2+n^2\eta_1(t)\eta_0(t)} \quad \text{is subharmonic on } \Delta, \\ &\frac{\eta_0(t)}{(m+n\xi(t))^2+n^2\eta_1(t)\eta_0(t)} \quad \text{is superharmonic on } \Delta.\end{aligned} \tag{4.5}$$

In fact, we can find $m',n'\in\mathbb{Z}$ such that $mn'-nm'=1$, and put $\{A',B'\}=\{mA+nB, m'A+n'B\}$, which is a canonical homology basis of $R(t)$. Then we uniquely have the L_1- (resp., L_0-) differential $\psi_1(t,z)$ (resp., $\psi_0(t,z)$) for

$(R(t), A')$, so that $\int_{A'} \psi_1(t,z) = 1$ and $\int_{A'} \psi_0(t,z) = 1$. We set

$$\int_{B'} \psi_1(t,z) = \tau_1'(t) := \alpha(t) + i\beta_1(t),$$

$$\int_{B'} \psi_0(t,z) = \tau_0'(t) := \alpha(t) + i\beta_0(t),$$

where $\alpha(t), \beta_1(t), \beta_0(t)$ are real numbers with $\beta_1(t) > \beta_0(t) > 0$ from Theorem 1.1(1) and (2). By Corollary 3.1(1) and (2), we see that $\beta_1(t)$ (resp., $\beta_0(t)$) is subharmonic (resp., superharmonic) on Δ. Thus, (4.2) yields (4.5).

We put

$$X = X(t,m,n) = \big(m + n\xi(t)\big)^2 + n^2\eta_1(t)\eta_0(t) > 0.$$

By straightforward calculation we have from (4.5)

$$X\Big(\frac{\partial^2\eta_1}{\partial t\partial\bar t}X - \eta_1\frac{\partial^2 X}{\partial t\partial\bar t}\Big) - 2X\Re\Big\{\frac{\partial\eta_1}{\partial\bar t}\frac{\partial X}{\partial t}\Big\} + 2\Big|\frac{\partial X}{\partial t}\Big|^2\eta_1 \geq 0 \quad \text{on } \Delta,$$

$$X\Big(\frac{\partial^2\eta_0}{\partial t\partial\bar t}X - \eta_0\frac{\partial^2 X}{\partial t\partial\bar t}\Big) - 2X\Re\Big\{\frac{\partial\eta_0}{\partial\bar t}\frac{\partial X}{\partial t}\Big\} + 2\Big|\frac{\partial X}{\partial t}\Big|^2\eta_0 \leq 0 \quad \text{on } \Delta.$$

Since $\eta_1(t) > \eta_0(t) > 0$ and $X > 0$ on Δ, we have

$$\Big(\eta_0\frac{\partial^2\eta_1}{\partial t\partial\bar t} - \eta_1\frac{\partial^2\eta_0}{\partial t\partial\bar t}\Big)X - 2\Re\Big\{\Big(\eta_0\frac{\partial\eta_1}{\partial\bar t} - \eta_1\frac{\partial\eta_0}{\partial\bar t}\Big)\frac{\partial X}{\partial t}\Big\} \geq 0 \quad \text{on } \Delta.$$

This is written into

$$A(t)\big(m + n\xi(t)\big)^2 + 2B(t)n\big(m + n\xi(t)\big) + n^2C(t) \geq 0, \tag{4.6}$$

where

$$A(t) := \eta_0\frac{\partial^2\eta_1}{\partial t\partial\bar t} - \eta_1\frac{\partial^2\eta_0}{\partial t\partial\bar t},$$

$$B(t) := -2\Re\Big\{\Big(\eta_0\frac{\partial\eta_1}{\partial\bar t} - \eta_1\frac{\partial\eta_0}{\partial\bar t}\Big)\frac{\partial\xi}{\partial t}\Big\},$$

$$C(t) := \eta_1\eta_0\Big(\eta_0\frac{\partial^2\eta_1}{\partial t\partial\bar t} - \eta_1\frac{\partial^2\eta_0}{\partial t\partial\bar t}\Big) - 2\Big(\eta_0^2\Big|\frac{\partial\eta_1}{\partial t}\Big|^2 - \eta_1^2\Big|\frac{\partial\eta_0}{\partial t}\Big|^2\Big),$$

which are all real numbers independent of m, n. By (4.6) we have

$$A(t)\Big(\frac{m}{n} + \xi(t)\Big)^2 + 2B(t)\Big(\frac{m}{n} + \xi(t)\Big) + C(t) \geq 0, \quad t \in \Delta.$$

This is true **for every** $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ with $(m,n) = \pm 1$. It follows that, for $t \in \Delta$,

$$A(t)x^2 + 2B(t)x + C(t) \geq 0 \quad \text{for all } x \in \mathbb{R},$$

$$\therefore A(t) \geq 0 \qquad \text{and} \qquad C(t) \geq 0 \quad \text{for } t \in \Delta.$$

Let us prove (4.3). Since $\Im\tau_1(t) = \eta_1(t) > 0$ and $\Im\tau_0(t) = \eta_0(t) > 0$, it suffices to show, for $t \in \Delta$,

$$L(t) := \eta_1\eta_0\Big(\eta_0\frac{\partial^2\eta_1}{\partial t\partial\bar t} - \eta_1\frac{\partial^2\eta_0}{\partial t\partial\bar t}\Big) - \Big(\eta_0^2\Big|\frac{\partial\eta_1}{\partial t}\Big|^2 - \eta_1^2\Big|\frac{\partial\eta_0}{\partial t}\Big|^2\Big) \geq 0.$$

In fact, we have two expressions of $L(t)$ such that

$$(l1)\quad L(t) = C(t) + \left(\eta_0^2\left|\frac{\partial\eta_1}{\partial t}\right|^2 - \eta_1^2\left|\frac{\partial\eta_0}{\partial t}\right|^2\right), \quad t \in \Delta,$$

$$(l2)\quad L(t) = \eta_1\eta_0 A(t) - \left(\eta_0^2\left|\frac{\partial\eta_1}{\partial t}\right|^2 - \eta_1^2\left|\frac{\partial\eta_0}{\partial t}\right|^2\right), \quad t \in \Delta.$$

These yield

$$L(t) = \frac{1}{2}\big(C(t) + \eta_1\eta_0 A(t)\big), \tag{4.7}$$

which is at least 0 on Δ. Theorem 1.2(1) is proved. $\square$

Proof of Theorem 1.2(2)

Step 1: Assertion (2) holds locally. That is, at each $t_0 \in \Delta$ there exists a small disk Δ_0 of center t_0 in Δ such that $\mathcal{R}|_{\Delta_0} \approx \Delta_0 \times R(t_0)$ if $\sigma_H(t)$ is harmonic on Δ_0. In fact, it suffices to prove the first step under the conditions (a) and (b) in Section 3 (cf. the proof of Theorem 1.2(1)). For simplicity, we write 0 (resp., Δ) for t_0 (resp., Δ_0). We use the same notations 0, Δ instead of t_0, Δ_0 in the first step. Since $\sigma_H(t) = \log\frac{\eta_1(t)}{\eta_0(t)}$ is harmonic on Δ, we have $L(t) \equiv 0$ on Δ.

We shall first prove

(i) $\phi_1(t,z)$ and $\phi_0(t,z)$ are holomorphic for $(t,z) \in \Delta \times \mathbf{R}$;
(ii) the moduli disk $\mathfrak{M}(R(t), \{A, B\})$ does not move with $t \in \Delta$.

In fact, since $C(t) \geq 0$, $A(t) \geq 0$, and $\eta_1(t) > \eta_0(t) > 0$ on Δ, it follows from $L(t) = 0$ on Δ and (4.7) that $C(t) = A(t) = 0$ on Δ. On the other hand, (3.5) yields

$$A(t) \geq \eta_0(t)\left\|\frac{\partial\phi_1}{\partial\bar{t}}\right\|_{R(t)}^2 + \eta_1(t)\left\|\frac{\partial\phi_0}{\partial\bar{t}}\right\|_{R(t)}^2 \geq 0 \quad \text{on } \Delta.$$

We have $\frac{\partial\phi_1(t,z)}{\partial\bar{t}} = \frac{\partial\phi_0(t,z)}{\partial\bar{t}} = 0$ on $\mathcal{R}|_\Delta$, which induces (i).

By (i), $\tau_1(t) = \int_B \phi_1(t,z)$ and $\tau_0(t) = \int_B \phi_0(t,z)$ are holomorphic on Δ. Since $\tau_1(t) - \tau_0(t) = i(\eta_1(t) - \eta_0(t))$ is pure imaginary, we have $\tau_1(t) - \tau_0(t) = \mathrm{const}\, i\rho$ on Δ, and hence, $\frac{1}{2i}\frac{\partial\tau_1}{\partial t} = \frac{\partial\eta_1}{\partial t} = \frac{\partial\eta_0}{\partial t} = \frac{1}{2i}\frac{\partial\tau_0}{\partial t}$ on Δ. It follows from $(l2)$ that $0 = (\eta_1(t)^2 - \eta_0(t)^2)|\frac{\partial\tau_1}{\partial t}|^2$ on Δ, so that $|\frac{\partial\tau_1}{\partial t}|^2 = 0$ on Δ. Consequently, neither $\tau_1(t)$ nor $\tau_0(t)$ depends on $t \in \Delta$:

$$\tau_1(t) = \tau_1(0), \qquad \tau_0(t) = \tau_0(0) \quad \text{on } \Delta. \tag{4.8}$$

This together with Theorem 2.1(2) yields (ii).

Using (i) and (ii) we next prove the first step: $\mathcal{R}|_\Delta \approx \Delta \times R(0)$. We set $\zeta_0 = A \cap B$ and use the notation $\mathbf{R}$ defined by (3.1). We consider the abelian integrals

$$Z = \Phi_1(t,z) := \int_{\zeta_0}^{z} \phi_1(t,\cdot), \quad (t,z) \in \Delta \times \mathbf{R},$$

$$W = \Phi_0(t,z) := \int_{\zeta_0}^{z} \phi_0(t,\cdot), \quad (t,z) \in \Delta \times \mathbf{R},$$

which are multivalued holomorphic functions for $(t,z) \in \Delta \times \mathbf{R}$ by (i).

Let us first fix $t \in \Delta$. Since $\phi_1(t,z)$ (resp., $\phi_0(t,z)$) is the L_1- (resp., L_0-) differential for $(R(t),A)$, which is holomorphic on $\mathbf{R}$, the branch of $\Phi_1(t,z)$ (resp., $\Phi_0(t,z)$) with $\Phi_1(t,\zeta_0)=0$ (resp., $\Phi_0(t,\zeta_0)=0$) is a single-valued holomorphic function on $\mathbf{R}\setminus(A\cup B)$ such that $\Phi_1(t,C_j(t))$ (resp., $\Phi_0(t,C_j(t))$), $j=1,\dots,\nu$, is a double *vertical* (resp., *horizontal*) segment:

$$\Phi_1\big(t,C_j(t)\big)=\big[a_j(t),a_j(t)+i\ell_j(t)\big]^{\pm},$$

$$\Phi_0\big(t,C_j(t)\big)=\big[b_j(t),b_j(t)+m_j(t)\big]^{\pm},$$

where $\ell_j(t),m_j(t)>0$. We set

$$\Sigma_1(t):=\Phi_1\big(t,\mathbf{R}\setminus(A\cup B)\big),\qquad \Sigma_0(t):=\Phi_0\big(t,\mathbf{R}\setminus(A\cup B)\big).$$

If necessary, take a thin tubular neighborhood $V_j\supset\partial R_j(t), t\in\Delta$. Then $\Sigma_1(t)$ (resp., $\Sigma_0(t)$) is a two-sheeted open Riemann surface over $\mathbb{C}_Z$ (resp., $\mathbb{C}_W$) with 2ν branch points $a_j(t),a_j(t)+i\ell_j(t)$ (resp., $b_j(t),b_j(t)+m_j(t)$) of order one.

Next let us move $t\in\Delta$. Since $\Phi_1(t,z)$ is a single-valued holomorphic function for two complex variables (t,z) in $\Delta\times(\mathbf{R}\setminus(A\cup B))$, it follows that

$$\mathbf{D}_1:=\bigcup_{t\in\Delta}(t,\Phi_1\big(t,\mathbf{R}\setminus(A\cup B)\big)=\bigcup_{t\in\Delta}\big(t,\Sigma_1(t)\big) \tag{4.9}$$

is a (two-dimensional) two-sheeted open *Riemann domain* over $\Delta\times\mathbb{C}_Z$ with 2ν holomorphic branch curves

$$C'_{1,j}=\bigcup_{t\in\Delta}\big(t,a_j(t)\big)\qquad\text{and}\qquad C''_{1,j}=\bigcup_{t\in\Delta}\big(t,a_j(t)+i\ell_j(t)\big),\quad j=1,\dots,\nu.$$

Therefore, $a_j(t),a_j(t)+i\ell_j(t)$ are holomorphic for $t\in\Delta$. Since $\ell_j(t)$ is a real number, it must be a constant on Δ; $\ell_j(t)=\ell_j>0$, $t\in\Delta$.

Similarly, we see that each $b_j(t)$ is holomorphic on Δ and that $m_j(t)$ is constant on Δ; $m_j(t)=m_j>0, t\in\Delta$. We set

$$\mathbf{D}_0:=\bigcup_{t\in\Delta}\big(t,\Phi_0\big(t,\mathbf{R}\setminus(A\cup B)\big)\big)=\bigcup_{t\in\Delta}\big(t,\Sigma_0(t)\big),$$

which is a two-sheeted open Riemann domain over $\Delta\times\mathbb{C}_W$ with 2ν holomorphic branch curves

$$C'_{0,j}=\bigcup_{t\in\Delta}\big(t,b_j(t)\big)\qquad\text{and}\qquad C''_{0,j}=\bigcup_{t\in\Delta}\big(t,b_j(t)+m_j\big),\quad j=1,\dots,\nu.$$

We consider the holomorphic function for two complex variables

$$W=\psi(t,Z):=\Phi_0\big(t,\Phi_1^{-1}(t,Z)\big),\quad (t,Z)\in\mathbf{D}_1. \tag{4.10}$$

Then $\mathbf{D}_1$ is biholomorphic to $\mathbf{D}_0$ namely, to $\Delta\times(\mathbf{R}\setminus(A\cup B)$ biholomorphic by $(t,Z)\to(t,W)=(t,\psi(t,Z))$ such that $\psi(t,0)=0$, $\psi(t,1)=1$, and

$$\psi\big(t,\big[a_j(t),a_j(t)+i\ell_j\big]^{\pm}\big)=\big[b_j(t),b_j(t)+m_j\big]^{\pm},\quad t\in\Delta. \tag{4.11}$$

Let $t\in\Delta$, and denote by $\widehat{R}(t)$ the covering Riemann surface of $R(t)$ with respect to $\{A,B\}$ modulo dividing cycles. We set $\widehat{\mathcal{R}}=\bigcup_{t\in\Delta}(t,\widehat{R}(t))$. We consider the abelian integral

$$Z = \Phi_1(t,z) = \int_{\zeta_0}^{z} \phi_1(t,\cdot) \quad \text{in } \widehat{R}(t),$$

which is univalent on $\widehat{R}(t)$. We put

$$\widehat{\Sigma}_1(t) := \Phi_1\big(t,\widehat{R}(t)\big) = \mathbb{C}_Z \setminus \Big\{\bigcup_{j=1}^{\nu} \big[a_j(t), a_j(t)+il_j(t)\big] + \sum_{m,n=-\infty}^{\infty} m+n\tau_1(t)\Big\}.$$

In our situation it becomes

$$\widehat{\Sigma}_1(t) = \mathbb{C}_Z \setminus \Big\{\bigcup_{j=1}^{\nu} \big[a_j(t), a_j(t)+il_j\big] + \sum_{m,n=-\infty}^{\infty} m+n\tau_1(0)\Big\},$$

where $a_j(t)$ is holomorphic on Δ and $l_j = l_j(0)$. We put $\alpha_j(t) = a_j(t) - a_1(t)$, $j = 1,\dots,\nu$, and define $\widetilde{\Sigma}_1(t) := \widehat{\Sigma}(t) - a_1(t)$, so that

$$\widetilde{\Sigma}_1(t) = \mathbb{C}_Z \setminus \Big\{[0, il_1] + \bigcup_{j=2}^{\nu} \big[\alpha_j(t), \alpha_j(t)+il_j\big] + \sum_{m,n=-\infty}^{\infty} m+n\tau_1(0)\Big\}.$$

Then $\widetilde{\Sigma}_1(t)/\{1,\tau_1(0)\}$ and $\widehat{\Sigma}_1(t)/\{1,\tau_1(0)\}$ are equivalent to $R(t)$ as Riemann surfaces, and hence,

$$\widetilde{\mathcal{R}}_1 := \bigcup_{t\in\Delta} \big(t, \widetilde{\Sigma}_1(t)/\{1,\tau_1(0)\}\big) \approx \mathcal{R} \quad \text{as a holomorphic family.}$$

Thus, for the first step, it suffices to show that, for $t \in \Delta$,

$$\alpha_j(t) = \alpha_j(0), \quad j = 2,\dots,\nu. \tag{4.12}$$

In the case in which $\nu = 1$, that is, $\partial R(t)$ consists of one component, the first step is true. In the case in which $\nu \geq 2$, we shall use the following elementary fact.

FACT 4.2

Let $f(t,z)$ be a holomorphic function for (t,z) in $\delta \times V (\subset \mathbb{C}_t \times \mathbb{C}_z)$, where $\delta = \{|t| < r_0\}$ and $V = \{|z| < r_1\}$. If there exists an open interval $I \subset (-r_1, r_1)$ such that, for any $t \in \delta$, $f(t,I)$ is a subset of the real axis, then $f(t,z) = f(0,z)$ for $(t,z) \in \delta \times V$.

Similarly to $\widetilde{\Sigma}_1(t)$ we define

$$W = \Phi_0(t,z) = \int_{\zeta_0}^{z} \phi_0(t,\cdot) \quad \text{in } \widehat{R}(t);$$

$$\begin{aligned}\widehat{\Sigma}_0(t) := \Phi_0\big(t,\widehat{R}(t)\big) &= \mathbb{C}_W \setminus \Big\{\bigcup_{j=1}^{\nu} \big[b_j(t)+m_j(t)\big] + \sum_{m,n=-\infty}^{\infty} m+n\tau_0(t)\Big\} \\ &= \mathbb{C}_W \setminus \Big\{\bigcup_{j=1}^{\nu} \big[b_j(t)+m_j\big] + \sum_{m,n=-\infty}^{\infty} m+n\tau_0(0)\Big\}.\end{aligned}$$

We put $\beta_j(t)=b_j(t)-b_1(0), j=1,\dots,\nu$, and $\widetilde{\Sigma}_0(t):=\widehat{\Sigma}_0(t)-b_1(t)$, so that

$$\widetilde{\Sigma}_0(t)=\mathbb{C}_W\setminus\Big\{[0,m_1]+\bigcup_{j=2}^{\nu}\big[\beta_j(t),\beta_j(t)+m_j\big]+\sum_{m,n=-\infty}^{\infty}m+n\tau_0(t)\Big\},$$

$$\widetilde{\Sigma}_0(t)/\big\{1,\tau_0(0)\big\}\sim R(t)\quad \text{as a Riemann surface,}$$

$$\widetilde{\mathcal{R}}_0:=\bigcup_{t\in\Delta}\big(t,\widetilde{\Sigma}_0(t)/\big\{1,\tau_0(0)\big\}\big)\approx\mathcal{R}\quad \text{as a holomorphic family.}$$

We thus have the automorphism

$$W=\widetilde{\psi}(t,Z):=\widetilde{\Sigma}_1(t)\to\widetilde{\Sigma}_0(t),\quad t\in\Delta,$$

such that, for $j=2,\dots,\nu$,

$$\widetilde{\psi}\big(t,[0,il_1]^{\pm}\big)=[0,m_1]^{\pm}\qquad\text{and}$$
$$\widetilde{\psi}\big(t\big[\alpha_j(t),\alpha_j(t)+il_j\big]^{\pm}\big)=\big[\beta_j(t),\beta_j(t)+m_j\big]^{\pm}.$$

Applying the above elementary fact to the first equation we have

$$\widetilde{\psi}(t,Z)=\widetilde{\psi}(0,Z),\quad t\in\Delta.$$

It follows from the second equation that, for each $j=2,\dots,\nu$,

$$\big[\beta_j(t),\beta_j(t)+m_j)\big]^{\pm}=\widetilde{\psi}\big(0,\big[\alpha_j(t),\alpha_j(t)+m_j\big]^{\pm}\big),\quad t\in\Delta.$$

This implies (4.12). In fact, if (4.12) were not true, we have $\alpha_j(t)\neq\alpha_j(0)$ for some j, $2\le j\le\nu$ and some sufficiently small $t\neq 0$. Hence $\widetilde{\psi}(0,Z)$ would be one-to-one.

Step 2: Assertion (2) holds. In fact, we have the L_1-differential $\phi_1(t,z)$ for $(R(t),A(t))$ and put $\tau_1(t):=\int_{B(t)}\phi_1(t,\cdot)$. Systematically applying the first step we see that $\phi_1(t,z)$ is holomorphic for $(t,z)\in\mathcal{R}$ and $\tau_1(t)=\tau_1(0)$ for $t\in\Delta$.

Since Δ is simply connected, we have a continuous section $\xi: t\in\Delta\to R(t)$ of $\mathcal{R}$ and a canonical homology basis $\{A(t),B(t)\}$ of $R(t)$ with $A(t)\cap B(t)=\xi(t)$, $t\in\Delta$, which moves continuously in $\mathcal{R}$ with $t\in\Delta$.

Let $t\in\Delta$, denote by $\widehat{R}(t)$ the covering Riemann surface of $R(t)$ with respect to $\{A(t),B(t)\}$ modulo dividing cycles, and put $\widehat{\mathcal{R}}=\bigcup_{t\in\Delta}(t,\widehat{R}(t))$. By the first step we find small disks $\Delta_k=\{|t-t_k|<r_k\}\Subset\Delta$, $k=1,2,\dots$, with $\Delta=\bigcup_{k=1}^{\infty}\Delta_k$ and $\lim_{k\to\infty}\partial\Delta_k=\partial\Delta$ such that the following statements hold.

(1) For $\Delta_k\cap\Delta_l\neq\emptyset$, $l,k=1,2,\dots$, we have a disk $\Delta_{kl}\supset\Delta_k\cup\Delta_l$ in Δ such that $\mathcal{R}|_{\Delta_{kl}}$ is holomorphically trivial and is realized as an unramified domain $\mathcal{D}_{kl}$ over $\Delta_{kl}\times\mathbb{C}_w$ such that $\mathcal{D}_{kl}$ contains the bidisk $\Delta_{kl}\times\{|w|<r\}$ in which $\xi|_{\Delta_{kl}}=\{\xi(t):t\in\Delta_{kl}\}$ is realized. We write $W_k(t)$ in $R(t)$, which corresponds to $\{t\}\times\{|w|<r\}$, $t\in\Delta_k$, and $\mathcal{W}_k=\bigcup_{t\in\Delta_k}(t,W_k(t))(\subset\mathcal{R}|_{\Delta_k})$.

(2) For $k=1,2,\dots$ we draw a holomorphic section $\zeta_k: t\in\Delta_k\to\zeta_k(t)$ of $\mathcal{R}|_{\Delta_k}$ such that $\zeta_k(t_k)=\xi(t_k)$ and $\zeta_k|_{\Delta_k}\subset\mathcal{W}_k$. If we put, for $t\in\Delta_k$,

$$\Phi_{1k}(t,z):=\int_{\zeta_k(t)}^{z}\phi_1(t,\cdot),\quad z\in\widehat{R}(t),$$

then $\widehat{\Sigma}_k(t) := \Phi_{1k}(t, \widehat{R}(t))$ is a (univalent) domain in $\mathbb{C}$ and

$$\begin{aligned}\widehat{\Sigma}_k(t) &= \widehat{\Sigma}_k(t_k) + \int_{\zeta_k(t)}^{\zeta_k(t_k)} \phi_1(t,\cdot) \quad \text{in } \mathbb{C}\\ &=: \widehat{\Sigma}_k(t_k) + h_k(t) \quad \text{in } \mathbb{C},\end{aligned}$$

where the integral path is an arc from $\zeta_k(t)$ to $\zeta_k(t_k)$ in $W_k(t)$. For a fixed $t \in \Delta_k$ we have

$$\widehat{\Sigma}_k(t)/\{1, \tau_1(t)\} \approx \big(\widehat{\Sigma}_k(t_k) + h_k(t)\big)/\{1, \tau_1(t_k)\} \approx \widehat{\Sigma}_k(t_k)/\{1, \tau_1(t_k)\},$$

which stands for an equality between the bordered tori. It follows that

$$\mathcal{R}|_{\Delta_k} \approx \bigcup_{t\in\Delta_k} \big(t, \widehat{\Sigma}_k(t_k)/\{1, \tau_1(t_k)\}\big).$$

Now let $\Delta_k \cap \Delta_l \neq \emptyset$, $k, l = 1, 2, \ldots$, and let $(t, z) \in \widehat{\mathcal{R}}|_{\Delta_k\cap\Delta_l}$. If we draw an arc $\gamma_{kl}(t)$ connecting $\zeta_k(t)$ and $\zeta_l(t)$ in $\mathcal{W}_k(t) \cup \mathcal{W}_l(t)$, then the condition $\zeta_k|_{\Delta_k} \subset \mathcal{W}_k$ yields

$$\Phi_{1k}(t, z) - \Phi_{1l}(t, z) = \int_{\gamma_{kl}(t)} \phi_1(t,\cdot) =: \alpha_{kl}(t), \quad z \in \widehat{R}(t). \tag{4.13}$$

We note that $\alpha_{kl}(t)$ is independent of the choice of $\gamma_{kl}(t)$ in $W_k(t) \cup W_l(t)$ and is a holomorphic function on $\Delta_k \cap \Delta_l$ such that $\alpha_{kl}(t) = -\alpha_{lk}(t)$.

Given any point $t \in \Delta_k \cap \Delta_l \cap \Delta_m \neq \emptyset$, since $\gamma_{kl}(t) \circ \gamma_{lm}(t) \circ \gamma_{mk}(t)$ is a closed curve in the simply connected domain $W_k(t) \cup W_l(t) \cup W_m(t)$, we have

$$\alpha_{kl}(t) + \alpha_{lm}(t) + \alpha_{mk}(t) = 0 \quad \text{on } \Delta_k \cap \Delta_l \cap \Delta_m. \tag{4.14}$$

Since the first Cousin problem is solvable on the disk Δ, we find a holomorphic function $\alpha_k(t)$ on $\Delta_k, k = 1, 2, \ldots$, such that $\alpha_{kl}(t) = \alpha_k(t) - \alpha_l(t)$ on Δ_{kl} for any pair $\{k, l\}$. Hence,

$$h(t, z) := \Phi_{1k}(t, z) - \alpha_k(t), \quad (t, z) \in \widehat{\mathcal{R}}|_{\Delta_k},$$

is independent of $k = 1, 2, \ldots$, that is, $h(t, z)$ is the (single-valued) holomorphic function for (t, z) in the *whole* $\widehat{\mathcal{R}}$. We put

$$\widehat{S}(t) := h\big(t, \widehat{R}(t)\big) = \mathbb{C}_Z \setminus \Big\{\bigcup_{j=1}^{\nu} \big[\widehat{\alpha}_j(t), \widehat{\alpha}_j(t) + i\ell_j\big]^{\pm} + \sum_{m,n=-\infty}^{\infty} m + n\tau_1(t)\Big\},$$

where $\widehat{\alpha}_j(t)$, $j = 1, \ldots, \nu$, is a holomorphic function on Δ. Then $\widehat{S}(t)/\{1, \tau_1(t)\} \sim R(t), t \in \Delta$. Since $R(t) \sim R(0), t \in \Delta$, we have $\tau_1(t) = \tau_1(0)$ and $\widehat{\alpha}_j(t) - \widehat{\alpha}_1(t) = \widehat{\alpha}_j(0) - \widehat{\alpha}_1(0), t \in \Delta, j = 1, \ldots, \nu$.

If we put $\mathbf{a}_j := \widehat{\alpha}_j(0) - \widehat{\alpha}_1(0)$, then the univalent function $H(t, z) := h(t, z) - \widehat{\alpha}_1(t)$ on $\widehat{R}(t)$ is written into

$$H\big(t, R(t)\big) = \mathbb{C}_Z \setminus \Big\{\bigcup_{j=1}^{\nu} [\mathbf{a}_j, \mathbf{a}_j + i\ell_j]^{\pm} + \sum_{m,n=-\infty}^{\infty} m + n\tau_1(0)\Big\}, \quad t \in \Delta. \tag{4.15}$$

It follows that

$$(4.16)\qquad R(t)\sim H\big(t,R(t)\big)/\big\{1,\tau_1(t)\big\}=H\big(0,R(0)\big)/\big\{1,\tau(0)\big\}\sim R(0),\quad t\in\Delta.$$

Since $H(t,z)$ is holomorphic for $(t,z)\in\widehat{\mathcal{R}}$ and $\tau_1(t)=\tau_1(0)$ for $t\in\Delta$, we prove the second step. □

Appendix

To prove Corollary 3.1(1) we fix $t_0\in\Delta$. By the assumption we have a ball $\widetilde{\delta}$ of center t_0 in Δ such that $\mathcal{R}|_{\widetilde{\delta}}$ is an $(n+1)$-dimensional Stein manifold. Let δ be a ball of center t_0 such that $\delta\Subset\widetilde{\delta}$. Then we have a strictly plurisubharmonic exhaustion function $\psi(t,z)$ on $\mathcal{R}|_{\widetilde{\delta}}$. For a sufficiently large integer $k>1$ we set

$$\mathcal{R}_k:=\big\{(t,z)\in\mathcal{R}|_\delta:\psi(t,z)<k\big\}=:\bigcup_{t\in\delta}\big(t,R_k(t)\big)\Subset\mathcal{R}.$$

Then $\mathcal{R}_k$ is a Stein manifold such that $\partial\mathcal{R}_k$ is smooth in $\mathcal{R}|_\delta$ and each $R_k(t)$, $t\in\delta$, is an open torus with ν smooth contours $C_{1k}(t),\dots,C_{\nu k}(t)$ in $R(t)$. We denote by $\sigma_{Hk}(t)$ the hyperbolic span for $R_k(t)$. By Theorem 1.2(1), $\sigma_{Hk}(t)$ is plurisubharmonic on δ. Since $\sigma_{Hk}(t)\searrow\sigma_H(t)$ as $k\to\infty$ for $t\in\delta$, it follows that $\sigma_H(t)$ is plurisubharmonic on δ, and hence, on Δ.

To prove Corollary 3.1(2) let $(\mathcal{R},\pi,\Delta)$ be a holomorphic family with conditions (i), (ii), and (iii). We use the exhaustion method as in the proof of Corollary 3.1(1). Then, by the standard argument under the pluriharmonicity of $\sigma_H(t)$ (which is the limit of the plurisubharmonic function $\sigma_{Hk}(t)$) and condition (iii), we may assume that there exists an $(n+1)$-dimensional manifold $\widetilde{\mathcal{R}}$ such that $\mathcal{R}=\bigcup_{t\in\Delta}(t,R(t))\subset\widetilde{\mathcal{R}}=\bigcup_{t\in\Delta}(t,\widetilde{R}(t))$ and $\partial R(t)$, $t\in\Delta$, consists of ν smooth contours $C_j(t)$ in $\widetilde{R}(t)$. Therefore, *Corollary 3.1(2) holds locally in* Δ by the same argument as that of the proof of the first step in Theorem 1.2 under the pluriharmonicity of $\sigma_H(t)$ in Δ. To go from *locally* in Δ to *globally* on Δ for Corollary 3.1(2), we introduce the κ*-cycle.*

Let R be a bordered torus with smooth contours $C_1,\dots,C_\nu$. We denote by R^κ the Kerékjártó–Stoïlow compactification of R; in short, we consider each C_j as one point in R^κ. Let γ be a closed curve in R or consist of a finite number of arcs $\{\gamma_k\}_{k=1,\dots,m}$ in R whose closure γ^* in R^κ is a closed curve in R^κ. We see that such γ in the second case yields a closed curve γ' in R which is homologous to γ^* in R^κ. If γ'' is another closed curve in R homologous to γ^* in R^κ, then it holds that $\gamma'\sim\gamma''$ in R *modulo dividing cycles* and vice versa. We call such γ in R the κ*-cycle* in R, which is identified with γ^* in R^κ or with the closed curve γ' in R stated above. For two κ-cycles γ_1 and γ_2 in R, if $\gamma_1^*\sim\gamma_2^*$ in R^κ, then we write $\gamma_1\sim_\kappa\gamma_2$ in R. For two κ-cycles A,B in R with $A\times B=1$ in R, if A^*,B^* is the canonical homology basis of R^κ, then we call $\{A,B\}$ the κ*-canonical homology basis* of R.

REMARK 2
The κ-canonical homology basis $\{A, B\}$ of R uniquely induces the L_1-differential $\phi_1(z)$ for (R, A) and the modulus $\tau_1 := \int_B \phi_1$.

By condition (ii) we have a continuous section $\xi : t \in \Delta \to R(t)$ of $\mathcal{R}$ and a κ-canonical homology basis $\{A(t), B(t)\}$ of $R(t)$ with $A(t) \cap B(t) = \xi(t)$, $t \in \Delta$, which moves continuously in $\mathcal{R}^\kappa$ with $t \in \Delta$. We uniquely have the L_1-differential $\phi_1(t, z)$ for $(R(t), A(t))$. We put $\tau_1(t) := \int_{B(t)} \phi_1(t, z)$. Since $\mathcal{R}$ is locally trivial, we see that $\phi_1(t, z)$ is holomorphic for $(t, z) \in \mathcal{R}$ and $\tau_1(t) = \tau_1(0)$ for $t \in \Delta$. Using Remark 2 and the same argument as that of the second step of the proof of Theorem 1.2(2), we have (4.13) and (4.14) for $\Delta(\subset \mathbb{C}_t^n)$. Then by the solvability of the first Cousin problem in the pseudoconvex domain Δ, we have (4.15) and (4.16) for Δ. Then, similar to Theorem 1.2(2) we have Corollary 1.3(2). $\square$

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