Homological aspects of the dual Auslander transpose, II

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Abstract Let R and S be rings, and let ${}_R\omega_S$ be a semidualizing bimodule. We prove that there exists a Morita equivalence between the class of ∞-ω-cotorsion-free modules and a subclass of the class of ω-adstatic modules. Also, we establish the relation between the relative homological dimensions of a module M and the corresponding standard homological dimensions of $\mathrm{Hom}(\omega, M)$. By investigating the properties of the Bass injective dimension of modules (resp., complexes), we get some equivalent characterizations of semitilting modules (resp., Gorenstein Artin algebras). Finally, we obtain a dual version of the Auslander–Bridger approximation theorem. As a consequence, we get some equivalent characterizations of Auslander n-Gorenstein Artin algebras.

1. Introduction

Semidualizing bimodules arise naturally in the investigation of various duality theories in commutative algebra. The study of such modules was initiated by Foxby [18] and Golod [20]. Then Holm and White [21] extended this notion to arbitrary associative rings, while Christensen [11] and Kubik [27] extended it to semidualizing complexes and quasidualizing modules, respectively. The study of semidualizing bimodules or complexes was connected to the so-called Auslander classes and Bass classes defined by Avramov and Foxby [5] and Christensen [11]. Semidualizing bimodules or complexes and the corresponding Auslander/Bass classes have been studied by many authors (see, e.g., [1], [5], [11]–[14], [16], [21], [33]). To dualize the important and useful notions of the Auslander transpose of modules and n-torsion-free modules, we [33] introduced the notions of the cotranspose of modules and n-cotorsion-free modules with respect to a semidualizing bimodule, and we obtained several dual counterparts of interesting results. Based on this previous work, we study further homological properties of the cotranspose of modules, n-cotorsion-free modules, and related modules.

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The article is organized as follows. In Section 2, we give some terminology and some preliminary results. In particular, we prove that if $(R, \mathfrak{m}, k)$ is a commutative Gorenstein complete local ring with $\dim R > 0$ and $\mathfrak{q}$ is a prime ideal of R with nonzero height, then the tensor product of the injective envelopes of $R/\mathfrak{q}$ and k is equal to zero. This gives a negative answer to an open question of Kubik [27, Question 3.12] about quasidualizing modules.

Let R and S be rings, and let ${}_R\omega_S$ be a semidualizing bimodule. In Section 3, we prove that if the projective dimension of ${}_R\omega$ is finite, then the class of ∞-ω-cotorsion-free modules is contained in the right orthogonal class of ${}_R\omega$; dually, if the projective dimension of ω_S is finite, then the above inclusion relation between these two classes of modules is reverse. Also, we prove that there exists a Morita equivalence between the class of ∞-ω-cotorsion-free modules and a subclass of the class of ω-adstatic modules. Finally, we establish the relation between the relative homological dimensions of a module M and the corresponding standard homological dimensions of $\mathrm{Hom}(\omega, M)$.

In Section 4, we first give some criteria for computing the Bass injective dimension of modules in terms of the vanishing of Ext-functors and some special approximations of modules. Then, motivated by the philosophy of [26], we introduce the notion of semitilting bimodules in the general case and prove that ${}_R\omega_S$ is right semitilting if and only if the Bass injective dimension of ${}_RR$ is finite.

In Section 5, we extend the Bass class and the Bass injective dimension of modules with respect to ω to that of homologically bounded complexes. We show that a homologically bounded complex has finite Bass injective dimension if and only if it admits a special quasi-isomorphism in the derived category of the category of modules. As an application of this result, we get some equivalent characterizations of Gorenstein–Artin algebras.

In Section 6, we first introduce the notions of the (strong) Ext-cograde and Tor-cograde of modules with respect to ω. Then we obtain a dual version of the Auslander–Bridger approximation theorem (see [17, Proposition 3.8]) as follows. For any left R-module M and $n \geq 1$, if the Tor-cograde of $\mathrm{Ext}^i_R(\omega, M)$ with respect to ω is at least i for any $1 \leq i \leq n$, then there exist a left R-module U and a homomorphism $f : U \to M$ of left R-modules satisfying the following properties: (1) the injective dimension of U relative to the class of ω-projective modules is at most n, and (2) $\mathrm{Ext}^i_R(\omega, f)$ is bijective for any $1 \leq i \leq n$. As an application of this result, we prove that, for any $n \geq 1$, the strong Ext-cograde of $\mathrm{Tor}^S_i(\omega, N)$ with respect to ω is at least i for any left S-module N and $1 \leq i \leq n$ if and only if the strong Tor-cograde of $\mathrm{Ext}^i_R(\omega, M)$ with respect to ω is at least i for any left R-module M and $1 \leq i \leq n$. Furthermore, we get some equivalent characterizations of Auslander n-Gorenstein–Artin algebras.

2. Preliminaries

Throughout this article, R and S are fixed associative rings with unity. We use $\mathrm{Mod}\, R$ (resp., $\mathrm{Mod}\, S^{op}$) to denote the class of left R-modules (resp., right S-

modules). Let $M \in \operatorname{Mod} R$. We use $\operatorname{pd}_R M$, $\operatorname{fd}_R M$, and $\operatorname{id}_R M$ to denote the projective, flat, and injective dimensions of M, respectively, and use $\operatorname{Add}_R M$ (resp., $\operatorname{Prod}_R M$) to denote the subclass of $\operatorname{Mod} R$ consisting of all direct summands of direct sums (resp., direct products) of copies of M. We use

$$0 \to M \to I^0(M) \xrightarrow{f^0} I^1(M) \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} I^i(M) \xrightarrow{f^i} \cdots \tag{2.1}$$

to denote a minimal injective resolution of M. For any $n \geq 1$, $\operatorname{co}\Omega^n(M) := \operatorname{Im} f^{n-1}$ is called the *nth cosyzygy* of M, and in particular, $\operatorname{co}\Omega^0(M) := M$.

DEFINITION 2.1 (SEE [21])

(1) An $(R\text{-}S)$-bimodule ${}_R\omega_S$ is called *semidualizing*[1] if the following conditions are satisfied.

(a1) ${}_R\omega$ admits a degreewise finite R-projective resolution.

(a2) ω_S admits a degreewise finite S-projective resolution.

(b1) The homothety map ${}_RR_R \xrightarrow{{}^R\gamma} \operatorname{Hom}_{S^{op}}(\omega,\omega)$ is an isomorphism.

(b2) The homothety map ${}_SS_S \xrightarrow{\gamma_S} \operatorname{Hom}_R(\omega,\omega)$ is an isomorphism.

(c1) $\operatorname{Ext}_R^{\geq 1}(\omega,\omega) = 0$.

(c2) $\operatorname{Ext}_{S^{op}}^{\geq 1}(\omega,\omega) = 0$.

(2) A semidualizing bimodule ${}_R\omega_S$ is called *faithful* if the following conditions are satisfied.

(f1) If $M \in \operatorname{Mod} R$ and $\operatorname{Hom}_R(\omega, M) = 0$, then $M = 0$.

(f2) If $N \in \operatorname{Mod} S^{op}$ and $\operatorname{Hom}_{S^{op}}(\omega, N) = 0$, then $N = 0$.

Typical examples of semidualizing bimodules include the free module of rank one, dualizing modules over a Cohen–Macaulay local ring, and the ordinary Matlis dual bimodule ${}_\Lambda D(\Lambda)_\Lambda$ of ${}_\Lambda\Lambda_\Lambda$ over an Artin algebra Λ. Any semidualizing bimodule over commutative rings is faithful (see [21, Proposition 3.1]). Semidualizing bimodules occur in the literature with several different names (e.g., in the work of [18], [20], [29], [35]).

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. According to [27], an Artinian R-module T is called *quasidualizing* if the homothety $\hat{R} \to \operatorname{Hom}_R(T,T)$ is an isomorphism (where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R) and $\operatorname{Ext}_R^{i\geq 1}(T,T) = 0$. It was proved in [27, Lemma 3.11] that if L and T are R-modules with T quasidualizing such that $\operatorname{Hom}_R(T,L) = 0$, then $L = 0$. Motivated by this result and [21, Lemma 3.1], an open question was posed in [27] as follows.

[1]In [33] and the original version of this article, we use C to denote the given semidualizing module. The referee suggests the following: "The notation $\operatorname{cTr}_C M$ (see Definition 2.5 below) is very confusing. I am not sure how the first 'c' is distinguished with the semidualizing module C, particularly when writing it on the blackboard. It would be better to change the notation or quit using C for the semidualizing module." Following this suggestion, we denote the given semidualizing module by substituting ω for C.

QUESTION 2.2 ([27, QUESTION 3.12])

Let R be a commutative Noetherian local ring. If L and T are R-modules with T quasidualizing such that $T \otimes_R L = 0$, then does $L = 0$?

The following result shows that the answer to this question is negative in general.

PROPOSITION 2.3

Let R be a commutative Noetherian complete local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. If R is Gorenstein (i.e., $\mathrm{id}_R R < \infty$) with $\dim R > 0$, then $E^0(R/\mathfrak{q}) \otimes_R E^0(k) = 0$ for any prime ideal $\mathfrak{q}$ with $\mathrm{ht}(\mathfrak{q}) > 0$, where $\mathrm{ht}(\mathfrak{q})$ is the height of $\mathfrak{q}$.

Proof

By [30, Theorem 4.2], $E^0(k)$ is quasidualizing. Since R is Gorenstein, it follows from [7, Fundamental Theorem] that $E^i(R) = \bigoplus_{\mathrm{ht}(\mathfrak{p})=i} E(R/\mathfrak{p})$ with $\mathfrak{p} \in \mathrm{Spec}(R)$ (the prime spectrum of R) for any $i \geq 0$. In particular, $E^0(R) = \bigoplus_{\mathrm{ht}(\mathfrak{p})=0} E(R/\mathfrak{p})$ with $\mathfrak{p} \in \mathrm{Spec}(R)$. On the other hand, for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = 0$ and $\mathrm{ht}(\mathfrak{q}) > 0$, we have $\mathrm{Hom}_R(E^0(R/\mathfrak{q}), E^0(R/\mathfrak{p})) = 0$. So $\mathrm{Hom}_R(E^0(R/\mathfrak{q}), E^0(R)) = 0$ and $\mathrm{Hom}_R(E^0(R/\mathfrak{q}), R) = 0$. Thus, we have

$$
\begin{aligned}
&\mathrm{Hom}_R\big(E^0(R/\mathfrak{q}) \otimes_R E^0(k), E^0(k)\big) \\
&\quad \cong \mathrm{Hom}_R\big(E^0(R/\mathfrak{q}), \mathrm{Hom}_R\big(E^0(k), E^0(k)\big)\big) \\
&\qquad \text{(by the adjoint isomorphism theorem)} \\
&\quad \cong \mathrm{Hom}_R\big(E^0(R/\mathfrak{q}), R\big) \quad \text{(by [30, Theorem 4.2])} \\
&\quad = 0.
\end{aligned}
$$

Because $E^0(k)$ is an injective cogenerator for $\mathrm{Mod}\, R$, $E^0(R/\mathfrak{q}) \otimes_R E^0(k) = 0$. □

From now on, ${}_R\omega_S$ is a semidualizing bimodule. For convenience, we write $(-)_* = \mathrm{Hom}_R(\omega, -)$ and ${}_R\omega^{\perp} = \{M \in \mathrm{Mod}\, R \mid \mathrm{Ext}_R^{i\geq 1}(\omega, M) = 0\}$.

Let $M \in \mathrm{Mod}\, R$ and $N \in \mathrm{Mod}\, S$. Then we have the following two canonical valuation homomorphisms:

$$\theta_M : \omega \otimes_S M_* \to M,$$

defined by $\theta_M(x \otimes f) = f(x)$ for any $x \in \omega$ and $f \in M_*$, and

$$\mu_N : N \to (\omega \otimes_S N)_*,$$

defined by $\mu_N(y)(x) = x \otimes y$ for any $y \in N$ and $x \in \omega$. Following [36], we call M (resp., N) *ω-static* (resp., *ω-adstatic*) if θ_M (resp., μ_N) is an isomorphism. We denote by $\mathrm{Stat}(\omega)$ and $\mathrm{Adst}(\omega)$ the class of all ω-static modules and the class of all ω-adstatic modules, respectively.

DEFINITION 2.4 (SEE [21])

The *Bass class* $\mathcal{B}_\omega(R)$ with respect to ω consists of all left R-modules M satisfying the following conditions:

(B1) $M \in {}_R\omega^\perp$,

(B2) $\mathrm{Tor}^S_{\geq 1}(\omega, M_*) = 0$, and

(B3) $M \in \mathrm{Stat}(\omega)$; that is, θ_M is an isomorphism in $\mathrm{Mod}\, R$.

The *Auslander class* $\mathcal{A}_\omega(S)$ with respect to ω consists of all left S-modules N satisfying the following conditions:

(A1) $\mathrm{Tor}^S_{i\geq 1}(\omega, N) = 0$,

(A2) $\omega \otimes_S N \in {}_R\omega^\perp$, and

(A3) $N \in \mathrm{Adst}(\omega)$; that is, μ_N is an isomorphism in $\mathrm{Mod}\, S$.

DEFINITION 2.5 (SEE [33])

Let $M \in \mathrm{Mod}\, R$, and let $n \geq 1$.

(1) $\mathrm{cTr}_\omega M := \mathrm{Coker}\, f^0_*$ is called the *cotranspose* of M with respect to ${}_R\omega_S$, where f^0 is as in (2.1).

(2) M is called *n-ω-cotorsion-free* if $\mathrm{Tor}^S_{1\leq i\leq n}(\omega, \mathrm{cTr}_\omega M) = 0$; and M is called *∞-ω-cotorsion-free* if it is n-ω-cotorsion-free for all n. The class of all ∞-ω-cotorsion-free modules is denoted by $\mathrm{c}\mathcal{T}(R)$. In particular, every module in $\mathrm{Mod}\, R$ is 0-ω-cotorsion-free.

By [33, Proposition 3.2], a module is 2-ω-cotorsion-free if and only if it is ω-static.

Let $\mathcal{W} \subseteq \mathcal{X}$ be subclasses of $\mathrm{Mod}\, R$. Recall from [2] that $\mathcal{W}$ is called a *generator* for $\mathcal{X}$ if, for any $X \in \mathcal{X}$, there exists an exact sequence $0 \to X' \to W \to X \to 0$ in $\mathrm{Mod}\, R$ with $W \in \mathcal{W}$ and $X' \in \mathcal{X}$; $\mathcal{W}$ is called an Ext-*projective generator* for $\mathcal{X}$ if $\mathcal{W}$ is a generator for $\mathcal{X}$ and $\mathrm{Ext}^{i\geq 1}_R(W, X) = 0$ for any $X \in \mathcal{X}$ and $W \in \mathcal{W}$. Also recall that $\mathcal{X}$ is called *coresolving* if it is closed under extensions and cokernels of monomorphisms and it contains all injective modules in $\mathrm{Mod}\, R$.

Let $M \in \mathrm{Mod}\, R$. An exact sequence (of finite or infinite length)

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

in $\mathrm{Mod}\, R$ is called an *$\mathcal{X}$-resolution* of M if all X_i's are in $\mathcal{X}$; furthermore, such an $\mathcal{X}$-resolution is called *proper* if it remains exact after applying the functor $\mathrm{Hom}_R(X, -)$ for any $X \in \mathcal{X}$. The *$\mathcal{X}$-projective dimension* $\mathcal{X}\text{-}\mathrm{pd}_R M$ of M is defined as $\inf\{n \mid \text{there exists an } \mathcal{X}\text{-resolution } 0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0 \text{ of } M \text{ in } \mathrm{Mod}\, R\}$. Dually, the notions of an *$\mathcal{X}$-coresolution*, an *$\mathcal{X}$-coproper coresolution*, and the *$\mathcal{X}$-injective dimension* $\mathcal{X}\text{-}\mathrm{id}_R M$ of M are defined.

DEFINITION 2.6 ([15])

A module $M \in \mathrm{Mod}\, R$ is called *Gorenstein projective* if there exists an exact sequence of projective modules

$$\mathbf{P} := \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\operatorname{Mod} R$ satisfying the following conditions: (1) it remains exact after applying the functor $\operatorname{Hom}_R(-,P)$ for any projective module P in $\operatorname{Mod} R$, and (2) $M \cong \operatorname{Im}(P_0 \to P^0)$. Dually, the notion of Gorenstein injective modules is defined. We use $\mathcal{GP}(R)$ (resp., $\mathcal{GI}(R)$) to denote the subclass of $\operatorname{Mod} R$ consisting of Gorenstein projective (resp., Gorenstein injective) modules.

FACT 2.7

(1) $\mathcal{B}_\omega(R)$ is coresolving and $\operatorname{Add}_R \omega$ is an Ext-projective generator for $\mathcal{B}_\omega(R)$ (see [21, Proposition 5.1(b), Theorem 6.2] and [33, Proposition 3.7]).

(2) When ω is a dualizing module over a local Cohen–Macaulay ring R, $\mathcal{B}_\omega(R)$ actually is exactly the class of modules admitting finite Gorenstein injective dimensions (see [16, Corollary 2.6]). However, the following example illustrates that these two classes of modules are different in general.

EXAMPLE 2.8

Let Λ be the finite-dimensional algebra over a field defined by the following quiver and relation:

$$\circ 1 \rightleftarrows \circ 2 \longleftarrow \circ 3 \longleftarrow \circ 4 \rightleftarrows \circ 5,$$

$$(\operatorname{rad} \Lambda)^2 = 0.$$

Take ${}_\Lambda\omega = {}^2_1 \oplus {}^1{}_2{}^3 \oplus 3 \oplus {}_3{}^4{}_5 \oplus {}^5_4$ and $M = {}_3{}^4{}_5$.Then by [34, Example 3.1] and [33, Theorem 3.9], ${}_\Lambda\omega_{\operatorname{End}({}_\Lambda\omega)}$ is a semidualizing bimodule and $M \in \mathcal{B}_\omega(\Lambda)$. But an easy computation shows that the Gorenstein injective dimension of M is infinite.

Let $\mathcal{E}$ be a subcategory of an abelian category $\mathcal{A}$. Recall from [15] that a sequence

$$\mathbb{S} : \cdots \to S_1 \to S_2 \to S_3 \to \cdots$$

in $\mathcal{A}$ is called $\operatorname{Hom}_{\mathcal{A}}(\mathcal{E},-)$-*exact* (resp., $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{E})$-*exact*) if $\operatorname{Hom}_{\mathcal{A}}(E,\mathbb{S})$ (resp., $\operatorname{Hom}_{\mathcal{A}}(\mathbb{S},E)$) is exact for any object E in $\mathcal{E}$. An epimorphism (resp., a monomorphism) in $\mathcal{A}$ is called $\mathcal{E}$-*proper* (resp., $\mathcal{E}$-*coproper*) if it is $\operatorname{Hom}_{\mathcal{A}}(\mathcal{E},-)$-exact (resp., $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{E})$-exact).

DEFINITION 2.9 ([24])

Let $\mathcal{E}$ and $\mathcal{T}$ be subcategories of an abelian category $\mathcal{A}$. Then $\mathcal{T}$ is called $\mathcal{E}$-*coresolving* in $\mathcal{A}$ if the following conditions are satisfied.

(1) $\mathcal{T}$ admits an $\mathcal{E}$-*coproper cogenerator* $\mathcal{C}$, that is, $\mathcal{C} \subseteq \mathcal{T}$, and for any object T in $\mathcal{T}$, there exists a $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{E})$-exact exact sequence $0 \to T \to C \to T' \to 0$ in $\mathcal{A}$ such that C is an object in $\mathcal{C}$ and T' is an object in $\mathcal{T}$.

(2) $\mathcal{T}$ is *closed under $\mathcal{E}$-coproper extensions*, that is, for any $\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{E})$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$, if both A_1 and A_3 are objects in $\mathcal{T}$, then A_2 is also an object in $\mathcal{T}$.

(3) $\mathcal{T}$ is *closed under cokernels of $\mathcal{E}$-coproper monomorphisms*, that is, for any $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{E})$-exact exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$, if both A_1 and A_2 are objects in $\mathcal{T}$, then A_3 is also an object in $\mathcal{T}$.

Dually, the notions of *$\mathcal{E}$-proper generators* and *$\mathcal{E}$-resolving subcategories* are defined.

3. Relative homological dimensions

Holm and White [21] obtained some equivalent characterizations of $\mathcal{B}_\omega(R)$ in terms of the so-called "ω-projective and ω-flat modules." Similar results were also proved by Enochs and Holm [14]. Recently, we proved in [33, Theorem 3.9] that $\mathcal{B}_\omega(R) = \mathrm{c}\mathcal{T}(R) \cap {}_R\omega^{\perp}$. In the beginning of this section, we investigate the further relations among $\mathrm{c}\mathcal{T}(R)$, ${}_R\omega^{\perp}$, and $\mathcal{B}_\omega(R)$.

PROPOSITION 3.1

(1) *If* $\mathrm{pd}_R\,\omega < \infty$, *then* $\mathrm{c}\mathcal{T}(R) \subseteq {}_R\omega^{\perp}$.

(2) *If* $\mathrm{pd}_{S^{op}}\,\omega < \infty$, *then* ${}_R\omega^{\perp} \subseteq \mathrm{c}\mathcal{T}(R)$.

Proof

(1) Let $M \in \mathrm{c}\mathcal{T}(R)$. Then by [33, Proposition 3.7], there exists an exact sequence

$$\cdots \to W_n \to W_{n-1} \to \cdots \to W_0 \to M \to 0$$

in $\mathrm{Mod}\,R$ with all $W_i \in \mathrm{Add}_R\,\omega$. Put $M_i = \mathrm{Im}(W_i \to W_{i-1})$ for any $i \geq 1$. We may assume $\mathrm{pd}_R\,\omega = n < \infty$ by assumption. Since $W_i \in {}_R\omega^{\perp}$ by [33, Lemma 2.5(1)], $\mathrm{Ext}^i_R(\omega, M) \cong \mathrm{Ext}^{i+n}_R(\omega, M_n) = 0$ for any $i \geq 1$ and $M \in {}_R\omega^{\perp}$.

(2) Let $M \in {}_R\omega^{\perp}$, and let $\mathrm{pd}_{S^{op}}\,\omega = n < \infty$. Then we get an exact sequence

$$0 \to \mathrm{co}\,\Omega^i(M)_* \to I^i(M)_* \to \mathrm{co}\,\Omega^{i+1}(M)_* \to 0$$

in $\mathrm{Mod}\,S$ for any $i \geq 0$. Note that $\mathrm{fd}_{S^{op}}\,\omega = \mathrm{pd}_{S^{op}}\,\omega = n$ because ω is finitely presented as a right S-module. Since $\mathrm{Tor}^S_{i\geq 1}(\omega, I_*) = 0$ for any injective left R-module I by [33, Lemma 2.5(2)], we have $\mathrm{Tor}^S_j(\omega, \mathrm{co}\,\Omega^i(M)_*) \cong \mathrm{Tor}^S_{j+n}(\omega, \mathrm{co}\,\Omega^{i+n}(M)_*) = 0$ for any $i \geq 0$ and $j \geq 1$; in particular, $\mathrm{Tor}^S_1(\omega, \mathrm{co}\,\Omega^2(M)_*) = 0$. Then we have the following diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & \omega \otimes_S \mathrm{co}\,\Omega^1(M)_* & \longrightarrow & \omega \otimes_S I^1(M)_* \\ & & \downarrow{\scriptstyle \theta_{\mathrm{co}\,\Omega^1(M)}} & & \downarrow{\scriptstyle \theta_{I^1(M)}} \\ 0 & \longrightarrow & \mathrm{co}\,\Omega^1(M) & \longrightarrow & I^1(M) \end{array}$$

Because $\theta_{I^1(M)}$ is an isomorphism by [33, Lemma 2.5(2)], $\theta_{\mathrm{co}\,\Omega^1(M)}$ is a monomorphism. So $\mathrm{co}\,\Omega^1(M)$ is 2-ω-cotorsion-free by [33, Lemma 4.1(1)]. On the other hand, because $\mathrm{Tor}^S_1(\omega, \mathrm{co}\,\Omega^1(M)_*) = 0$ by the above argument, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega\otimes_S M_* & \longrightarrow & \omega\otimes_S I^0(M)_* & \longrightarrow & \omega\otimes_S \operatorname{co}\Omega^1(M)_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_{I^0(M)}} & & \downarrow{\scriptstyle \theta_{\operatorname{co}\Omega^1(M)}} & & \\ 0 & \longrightarrow & M & \longrightarrow & I^0(M) & \longrightarrow & \operatorname{co}\Omega^1(M) & \longrightarrow & 0 \end{array}$$

Because $\theta_{I^0(M)}$ is an isomorphism by [33, Lemma 2.5(2)], applying the snake lemma we have that θ_M is also an isomorphism and that M is 2-ω-cotorsion-free. So by [33, Corollary 3.8], there exists an exact sequence $0\to M_1\to W_0\to M\to 0$ in $\operatorname{Mod} R$ with $W_0\in\operatorname{Add}_R\omega$ and $\operatorname{Ext}^1_R(\omega,M_1)=0$. Thus, $M_1\in {}_R\omega^{\perp}$ since $M\in {}_R\omega^{\perp}$. Then by an argument similar to that above, we get an exact sequence $0\to M_2\to W_1\to M_1\to 0$ in $\operatorname{Mod} R$ with $W_1\in\operatorname{Add}_R\omega$ and $M_2\in {}_R\omega^{\perp}$. Continuing this procedure, we get a proper $\operatorname{Add}_R\omega$-resolution

$$\cdots\to W_n\to W_{n-1}\to\cdots\to W_0\to M\to 0$$

of M in $\operatorname{Mod} R$. Thus, $M\in c\mathcal{T}(R)$ by [33, Proposition 3.7]. □

The following result extends [34, Corollary 2.16].

COROLLARY 3.2

(1) *If* $\operatorname{pd}_R\omega<\infty$, *then* $\mathcal{B}_\omega(R)=c\mathcal{T}(R)$.
(2) *If* $\operatorname{pd}_{S^{op}}\omega<\infty$, *then* $\mathcal{B}_\omega(R)={}_R\omega^{\perp}$.

Proof
It is an immediate consequence of Proposition 3.1 and [33, Theorem 3.9]. □

We write $\operatorname{Ker}\operatorname{Ext}^{i\geq 1}_S(-,\omega^+)=\{N\in\operatorname{Mod} S\mid \operatorname{Ext}^{i\geq 1}_S(N,\omega^+)=0\}$ and $\mathcal{H}(\omega)=\operatorname{Adst}(\omega)\cap\operatorname{Ker}\operatorname{Ext}^{i\geq 1}_S(-,\omega^+)$, where $(-)^+=\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$ with $\mathbb{Z}$ the additive group of integers and $\mathbb{Q}$ the additive group of rational numbers. In the following result, we provide a viewpoint from Morita equivalence for $c\mathcal{T}(R)$.

THEOREM 3.3

There exists an equivalence of categories

$$c\mathcal{T}(R)\ \underset{\omega\otimes_S -}{\overset{(-)_*}{\rightleftarrows}}\ \mathcal{H}(\omega).$$

Proof
According to [36, Section 2.4], the functors $(-)_*$ and $\omega\otimes_S-$ induce an equivalence between the category of all 2-ω-cotorsion-free modules and $\operatorname{Adst}(\omega)$. So it suffices to show that $(-)_*$ (resp., $\omega\otimes_S-$) maps $c\mathcal{T}(R)$ (resp., $\mathcal{H}(\omega)$) to $\mathcal{H}(\omega)$ (resp., $c\mathcal{T}(R)$).

Let $M \in \mathrm{c}\mathcal{T}(R)$. Then by [36, 2.4], we have $M_* \in \mathrm{Adst}(\omega)$. By [33, Proposition 3.7] there exists a proper $\mathrm{Add}_R\,\omega$-resolution

$$\cdots \to W_n \to W_{n-1} \to \cdots \to W_0 \to M \to 0 \tag{3.1}$$

of M in $\mathrm{Mod}\,R$. Thus, we get an exact sequence

$$\cdots \to W_{n*} \to W_{n-1*} \to \cdots \to W_{0*} \to M_* \to 0$$

in $\mathrm{Mod}\,S$. Applying $\omega \otimes_S -$ to this exact sequence gives back the sequence (3.1). Then we easily obtain that $\mathrm{Tor}^S_{i\geq 1}(\omega, M_*) = 0$ because $\mathrm{Tor}^S_{i\geq 1}(\omega, W_{j*}) = 0$ for any $j \geq 0$ by [33, Lemma 2.5(2)]. It follows from the mixed isomorphism theorem that $\mathrm{Ext}^{i\geq 1}_S(M_*, \omega^+) \cong [\mathrm{Tor}^S_{i\geq 1}(\omega, M_*)]^+ = 0$. So $M_* \in \mathrm{Ker}\,\mathrm{Ext}^{i\geq 1}_S(-, \omega^+)$ and $M_* \in \mathcal{H}(\omega)$.

Conversely, let $N \in \mathcal{H}(\omega)$. Then $(\omega \otimes_S N)_* \cong N$. It follows from the mixed isomorphism theorem that $[\mathrm{Tor}^S_{i\geq 1}(\omega, (\omega \otimes_S N)_*)]^+ \cong [\mathrm{Tor}^S_{i\geq 1}(\omega, N)]^+ \cong \mathrm{Ext}^{i\geq 1}_S(N, \omega^+) = 0$ and $\mathrm{Tor}^S_{i\geq 1}(\omega, (\omega \otimes_S N)_*) = 0$. In addition, $\omega \otimes_S N$ is 2-ω-cotorsion-free by [36, 2.4]. Thus, we conclude that $\omega \otimes_S N$ is ∞-ω-cotorsion-free by [33, Corollary 3.4]. □

Following [21], set

$$\mathcal{F}_\omega(R) = \{\omega \otimes_S F \mid F \text{ is flat in } \mathrm{Mod}\,S\},$$

$$\mathcal{P}_\omega(R) = \{\omega \otimes_S P \mid P \text{ is projective in } \mathrm{Mod}\,S\},$$

$$\mathcal{I}_\omega(S) = \big\{\mathrm{Hom}_R(\omega, I) \,\big|\, I \text{ is injective in } \mathrm{Mod}\,R\big\}.$$

The modules in $\mathcal{F}_\omega(R)$, $\mathcal{P}_\omega(R)$, and $\mathcal{I}_\omega(S)$ are called *ω-flat*, *ω-projective*, and *ω-injective*, respectively. For a module $M \in \mathrm{Mod}\,R$, we use $\lim_M(R)$ to denote the subcategory of $\mathrm{Mod}\,R$ consisting of all modules isomorphic to direct summands of a direct limit of a family of modules in which each is a finite direct sum of copies of M.

PROPOSITION 3.4

(1) $\mathcal{F}_\omega(R) = \lim_\omega(R)$.
(2) $\mathcal{P}_\omega(R) = \mathrm{Add}_R\,\omega$.
(3) $\mathcal{I}_\omega(S) = \mathrm{Prod}_S\,E_*$ *with* ${}_RE$ *an injective cogenerator for* $\mathrm{Mod}\,R$.

Proof
(1) It is well known that a module in $\mathrm{Mod}\,S$ is flat if and only if it is in $\lim_S(S)$. Because the functor $\omega \otimes_S -$ commutes with direct limits, we easily obtain $\mathcal{F}_\omega(R) \subseteq \lim_\omega(R)$. Now let $M \in \lim_\omega(R)$. Then $M \in \mathcal{B}_\omega(R)$ by [21, Proposition 4.2(a)]. Because ${}_R\omega$ admits a degreewise finite R-projective resolution, $\mathrm{Hom}_R(\omega, -)$ commutes with direct limits. So $\mathrm{Hom}_R(\omega, M)$ is in $\lim_S(S)$, that is, $\mathrm{Hom}_R(\omega, M)$ is a flat left S-module. Then by [21, Lemma 5.1(a)], we have $M \in \mathcal{F}_\omega(R)$, and thus, $\lim_\omega(R) \subseteq \mathcal{F}_\omega(R)$.

For (2) and (3), see [28, Proposition 2.4]. □

The following result establishes the relation between the relative homological dimensions of a module M and the corresponding standard homological dimensions of M_*. It extends [31, Theorem 2.11].

THEOREM 3.5

(1) $\operatorname{fd}_S M_* \leq \mathcal{F}_\omega(R)\text{-}\operatorname{pd}_R M$ *for any* $M \in \operatorname{Mod} R$*; the equality holds if* $M \in \mathrm{c}\mathcal{T}(R)$.

(2) $\operatorname{pd}_S M_* \leq \mathcal{P}_\omega(R)\text{-}\operatorname{pd}_R M$ *for any* $M \in \operatorname{Mod} R$*; the equality holds if* $M \in \mathrm{c}\mathcal{T}(R)$.

(3) $\operatorname{id}_R \omega \otimes_S N \leq \mathcal{I}_\omega(S)\text{-}\operatorname{id}_S N$ *for any* $N \in \operatorname{Mod} S$*; the equality holds if* $N \in \mathcal{A}_\omega(S)$.

Proof

(1) Let $M \in \operatorname{Mod} R$ with $\mathcal{F}_\omega(R)\text{-}\operatorname{pd}_R M = n < \infty$. Then there exists an exact sequence

$$0 \to L_n \to \cdots \to L_1 \to L_0 \to M \to 0 \tag{3.2}$$

in $\operatorname{Mod} R$ with all L_i's in $\lim_\omega(R)$ by Proposition 3.4(1). Because ${}_R\omega$ admits a degreewise finite R-projective resolution, $\operatorname{Ext}^i_R(\omega, -)$ commutes with direct limits for any $i \geq 0$. Also note that $({}_R\omega)_* \cong S$ and $\omega \in {}_R\omega^\perp$, so we have that L_{i*} is in $\lim_S(S)$ (i.e., L_{i*} is left S-flat) and $L_i \in {}_R\omega^\perp$ for any $0 \leq i \leq n$. Applying the functor $\operatorname{Hom}_R(\omega, -)$ to the exact sequence (3.2), we obtain the exact sequence

$$0 \to L_{n*} \to \cdots \to L_{1*} \to L_{0*} \to M_* \to 0$$

in $\operatorname{Mod} S$, and so $\operatorname{fd}_S M_* \leq n$.

(2) Let $M \in \operatorname{Mod} R$ with $\mathcal{P}_\omega(R)\text{-}\operatorname{pd}_R M = n < \infty$. Then there exists an exact sequence

$$0 \to \omega_n \to \cdots \to \omega_1 \to \omega_0 \to M \to 0 \tag{3.3}$$

in $\operatorname{Mod} R$ with all $\omega_i \in \operatorname{Add}_R \omega$ by Proposition 3.4(2). Because all the ω_{i*}'s are projective left S-modules and $\operatorname{Add}_R \omega \subseteq {}_R\omega^\perp$ by [33, Lemma 2.5(1)], applying the functor $(-)_*$ to the exact sequence (3.3), we get the exact sequence

$$0 \to \omega_{n*} \to \cdots \to \omega_{1*} \to \omega_{0*} \to M_* \to 0$$

in $\operatorname{Mod} S$, and so $\operatorname{pd}_S M_* \leq n$.

Now suppose $M \in \mathrm{c}\mathcal{T}(R)$. Then $\omega \otimes_S M_* \cong M$. By [33, Corollary 3.4(3)], we have $\operatorname{Tor}^S_{i\geq 1}(\omega, M_*) = 0$. We will prove that the equalities in (1) and (2) hold.

(1) Assume $\operatorname{fd}_S M_* = n < \infty$. Then there exists an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M_* \to 0$$

in $\operatorname{Mod} S$ with all F_i's flat. Applying the functor $\omega \otimes_S -$ to it, we get an exact sequence

$$0 \to \omega \otimes_S F_n \to \cdots \to \omega \otimes_S F_1 \to \omega \otimes_S F_0 \to \omega \otimes_S M_* (\cong M) \to 0$$

in $\operatorname{Mod} R$ with all $\omega \otimes_S F_i$'s in $\mathcal{F}_\omega(R)$, so we have $\mathcal{F}_\omega(R)\text{-}\operatorname{pd}_R M \leq n$.

(2) Assume $\mathrm{pd}_S M_* = n < \infty$. Then there exists an exact sequence

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M_* \to 0$$

in $\mathrm{Mod}\, S$ with all P_i's projective. Applying the functor $\omega \otimes_S -$ to it, we get an exact sequence

$$0 \to \omega \otimes_S P_n \to \cdots \to \omega \otimes_S P_1 \to \omega \otimes_S P_0 \to \omega \otimes_S M_* (\cong M) \to 0$$

in $\mathrm{Mod}\, R$ with all $\omega \otimes_S P_i$'s in $\mathcal{P}_\omega(R)$, and so $\mathcal{P}_\omega(R)$-$\mathrm{pd}_R M \leq n$.

(3) Let $N \in \mathrm{Mod}\, S$ with $\mathcal{I}_\omega(S)$-$\mathrm{id}_S N = n < \infty$, and let ${}_RE$ be an injective cogenerator for $\mathrm{Mod}\, R$. Then there exists an exact sequence

$$0 \to N \to I^0 \to I^1 \to \cdots \to I^n \to 0 \tag{3.4}$$

in $\mathrm{Mod}\, S$ with all I^i's in $\mathrm{Prod}_S E_*$ by Proposition 3.4(3). Because ω_S admits a degreewise finite S-projective resolution, $\mathrm{Tor}_j^S(\omega, -)$ commutes with direct products for any $j \geq 0$. Then by [33, Lemma 2.5(2)], $\omega \otimes_S I^i (\in \mathrm{Prod}_R E)$ is injective in $\mathrm{Mod}\, R$ and $\mathrm{Tor}_{j\geq 1}^S(\omega, I^i) = 0$ for any $0 \leq i \leq n$. Applying the functor $\omega \otimes_S -$ to the exact sequence (3.4), we obtain the exact sequence

$$0 \to \omega \otimes_S N \to \omega \otimes_S I^0 \to \omega \otimes_S I^1 \to \cdots \to \omega \otimes_S I^n \to 0$$

in $\mathrm{Mod}\, R$, and so $\mathrm{id}_R\, \omega \otimes_S N \leq n$.

Now suppose $N \in \mathcal{A}_\omega(S)$. Then $N \cong (\omega \otimes_S N)_*$ and $\omega \otimes_S N \in {}_R\omega^\perp$. If $\mathrm{id}_R\, \omega \otimes_S N = n < \infty$, then there exists an exact sequence

$$0 \to \omega \otimes_S N \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

in $\mathrm{Mod}\, R$ with all E^i's injective. Applying the functor $\mathrm{Hom}_R(\omega, -)$ to it, we get an exact sequence

$$0 \to (\omega \otimes_S N)_* (\cong N) \to E^0_* \to {E^1}_* \to \cdots \to {E^n}_* \to 0$$

in $\mathrm{Mod}\, S$ with all ${E^i}_* \in \mathcal{I}_\omega(S)$, and so $\mathcal{I}_\omega(S)$-$\mathrm{id}_S N \leq n$. □

For a subclass $\mathcal{X}$ of $\mathrm{Mod}\, R$, we write $\mathrm{id}_R \mathcal{X} := \sup\{\mathrm{id}_R X \mid X \in \mathcal{X}\}$. As an application of Theorem 3.5, we get the following result.

PROPOSITION 3.6

(1) $\sup\{\mathcal{F}_\omega(R)\text{-}\mathrm{pd}_R M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{F}_\omega(R)\text{-}\mathrm{pd}_R M < \infty\} \leq \mathrm{id}_R \mathcal{F}_\omega(R)$.
(2) $\sup\{\mathcal{P}_\omega(R)\text{-}\mathrm{pd}_R M \mid M \in \mathrm{c}\mathcal{T}(R)$ *with* $\mathcal{P}_\omega(R)\text{-}\mathrm{pd}_R M < \infty\} \leq \mathrm{id}_R \mathcal{P}_\omega(R)$.

Proof
(1) Let $\mathrm{id}_R \mathcal{F}_\omega(R) = n < \infty$, and let $M \in \mathrm{c}\mathcal{T}(R)$ with $\mathcal{F}_\omega(R)$-$\mathrm{pd}_R M = m < \infty$. By Theorem 3.5(1), $\mathrm{fd}_S M_* = m$ and there exists an exact sequence

$$0 \to F_m \to Q_{m-1} \to \cdots \to Q_1 \to Q_0 \to M_* \to 0 \tag{3.5}$$

in $\mathrm{Mod}\, S$ with F_m flat and all Q_i's projective. Because $\omega \otimes_S M_* \cong M$ and $\mathrm{Tor}_{j\geq 1}^S(\omega, M_*) = 0$ by [33, Corollary 3.4(3)], applying the functor $\omega \otimes_S -$ to the exact sequence (3.5), we get the exact sequence

$$0 \to \omega \otimes_S F_m \to \omega \otimes_S Q_{m-1} \to \cdots \to \omega \otimes_S Q_1 \to \omega \otimes_S Q_0 \to \omega \otimes_S M_*(\cong M) \to 0$$

(3.6)

in $\operatorname{Mod} R$ with $\omega \otimes_S F_m$ in $\mathcal{F}_\omega(R)$ (which equals $\lim_\omega(R)$ by Proposition 3.4(1)) and all $\omega \otimes_S Q_i$'s in $\mathcal{P}_\omega(R)$ (which equals $\operatorname{Add}_R \omega$ by Proposition 3.4(2)). Notice that ${}_R\omega$ admits a degreewise finite R-projective resolution and $\omega \in {}_R\omega^\perp$, so $\operatorname{Ext}_R^{j\geq 1}(\omega \otimes_S Q_i, \omega \otimes_S F_m) = 0$ for any $0 \leq i \leq m-1$.

Suppose $m > n$. Because $\operatorname{id}_R \omega \otimes_S F_m \leq n$, it follows from the exact sequence (3.6) that $\operatorname{Ext}_R^1(K, \omega \otimes_S F_m) \cong \operatorname{Ext}_R^m(M, \omega \otimes_S F_m) = 0$, where $K = \operatorname{Coker}(\omega \otimes_S F_m \to \omega \otimes_S Q_{m-1})$. Thus, the exact sequence $0 \to \omega \otimes_S F_m \to \omega \otimes_S Q_{m-1} \to K \to 0$ splits and $K \in \mathcal{P}_\omega(R)$ $(\subseteq \mathcal{F}_\omega(R))$. It induces that $\mathcal{F}_\omega(R)$-$\operatorname{pd}_R M \leq m-1$, which is a contradiction. Thus, we conclude that $m \leq n$.

(2) It is similar to the proof of (1), so we omit it. □

Note that ${}_RR_R$ is a semidualizing bimodule. Let R be a left Noetherian ring, and let ${}_R\omega_S = {}_RR_R$. Then we have the following facts:

(1) $\mathcal{F}_\omega(R)$ and $\mathcal{P}_\omega(R)$ are the subclasses of $\operatorname{Mod} R$ consisting of flat modules and projective modules, respectively, and $\mathcal{F}_\omega(R)$-$\operatorname{pd}_R M = \operatorname{fd}_R M$ and $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M = \operatorname{pd}_R M$ for any $M \in \operatorname{Mod} R$;

(2) $\operatorname{id}_R \mathcal{F}_\omega(R) = \operatorname{id}_R R$ and $\operatorname{id}_R \mathcal{P}_\omega(R) = \operatorname{id}_R R$ by [6, Theorem 1.1];

(3) $\mathrm{c}\mathcal{T}(R) = \operatorname{Mod} R$ by [33, Proposition 3.7].

So by Proposition 3.6, we immediately have the following result.

COROLLARY 3.7

For a left Noetherian ring R, we have

(1) $\sup\{\operatorname{fd}_R M \mid M \in \operatorname{Mod} R$ *with* $\operatorname{fd}_R M < \infty\} \leq \operatorname{id}_R R$, *and*

(2) $\sup\{\operatorname{pd}_R M \mid M \in \operatorname{Mod} R$ *with* $\operatorname{pd}_R M < \infty\} \leq \operatorname{id}_R R$ *(see [6, Proposition 4.3]).*

In the rest of this section, for a module $M \in \operatorname{Mod} R$, in the case in which $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M < \infty$, we establish the relation between $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M$ and some standard homological dimensions of related modules.

LEMMA 3.8

If $M \in \mathrm{c}\mathcal{T}(R)$ *and* $N \in {}_R\omega^\perp$, *then for any* $i \geq 0$, *we have an isomorphism of abelian groups*

$$\operatorname{Ext}_R^i(M, N) \cong \operatorname{Ext}_S^i(M_*, N_*).$$

Proof

We proceed by induction on i. Let $i = 0$. Since $M \in \mathrm{c}\mathcal{T}(R)$, $\omega \otimes_S M_* \cong M$. It follows from the adjoint isomorphism theorem that $\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_R(\omega \otimes_S M_*, N) \cong \operatorname{Hom}_S(M_*, N_*)$. Indeed, the isomorphism is natural in M and N.

Now suppose $i \geq 1$. The induction hypothesis implies that there exists a natural isomorphism

$$\operatorname{Ext}_R^j(L,H) \cong \operatorname{Ext}_S^j(L_*, H_*)$$

for any $L \in \mathrm{c}\mathcal{T}(R)$, $H \in {}_R\omega^{\perp}$, and $0 \leq j \leq i-1$. Because $N \in {}_R\omega^{\perp}$ by assumption, $\mathrm{co}\,\Omega^1(N) \in {}_R\omega^{\perp}$ and we have an exact sequence

$$0 \to N_* \to I^0(N)_* \to \mathrm{co}\,\Omega^1(N)_* \to 0.$$

Applying the functor $\operatorname{Hom}_S(M_*, -)$ to it yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{Ext}_R^{i-1}(M, I^0(N)) & \longrightarrow & \operatorname{Ext}_R^{i-1}(M, \mathrm{co}\,\Omega^1(N)) & \longrightarrow & \operatorname{Ext}_R^{i}(M, N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Ext}_S^{i-1}(M_*, I^0(N)_*) & \to & \operatorname{Ext}_S^{i-1}(M_*, \mathrm{co}\,\Omega^1(N)_*) & \to & \operatorname{Ext}_S^{i}(M_*, N_*) & \to & \operatorname{Ext}_S^{i}(M_*, I^0(N)_*)
\end{array}$$

By the induction hypothesis, the first two columns in the above diagram are natural isomorphisms. Since $M \in \mathrm{c}\mathcal{T}(R)$ by assumption, we have $\operatorname{Ext}_S^i(M_*, I^0(N)_*) \cong \operatorname{Hom}_R(\operatorname{Tor}_S^i(\omega, M_*), I^0(N)) = 0$ by the mixed isomorphism theorem and [33, Corollary 3.4(3)]. It follows that $\operatorname{Ext}_R^i(M,N) \cong \operatorname{Ext}_S^i(M_*, N_*)$ naturally. □

We also need the following criterion.

LEMMA 3.9

Let $M \in \operatorname{Mod} R$ admit a degreewise finite R-projective resolution. If $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M < \infty$, then $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M = \sup\{i \geq 0 \mid \operatorname{Ext}_R^i(M,\omega) \neq 0\}$.

Proof

Let $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M = n < \infty$, and let

$$0 \to \omega_n \to \cdots \to \omega_1 \to \omega_0 \to M \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all ω_i's in $\mathcal{P}_\omega(R)$ (which equals $\operatorname{Add}_R \omega$). It is easy to see that $\operatorname{Ext}_R^i(M,\omega) = 0$ for $i \geq n+1$. Put $M_{n-1} = \operatorname{Coker}(\omega_n \to \omega_{n-1})$.

If $\operatorname{Ext}_R^n(M,\omega) = 0$, then by [19, Lemma 3.1.6], we have that $\operatorname{Ext}_R^n(M,\omega_i) = 0$ and $\operatorname{Ext}_R^{\geq 1}(\omega_j, \omega_i) = 0$ for any $0 \leq i,j \leq n$. So $\operatorname{Ext}_R^1(M_{n-1}, \omega_n) \cong \operatorname{Ext}_R^n(M, \omega_n) = 0$ and the exact sequence

$$0 \to \omega_n \to \omega_{n-1} \to M_{n-1} \to 0$$

splits. It implies that $M_{n-1} \in \mathcal{P}_\omega(R)$ and $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M \leq n-1$, which is a contradiction. So we conclude that $\operatorname{Ext}_R^n(M,\omega) \neq 0$. □

Now we are in a position to give the following result.

PROPOSITION 3.10

Let $M \in \operatorname{Mod} R$ admit a degreewise finite R-projective resolution. If $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M < \infty$, then $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M \le \min\{\operatorname{id}_R \omega, \operatorname{id}_S S, \operatorname{pd}_R M, \operatorname{pd}_S M_*\}$.

Proof

Let $M \in \operatorname{Mod} R$ with $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M < \infty$. Then $M \in \mathrm{c}\mathcal{T}(R)$ by [33, Proposition 3.7]. So $\operatorname{Ext}^i_R(M,\omega) \cong \operatorname{Ext}^i_S(M_*,\omega_*) \cong \operatorname{Ext}^i_S(M_*,S)$ for any $i \ge 0$ by Lemma 3.8, and hence, $\sup\{i \ge 0 \mid \operatorname{Ext}^i_R(M,\omega) \ne 0\} \le \min\{\operatorname{id}_R \omega, \operatorname{id}_S S, \operatorname{pd}_R M, \operatorname{pd}_S M_*\}$. Now the assertion follows from Lemma 3.9. □

The following example shows that the finiteness of $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M$ is necessary for the conclusion of Proposition 3.10.

EXAMPLE 3.11

Let G be a finite group, and let k be a field such that the characteristic of k divides $|G|$. Take $R = S = \omega = kG$. By [4, Theorem 3.3 and Proposition 3.10], the group algebra kG is a nonsemisimple symmetric Artin algebra. Then $\operatorname{id}_R \omega = 0$ and there exists a kG-module M with $\mathcal{P}_\omega(R)$-$\operatorname{pd}_R M$ infinite.

4. The Bass injective dimension of modules

For a module M in $\operatorname{Mod} R$, we study in this section the properties of the *Bass injective dimension* $\mathcal{B}_\omega(R)$-$\operatorname{id}_R M$ of M. We begin with the following easy observation.

LEMMA 4.1

For any $M \in \operatorname{Mod} R$, if $\mathcal{B}_\omega(R)$-$\operatorname{id}_R M < \infty$ and $M \in {}_R\omega^{\perp}$, then $M \in \mathcal{B}_\omega(R)$.

Proof

It is easy to get the assertion by using induction on $\mathcal{B}_\omega(R)$-$\operatorname{id}_R M$. □

Now we give some criteria for computing $\mathcal{B}_\omega(R)$-$\operatorname{id}_R M$ in terms of the vanishing of Ext-functors and some special approximations of M.

THEOREM 4.2

Let $M \in \operatorname{Mod} R$ with $\mathcal{B}_\omega(R)$-$\operatorname{id}_R M < \infty$, and let $n \ge 0$. Then the following statements are equivalent.

(1) $\mathcal{B}_\omega(R)$-$\operatorname{id}_R M \le n$.
(2) $\operatorname{co}\Omega^m(M) \in \mathcal{B}_\omega(R)$ *for* $m \ge n$.
(3) $\operatorname{Ext}^{\ge n+1}_R(\omega, M) = 0$.
(4) *There exists an exact sequence*

$$0 \to M \to X^M \to W^M \to 0$$

in $\operatorname{Mod} R$ *such that* $X^M \in \mathcal{B}_\omega(R)$ *and* $\mathcal{P}_\omega(R)$-$\operatorname{id}_R W^M \le n-1$.

(5) *There exists an exact sequence*

$$0 \to X_M \to W_M \to M \to 0$$

in $\operatorname{Mod} R$ *such that* $X_M \in \mathcal{B}_\omega(R)$ *and* $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, W_M \leq n$.

Proof
We have that (1) ⇒ (2) follows from [21, Theorem 6.2] and [24, Theorem 4.8], (2) ⇒ (3) follows from the dimension shifting, and (4) ⇒ (1) follows from the fact that $\mathcal{P}_\omega(R) \subseteq \mathcal{B}_\omega(R)$.

(3) ⇒ (1) Let $M \in \operatorname{Mod} R$ with $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\, M < \infty$. Then $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R \operatorname{co}\Omega^n(M) < \infty$ by [21, Theorem 6.2] and [24, Theorem 4.8]. If $\operatorname{Ext}_R^{\geq n+1}(\omega, M) = 0$, then $\operatorname{co}\Omega^n(M) \in {}_R\omega^{\perp}$, and so $\operatorname{co}\Omega^n(M) \in \mathcal{B}_\omega(R)$ by Lemma 4.1. It follows that $\mathcal{B}_\omega(R)\text{-}\mathrm{id}_R\, M \leq n$.

(1) ⇒ (4) By [21, Theorem 6.2], $\mathcal{B}_\omega(R)$ is closed under extensions. By [33, Proposition 3.7], it is easy to see that $\mathcal{P}_\omega(R)$ (which equals $\operatorname{Add}_R \omega$) is a $\mathcal{P}_\omega(R)$-proper generator for $\mathcal{B}_\omega(R)$. Then the assertion follows from [24, Theorem 3.7].

(4) ⇒ (5) Assume that there exists an exact sequence

$$0 \to M \to X^M \to W^M \to 0$$

in $\operatorname{Mod} R$ such that $X^M \in \mathcal{B}_\omega(R)$ and $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R\, W^M \leq n-1$. By [33, Proposition 3.7], there exists an exact sequence

$$0 \to X' \to W_0 \to X^M \to 0$$

in $\operatorname{Mod} R$ with $W_0 \in \mathcal{P}_\omega(R)$ and $X' \in \mathcal{B}_\omega(R)$. Now consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X' & = & X' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & W_M & \to & W_0 & \to & W^M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & X^M & \to & W^M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then the leftmost column in the above diagram is the desired sequence.

(5) ⇒ (4) Assume that there exists an exact sequence

$$0 \to X_M \to W_M \to M \to 0$$

in $\operatorname{Mod} R$ such that $X_M \in \mathcal{B}_\omega(R)$ and $\mathcal{P}_\omega(R)\text{-id}_R W_M \leq n$. Then there exists an exact sequence

$$0 \to W_M \to W^0 \to W' \to 0$$

in $\operatorname{Mod} R$ with $W^0 \in \mathcal{P}_\omega(R)$ and $\mathcal{P}_\omega(R)\text{-id}_R W' \leq n-1$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_M & \longrightarrow & W_M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_M & \longrightarrow & W^0 & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W' & = & W' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It follows from [21, Theorem 6.2] and the exactness of the middle row in the above diagram that $X \in \mathcal{B}_\omega(R)$. So the rightmost column in the above diagram is the desired sequence. □

REMARK 4.3

The only place where the assumption that $\mathcal{B}_\omega(R)\text{-id}_R M < \infty$ in Theorem 4.2 is used is in showing $(3) \Rightarrow (1)$.

If the given semidualizing module ${}_R\omega_S$ is faithful, then a module in $\operatorname{Mod} R$ with finite Bass injective dimension is in $\mathcal{B}_\omega(R)$ by [21, Theorem 6.3]. However, this property does not hold true in general.

EXAMPLE 4.4

Let Λ be a finite-dimensional algebra over an algebraically closed field given by the quiver:

$$1\circ \longrightarrow \circ 2.$$

Put $\omega = I(1) \oplus I(2)$. Then ${}_\Lambda\omega_\Lambda$ is a semidualizing bimodule, but is nonfaithful since $\operatorname{Hom}_\Lambda(\omega, S(2)) = 0$. We have an exact sequence $0 \to S(2) \to I(2) \to I(1) \to 0$ in $\operatorname{Mod}\Lambda$. Both $I(1)$ and $I(2)$ are obviously in $\mathcal{B}_\omega(\Lambda)$. But $S(2)$ is not in $\mathcal{B}_\omega(\Lambda)$ because $S(2)$ is not 2-ω-cotorsion-free.

Motivated by [26, Definition 2.4 and Lemma 2.5], we introduce the following.

DEFINITION 4.5

A semidualizing bimodule ${}_R\omega_S$ is called *left* (resp., *right*) *semitilting* if $\mathrm{pd}_R\,\omega < \infty$ (resp., $\mathrm{pd}_{S^{op}}\,\omega < \infty$).

In the following, we will give an equivalent characterization of right semitilting bimodules in terms of the finiteness of the Bass injective dimension of ${}_RR$. We need the following two lemmas.

LEMMA 4.6

Let $M \in \mathrm{Mod}\,R$ with $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\,M \leq n(<\infty)$. If $K \in \mathrm{Mod}\,R$ is isomorphic to a direct summand of M, then $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\,K \leq n$.

Proof

Note that $\mathcal{P}_\omega(R) = \mathrm{Add}_R\,\omega$ by Proposition 3.4(2). It is clear that $\mathcal{P}_\omega(R) \subseteq {}^{\perp}\mathcal{P}_\omega(R)$. In addition, it is not difficult to verify that $\mathcal{P}_\omega(R)$ is $\mathcal{P}_\omega(R)$-coresolving in $\mathrm{Mod}\,R$ with $\mathcal{P}_\omega(R)$ a $\mathcal{P}_\omega(R)$-coproper cogenerator in the sense of [24]. Now the assertion follows from [24, Corollary 4.9]. □

We use $\mathrm{add}_R\,\omega$ to denote the subclass of $\mathrm{Mod}\,R$ consisting of direct summands of finite direct sums of copies of ω.

LEMMA 4.7

Let $M \in \mathrm{Mod}\,R$ be finitely generated, and let $n \geq 0$. If $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\,M \leq n$, then there exists an exact sequence

$$0 \to M \to \omega^0 \to \omega^1 \to \cdots \to \omega^n \to 0$$

in $\mathrm{Mod}\,R$ with all ω^i's in $\mathrm{add}_R\,\omega$.

Proof

Let $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\,M \leq n$, and let

$$0 \to M \xrightarrow{\alpha^0} D^0 \xrightarrow{\alpha^1} D^1 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^n} D^n \to 0 \tag{4.1}$$

be an exact sequence in $\mathrm{Mod}\,R$ with all D^i's in $\mathrm{Add}_R\,\omega$ (which equals $\mathcal{P}_\omega(R)$). Put $K^i = \mathrm{Im}\,\alpha^i$ for any $0 \leq i \leq n$. There exists a module $G^0 \in \mathrm{Add}_R\,\omega$ such that $D^0 \oplus G^0$ is a direct sum of copies of ω, so we get a $\mathrm{Hom}_R(-, \mathcal{P}_\omega(R))$-exact exact sequence

$$0 \to M \xrightarrow{\beta^0} D^0 \oplus G^0 \xrightarrow{\beta^1} D^1 \oplus G^0 \xrightarrow{\beta^2} D^2 \xrightarrow{\alpha^3} \cdots \xrightarrow{\alpha^n} D^n \to 0,$$

where $\beta^0 = \binom{\alpha^0}{0}$, $\beta^1 = \begin{pmatrix} \alpha^1 & 0 \\ 0 & 1_{G^0} \end{pmatrix}$, and $\beta^2 = (\alpha^2, 0)$. Then $\mathrm{Im}\,\beta^1 = K^1 \oplus G^0$ and $\mathrm{Im}\,\beta^2 = K^2$. Because M is finitely generated by assumption, there exist $\omega^0 \in \mathrm{add}_R\,\omega$ and $H^0 \in \mathrm{Add}_R\,\omega$ such that $D^0 \oplus G^0 = \omega^0 \oplus H^0$ and $\mathrm{Im}\,\alpha^0 \subseteq \omega^0$. So we get an exact sequence

$$(4.2) \qquad 0 \to M \to \omega^0 \to L^0 \to 0$$

in $\operatorname{Mod} R$ with $L^0 \oplus H^0 = \operatorname{Im} \beta^1$.

Consider the following pushout diagram with the middle row $\operatorname{Hom}_R(-, \mathcal{P}_\omega(R))$-exact exact and the leftmost column splitting:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^0 & = & H^0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Im} \beta^1 & \longrightarrow & D^1 \oplus G^0 & \longrightarrow & K^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L^0 & \longrightarrow & X^1 & \longrightarrow & K^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then the middle column in the above diagram is $\operatorname{Hom}_R(-, \mathcal{P}_\omega(R))$-exact exact. From the proof of Lemma 4.6, we know that $\operatorname{Add}_R \omega$ (which is equal to $\mathcal{P}_\omega(R)$) is $\mathcal{P}_\omega(R)$-coresolving in $\operatorname{Mod} R$. So $X^1 \in \operatorname{Add}_R \omega$. Combining the exact sequences (4.1) and (4.2) with the bottom row in the above diagram, we get an exact sequence

$$0 \to M \to \omega^0 \to X^1 \to D^2 \xrightarrow{\alpha^3} \cdots \xrightarrow{\alpha^n} D^n \to 0$$

in $\operatorname{Mod} R$ with $\omega^0 \in \operatorname{add}_R \omega$ and $X^1 \in \operatorname{Add}_R \omega$. Repeating the above argument with $\operatorname{Im}(\omega^0 \to X^1)$ replacing M, we get an exact sequence

$$0 \to M \to \omega^0 \to \omega^1 \to X^2 \to D^3 \xrightarrow{\alpha^4} \cdots \xrightarrow{\alpha^n} D^n \to 0$$

in $\operatorname{Mod} R$ with $\omega^0, \omega^1 \in \operatorname{add}_R \omega$ and $X^2 \in \operatorname{Add}_R \omega$. Continuing this procedure, we finally get an exact sequence

$$0 \to M \to \omega^0 \to \omega^1 \to \cdots \to \omega^n \to 0$$

in $\operatorname{Mod} R$ with all ω^i's in $\operatorname{add}_R \omega$. □

We are now in a position to prove the following result.

THEOREM 4.8

(1) *If ${}_R\omega_S$ is right semitilting, then* $\mathcal{B}_\omega(R) = {}_R\omega^{\perp}$.

(2) *If S is a left coherent ring, then ${}_R\omega_S$ is right semitilting with* $\operatorname{pd}_{S^{op}} \omega \leq n$ *if and only if* $\mathcal{B}_\omega(R)$-$\operatorname{id}_R R \leq n$.

Proof

(1) It follows from Corollary 3.2 and [33, Theorem 3.9].

(2) It is easy to see that $\mathcal{B}_\omega(R)$-$\mathrm{id}_R R \leq \mathcal{P}_\omega(R)$-$\mathrm{id}_R R = \mathrm{pd}_{S^{op}} \omega$. Now the necessity is clear. Conversely, if $\mathcal{B}_\omega(R)$-$\mathrm{id}_R R = n < \infty$, then by Theorem 4.2, there exists a split exact sequence

$$0 \to X \to W \to R \to 0$$

in $\mathrm{Mod}\, R$ such that $X \in \mathcal{B}_\omega(R)$ and $\mathcal{P}_\omega(R)$-$\mathrm{id}_R W \leq n$. So $W \cong X \oplus R$ and $\mathcal{P}_\omega(R)$-$\mathrm{id}_R R \leq n$ by Lemma 4.6. It follows from Lemma 4.7 that there exists an exact sequence

$$0 \to R \to \omega^0 \to \omega^1 \to \cdots \to \omega^n \to 0$$

in $\mathrm{Mod}\, R$ with all ω^i's in $\mathrm{add}_R\, \omega$. Applying the functor $\mathrm{Hom}_R(-,\omega)$ to it, we get the exact sequence

$$0 \to \mathrm{Hom}_R(\omega^n,\omega) \to \cdots \to \mathrm{Hom}_R(\omega^1,\omega) \to \mathrm{Hom}_R(\omega^0,\omega) \to \omega \to 0$$

in $\mathrm{Mod}\, S^{op}$ with all $\mathrm{Hom}_R(\omega^i,\omega)$'s projective. So ${}_R\omega_S$ is right semitilting with $\mathrm{pd}_{S^{op}}\, \omega \leq n$. □

Compare the following result with Lemma 3.9.

COROLLARY 4.9

If ${}_R\omega_S$ is left and right semitilting, then for every $M \in \mathrm{Mod}\, R$, $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M = \sup\{i \geq 0 \mid \mathrm{Ext}^i_R(\omega, M) \neq 0\} < \infty$.

Proof

Let ${}_R\omega_S$ be left and right semitilting. Then $\mathrm{pd}_R\, \omega < \infty$ and $\mathrm{pd}_{S^{op}}\, \omega < \infty$. Put $\sup\{i \geq 0 \mid \mathrm{Ext}^i_R(\omega, M) \neq 0\} = n$. Then $n < \infty$. It is easy to see that ${}_R\omega^\perp$-$\mathrm{id}_R M \geq n$. So $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M \geq n$ by Theorem 4.8(1).

We will use induction on n to prove $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M \leq n$. If $n = 0$, then $M \in {}_R\omega^\perp$. It follows from Theorem 4.8(1) that $M \in \mathcal{B}_\omega(R)$. Now suppose $n \geq 1$. Then $\sup\{i \geq 0 \mid \mathrm{Ext}^i_R(\omega, \mathrm{co}\,\Omega^1(M)) \neq 0\} = n - 1$. So $\mathcal{B}_\omega(R)$-$\mathrm{id}_R\, \mathrm{co}\,\Omega^1(M) = n - 1$ by the induction hypothesis, and hence, $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M \leq n$. □

5. The Bass injective dimension of complexes

In this section, we extend the Bass injective dimension of modules to that of complexes in derived categories. A *cochain complex* $M^\bullet$ is a sequence of modules and morphisms in $\mathrm{Mod}\, R$ of the form

$$\cdots \to M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \to \cdots$$

such that $d^n d^{n-1} = 0$ for any $n \in \mathbb{Z}$, and the *shifted complex* $M^\bullet[m]$ is the complex with $M^\bullet[m]^n = M^{m+n}$ and $d^n_{M^\bullet[m]} = (-1)^m d^n_{m+n}$. Any $M \in \mathrm{Mod}\, R$ can be considered as a complex having M in its 0th spot and 0 in its other spots. We use $\mathbf{C}(R)$ and $\mathbf{D}^b(R)$ to denote the category of cochain complexes and the derived category of complexes with bounded finite homologies of $\mathrm{Mod}\, R$, respectively. According to [10, Appendix], the *supremum*, the *infimum*, and the *amplitude* of a complex $M^\bullet$ are defined as follows:

$$\sup M^\bullet = \sup\{n \in \mathbb{Z} \mid H^n(M^\bullet) \neq 0\},$$
$$\inf M^\bullet = \inf\{n \in \mathbb{Z} \mid H^n(M^\bullet) \neq 0\},$$
$$\operatorname{amp} M^\bullet = \sup M^\bullet - \inf M^\bullet.$$

The Auslander category with respect to a dualizing complex was defined in [12]. Dually we define the Bass class of complexes with respect to ω as follows.

DEFINITION 5.1
A full subcategory $\mathcal{B}^\bullet_\omega(R)$ of $\mathbf{D}^b(R)$ consisting of complexes $M^\bullet$ is called the *Bass class* with respect to ω if the following conditions are satisfied:

(1) $\mathbf{R}\operatorname{Hom}_R(\omega, M^\bullet) \in \mathbf{D}^b(R)$;
(2) $\omega \otimes^{\mathbf{L}}_S \mathbf{R}\operatorname{Hom}_R(\omega, M^\bullet) \to M^\bullet$ is an isomorphism in $\mathbf{D}^b(R)$.

Let $M^\bullet \in \mathbf{C}(R)$ and $n \in \mathbb{Z}$. The *hard left-truncation* $\sqsubset^n M^\bullet$ of $M^\bullet$ at n is given by

$$\sqsubset^n M^\bullet := \cdots \to 0 \to 0 \to M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} M^{n+2} \to \cdots.$$

Let $M^\bullet \in \mathbf{D}^b(R)$ with $H(M^\bullet) \neq 0$, and let $\inf M^\bullet = i$. Taking an injective resolution $I^\bullet$ of $M^\bullet$, we define the *injective complex* $vI^\bullet = (\sqsubset^{i+1} I^\bullet)[1]$, which is unique up to an injective summand in degree i. In general, we have that $H^t(vI^\bullet) \cong H^t(I^\bullet[1])$ if $t \geq i+1$. In particular, when $M^\bullet$ is a module M, $vI^\bullet$ is isomorphic to $\operatorname{co}\Omega^1(M)$ in $\mathbf{D}^b(R)$.

REMARK 5.2
(1) Let $M^\bullet \in \mathbf{D}^b(R)$. We see from the definition of $vI^\bullet$ that there exists a distinguished triangle in $\mathbf{D}^b(R)$ of the form

$$vI^\bullet[-1] \to M^\bullet \to I^i[-i] \to vI^\bullet.$$

(2) It is routine to check that $\mathcal{B}^\bullet_\omega(R)$ forms a triangulated subcategory of $\mathbf{D}^b(R)$. Thus, for an injective complex $I^\bullet$, $I^\bullet \in \mathcal{B}^\bullet_\omega(R)$ if and only if $vI^\bullet \in \mathcal{B}^\bullet_\omega(R)$.

LEMMA 5.3
Let $M \in \operatorname{Mod} R$. Then the following statements are equivalent:

(1) $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M < \infty$;
(2) $M \in \mathcal{B}^\bullet_\omega(R)$.

Proof
(1) $\Rightarrow$ (2) Let $\mathcal{B}_\omega(R)$-$\mathrm{id}_R M < \infty$, and let

$$0 \to M \to Y^0 \to Y^1 \to \cdots \to Y^n \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all Y^i's in $\mathcal{B}_\omega(R)$. Then by Remark 5.2(2) and [22, p. 41, Corollary 7.22], we have $M \in \mathcal{B}^\bullet_\omega(R)$.

(2) $\Rightarrow$ (1) Let $M \in \mathcal{B}_\omega^\bullet(R)$, and let $I^\bullet$ be an injective resolution of M. Then $I^\bullet \in \mathcal{B}_\omega^\bullet(R)$ and $\mathbf{R}\operatorname{Hom}_R(\omega, M) \in \mathbf{D}^b(R)$. Put $s = \sup \mathbf{R}\operatorname{Hom}_R(\omega, M)$. Because $H^i(\mathbf{R}\operatorname{Hom}_R(\omega, v^s I^\bullet)) \cong H^{i+s}(\mathbf{R}\operatorname{Hom}_R(\omega, I^\bullet)) = 0$ for any $i \geq 1$, it implies that $\operatorname{co}\Omega^s(M) \in {}_R\omega^\perp$. By Remark 5.2(2) we have that $v^s I^\bullet \cong \operatorname{co}\Omega^s(M)$ and $v^s I^\bullet \in \mathcal{B}_\omega^\bullet(R)$, so $\operatorname{co}\Omega^s(M) \in \mathcal{B}_\omega^\bullet(R)$, and hence, $\omega \otimes_S^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(\omega, \operatorname{co}\Omega^s(M)) \to \operatorname{co}\Omega^s(M)$ is an isomorphism in $\mathbf{D}^b(R)$. Equivalently, we have $\omega \otimes_S \operatorname{co}\Omega^s(M)_* \cong \operatorname{co}\Omega^s(M)$ and $\operatorname{Tor}_{i\geq 1}^S(\omega, \operatorname{co}\Omega^s(M)_*) = 0$. It follows that $\operatorname{co}\Omega^s(M) \in \mathcal{B}_\omega(R)$ and $\mathcal{B}_\omega(R)\text{-id}_R M \leq s$. □

We define the Bass injective dimension of complexes in $\mathbf{D}^b(R)$ as follows.

DEFINITION 5.4

Let $M^\bullet$ be a complex in $\mathbf{D}^b(R)$. We define the *Bass injective dimension* of $M^\bullet$ as

$$\mathcal{B}_\omega^\bullet(R)\text{-id}\, M^\bullet := \begin{cases} \sup \mathbf{R}\operatorname{Hom}_R(\omega, M^\bullet) & \text{if } M^\bullet \in \mathcal{B}_\omega^\bullet(R), \\ +\infty & \text{if } M^\bullet \notin \mathcal{B}_\omega^\bullet(R). \end{cases}$$

In the following result, we give an equivalent characterization when the Bass injective dimension of complexes is finite.

THEOREM 5.5

Let $M^\bullet$ be a complex in $\mathbf{D}^b(R)$. Then the following statements are equivalent.

(1) $\mathcal{B}_\omega^\bullet(R)\text{-id}\, M^\bullet < \infty$.

(2) *There exists an isomorphism $M^\bullet \to Y^\bullet$ in $\mathbf{D}^b(R)$ with $Y^\bullet$ a bounded complex consisting of modules in $\mathcal{B}_\omega(R)$.*

Proof

(2) $\Rightarrow$ (1) The assertion follows from the fact that a complex $Y^\bullet$ of finite length consisting of modules in $\mathcal{B}_\omega(R)$ is in $\mathcal{B}_\omega^\bullet(R)$.

(1) $\Rightarrow$ (2) Let $\mathcal{B}_\omega^\bullet(R)\text{-id}\, M^\bullet < \infty$. Then $M^\bullet \in \mathcal{B}_\omega^\bullet(R)$. We will proceed by induction on $\operatorname{amp} M^\bullet$. If $\operatorname{amp} M^\bullet = 0$, then there exists $T \in \operatorname{Mod} R$ such that $M^\bullet \cong T[-s]$, where $s = \sup M^\bullet$. Since $\mathcal{B}_\omega(R)\text{-id}_R T < \infty$ by Lemma 5.3, we have a quasi-isomorphism $T \to Y^\bullet$ with

$$Y^\bullet := \cdots \to 0 \to Y^0 \to Y^1 \to \cdots \to Y^n \to 0 \to \cdots$$

a bounded complex and all Y^i's in $\mathcal{B}_\omega(R)$. Then the complex $Y^\bullet[-s]$ is the desired complex.

Now suppose $\operatorname{amp} M^\bullet \geq 1$. By Remark 5.2(1), there exists a distinguished triangle

$$vI^\bullet[-1] \to M^\bullet \to I^i[-i] \xrightarrow{\alpha} vI^\bullet$$

in $\mathbf{D}^b(R)$. Since $\operatorname{amp} vI^\bullet < \operatorname{amp} M^\bullet$, by the induction hypothesis, there exists an isomorphism $\beta : vI^\bullet \to Y_1^\bullet$ in $\mathbf{D}^b(R)$ with $Y_1^\bullet$ a bounded complex consisting of modules in $\mathcal{B}_\omega(R)$. Thus, we get another triangle

$$vI^\bullet[-1] \to M^\bullet \to I^i[-i] \xrightarrow{\beta\alpha} Y_1^\bullet$$

in $\mathbf{D}^b(R)$. Furthermore, we have a triangle

$$I^i[-i] \stackrel{\beta\alpha}{\to} Y_1^\bullet \to M^\bullet[1] \to I^i[-i+1]$$

in $\mathbf{D}^b(R)$. Let $Y_2^\bullet$ be the mapping cone of $\beta\alpha$. Then there exists an isomorphism $M^\bullet[1] \to Y_2^\bullet$ in $\mathbf{D}^b(R)$. Put $Y^\bullet = Y_2^\bullet[-1]$. Then $Y^\bullet$ has finite length and all spots in $Y^\bullet$ are in $\mathcal{B}_\omega(R)$, and so $Y^\bullet$ is the desired complex. □

Let Λ be an Artin R-algebra over a commutative Artin ring R. We denote by D the ordinary Matlis duality, that is, $D(-) := \mathrm{Hom}_R(-, E^0(R/J(R)))$, where $J(R)$ is the Jacobson radical of R and $E^0(R/J(R))$ is the injective envelope of $R/J(R)$. It is easy to verify that (Λ,Λ)-bimodule $D(\Lambda)$ is semidualizing. Recall that Λ is called *Gorenstein* if $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda < \infty$. As an application of Theorem 5.5, we get the following result.

COROLLARY 5.6

Let Λ be an Artin algebra. Then the following statements are equivalent for any $n \geq 0$.

(1) *Λ is Gorenstein with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda \leq n$.*

(2) *For any simple module $T \in \mathrm{Mod}\,\Lambda$, $\mathcal{B}^\bullet_{D(\Lambda)}(\Lambda)$-$\mathrm{id}_\Lambda T \leq n$.*

(3) *For any simple module $T \in \mathrm{Mod}\,\Lambda$, there exists a quasi-isomorphism $T \to Y^\bullet$ with $Y^\bullet$ a bounded complex of length at most $n+1$ consisting of modules in $\mathcal{B}_{D(\Lambda)}(\Lambda)$.*

(4) *For any simple module $T \in \mathrm{Mod}\,\Lambda$, there exists an exact sequence*

$$0 \to T \to X^T \to W^T \to 0$$

in $\mathrm{Mod}\,\Lambda$ such that $X^T \in \mathcal{B}_{D(\Lambda)}(\Lambda)$ and $\mathrm{id}_\Lambda W^T \leq n-1$.

(5) *For any simple module $T \in \mathrm{Mod}\,\Lambda$, there exists an exact sequence*

$$0 \to X_T \to W_T \to T \to 0$$

in $\mathrm{Mod}\,\Lambda$ such that $X^T \in \mathcal{B}_{D(\Lambda)}(\Lambda)$ and $\mathrm{id}_\Lambda W_T \leq n$.

Proof

By Theorem 4.2, we have (2) ⇔ (4) ⇔ (5). By Theorem 5.5, we have (2) ⇔ (3).

(1) ⇒ (2) Let $T \in \mathrm{Mod}\,\Lambda$ be simple. Since Λ is Gorenstein with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda \leq n$, it follows from [15, Theorem 12.3.1] that $\mathrm{co}\,\Omega^n(T)$ is Gorenstein injective. Then $\mathrm{co}\,\Omega^n(T) \in \mathcal{B}_{D(\Lambda)}(\Lambda)$ by [33, Corollary 5.2 and Theorem 3.9]. Now the assertion follows from Lemma 5.3.

(4) ⇒ (1) Let $T \in \mathrm{Mod}\,\Lambda$ be simple. Then by (4) and [33, Theorem 3.9 and Corollary 4.2], $\mathcal{GI}(\Lambda)$-$\mathrm{id}_\Lambda T \leq n$. So $\sup\{\mathcal{GP}(\Lambda)\text{-}\mathrm{pd}_\Lambda M \mid M \in \mathrm{Mod}\,\Lambda\} = \sup\{\mathcal{GI}(\Lambda)\text{-}\mathrm{id}_\Lambda M \mid M \in \mathrm{Mod}\,\Lambda\} \leq n$ by [8, Theorem 1.1] and [32, Theorem 2.1].

It follows from [25, Theorem 1.4] that Λ is Gorenstein with $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{op}} \Lambda \leq n$. □

6. A dual of the Auslander–Bridger approximation theorem

In this section, we first obtain a dual version of the Auslander–Bridger approximation theorem and then give several applications. We begin with the following.

LEMMA 6.1 ([36, PROPOSITION 2.2])

(1) *For any* $X \in \mathrm{Mod}\, R$, *we have* $(\theta_X)_* \cdot \mu_{X_*} = 1_{X_*}$.
(2) *For any* $Y \in \mathrm{Mod}\, S$, *we have* $\theta_{\omega \otimes_S Y} \cdot (1_\omega \otimes \mu_Y) = 1_{\omega \otimes_S Y}$.

For any $n \geq 0$, recall from [3] that the *grade* of a finitely generated R-module M is defined as $\mathrm{grade}_R M := \inf\{i \geq 0 \mid \mathrm{Ext}_R^i(M,R) \neq 0\}$; the *strong grade* of M, denoted by $\mathrm{s.grade}_R M$, is said to be at least n if $\mathrm{grade}_R X \geq n$ for any submodule X of M. We introduce two dual versions of these notions as follows.

DEFINITION 6.2

Let $M \in \mathrm{Mod}\, R$ and $N \in \mathrm{Mod}\, S$, and let $n \geq 0$.

(1) The Ext-*cograde* of M with respect to ω is defined as $\text{E-cograde}_\omega M := \inf\{i \geq 0 \mid \mathrm{Ext}_R^i(\omega, M) \neq 0\}$; the *strong* Ext-*cograde* of M with respect to ω, denoted by $\text{s.E-cograde}_\omega M$, is said to be at least n if $\text{E-cograde}_\omega X \geq n$ for any quotient module X of M.

(2) The Tor-*cograde* of N with respect to ω is defined as $\text{T-cograde}_\omega N := \inf\{i \geq 0 \mid \mathrm{Tor}_i^S(\omega, N) \neq 0\}$; the *strong* Tor-*cograde* of N with respect to ω, denoted by $\text{s.T-cograde}_\omega N$, is said to be at least n if $\text{T-cograde}_\omega Y \geq n$ for any submodule Y of N.

We remark that the Tor-cograde of N with respect to ω is called the *cograde* of N with respect to ω in [33].

The following result can be regarded as a dual version of the Auslander–Bridger approximation theorem (see [17, Proposition 3.8]).

THEOREM 6.3

Let $M \in \mathrm{Mod}\, R$, *and let* $n \geq 1$. *If* $\text{T-cograde}_\omega \mathrm{Ext}_R^i(\omega, M) \geq i$ *for any* $1 \leq i \leq n$, *then there exist a module* $U \in \mathrm{Mod}\, R$ *and a homomorphism* $f: U \to M$ *in* $\mathrm{Mod}\, R$ *satisfying the following properties:*

(1) $\mathcal{P}_\omega(R)\text{-}\mathrm{id}_R U \leq n$, *and*
(2) $\mathrm{Ext}_R^i(\omega, f)$ *is bijective for any* $1 \leq i \leq n$.

Proof

We proceed by induction on n. Let $n = 1$, and let

$$Q_1 \xrightarrow{f_1} Q_0 \to \mathrm{Ext}^1_R(\omega, M) \to 0$$

be a projective presentation of $\mathrm{Ext}^1_R(\omega, M)$ in $\mathrm{Mod}\, S$. Then we get the exact sequence

$$\omega \otimes_S Q_1 \xrightarrow{1_\omega \otimes f_1} \omega \otimes_S Q_0 \to \omega \otimes_S \mathrm{Ext}^1_R(\omega, M) \to 0$$

in $\mathrm{Mod}\, R$ with both $\omega \otimes_S Q_1$ and $\omega \otimes_S Q_0$ in $\mathcal{P}_\omega(R)$ (which equals $\mathrm{Add}_R\, \omega$). Put $U = \mathrm{Ker}(1_\omega \otimes f_1)$. Because $\omega \otimes_S \mathrm{Ext}^1_R(\omega, M) = 0$ by assumption, $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\, U \leq 1$.

Next we show that there exists a homomorphism $f : U \to M$ in $\mathrm{Mod}\, R$ such that $\mathrm{Ext}^1_R(\omega, f)$ is bijective. Since Q_1 and Q_0 are projective, there exist two homomorphisms g_0 and g_1 such that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
Q_1 & \xrightarrow{f_1} & Q_0 & \xrightarrow{\delta'} & \mathrm{Ext}^1_R(\omega, M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \Vert & & \\
I^0(M)_* & \longrightarrow & \mathrm{co}\,\Omega^1(M)_* & \xrightarrow{\delta} & \mathrm{Ext}^1_R(\omega, M) & \longrightarrow & 0
\end{array} \tag{6.1}$$

Then there exists a homomorphism f such that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & \omega \otimes_S Q_1 & \xrightarrow{1_\omega \otimes f_1} & \omega \otimes_S Q_0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \\
0 & \longrightarrow & M & \longrightarrow & I^0(M) & \longrightarrow & \mathrm{co}\,\Omega^1(M) & \longrightarrow & 0
\end{array} \tag{6.2}$$

where $h_1 = \theta_{I^0(M)} \cdot (1_\omega \otimes g_1)$ and $h_0 = \theta_{\mathrm{co}\,\Omega^1(M)} \cdot (1_\omega \otimes g_0)$. Applying the functor $(-)_*$ to diagram (6.2), we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
(\omega \otimes_S Q_1)_* & \xrightarrow{(1_\omega \otimes f_1)_*} & (\omega \otimes_S Q_0)_* & \xrightarrow{\delta''} & \mathrm{Ext}^1_R(\omega, U) & \longrightarrow & 0 \\
\downarrow{\scriptstyle h_{1*}} & & \downarrow{\scriptstyle h_{0*}} & & \downarrow{\scriptstyle \mathrm{Ext}^1_R(\omega, f)} & & \\
I^0(M)_* & \longrightarrow & \mathrm{co}\,\Omega^1(M)_* & \xrightarrow{\delta} & \mathrm{Ext}^1_R(\omega, M) & \longrightarrow & 0
\end{array} \tag{6.3}$$

Because the diagram

$$\begin{array}{ccc}
Q_0 & \xrightarrow{g_0} & \mathrm{co}\,\Omega^1(M)_* \\
\downarrow{\scriptstyle \mu_{Q_0}} & & \downarrow{\scriptstyle \mu_{\mathrm{co}\,\Omega^1(M)_*}} \\
(\omega \otimes_S Q_0)_* & \xrightarrow{(1_\omega \otimes g_0)_*} & \big(\omega \otimes_S \mathrm{co}\,\Omega^1(M)_*\big)_*
\end{array}$$

is commutative, $\mu_{\mathrm{co}\,\Omega^1(M)_*}\cdot g_0=(1_\omega\otimes g_0)_*\cdot\mu_{Q_0}$. Then we have

$$\begin{aligned}
&h_{0*}\cdot\mu_{Q_0}\\
&\quad=\big(\theta_{\mathrm{co}\,\Omega^1(M)}\cdot(1_\omega\otimes g_0)\big)_*\cdot\mu_{Q_0}\\
&\quad=(\theta_{\mathrm{co}\,\Omega^1(M)})_*\cdot(1_\omega\otimes g_0)_*\cdot\mu_{Q_0}\\
&\quad=(\theta_{\mathrm{co}\,\Omega^1(M)})_*\cdot\mu_{\mathrm{co}\,\Omega^1(M)_*}\cdot g_0\\
&\quad=1_{\mathrm{co}\,\Omega^1(M)_*}\cdot g_0\quad\text{(by Lemma 6.1(1))}\\
&\quad=g_0.
\end{aligned}$$

On the other hand, from diagrams (6.1) and (6.3), we get that $\delta'=\delta\cdot g_0$ and $\mathrm{Ext}^1_R(\omega,f)\cdot\delta''=\delta\cdot h_{0*}$. So we have

$$\begin{aligned}
&\mathrm{Ext}^1_R(\omega,f)\cdot\delta''\cdot\mu_{Q_0}\\
&\quad=\delta\cdot h_{0*}\cdot\mu_{Q_0}\\
&\quad=\delta\cdot g_0\\
&\quad=\delta',
\end{aligned}$$

and we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
(\omega\otimes_S Q_1)_* & \xrightarrow{(1_\omega\otimes f_1)_*} & (\omega\otimes_S Q_0)_* & \xrightarrow{\delta''} & \mathrm{Ext}^1_R(\omega,U) & \longrightarrow & 0\\
\cong\big\downarrow{\scriptstyle (\mu_{Q_1})^{-1}} & & \cong\big\downarrow{\scriptstyle (\mu_{Q_0})^{-1}} & & \big\downarrow{\scriptstyle \mathrm{Ext}^1_R(\omega,f)} & & \\
Q_1 & \xrightarrow{f_1} & Q_0 & \xrightarrow{\delta'} & \mathrm{Ext}^1_R(\omega,M) & \longrightarrow & 0
\end{array}$$

Thus, $\mathrm{Ext}^1_R(\omega,f)$ is bijective.

Now suppose $n\geq 2$. By the induction hypothesis, there exists a homomorphism $f':U'\to M$ in $\mathrm{Mod}\,R$ such that $\mathcal{P}_\omega(R)$-$\mathrm{id}_R\,U'\leq n-1$ and $\mathrm{Ext}^i_R(\omega,f')$ is bijective for any $1\leq i\leq n-1$. Then there exists a $\mathrm{Hom}_R(-,\mathcal{P}_\omega(R))$-exact exact sequence

$$0\to U'\xrightarrow{g'}W\to X\to 0$$

in $\mathrm{Mod}\,R$ with W in $\mathcal{P}_\omega(R)$, and we get the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M & = & M & & \\
 & & & & \downarrow {\scriptstyle \binom{1_M}{0}} & & \downarrow & & \\
0 & \longrightarrow & U' & \xrightarrow{\binom{f'}{g'}} & M \oplus W & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow {\scriptstyle (0,1_W)} & & \downarrow & & \\
0 & \longrightarrow & U' & \xrightarrow{g'} & W & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $L = \operatorname{Coker} \binom{f'}{g'}$. It is easy to see that the exact sequence

$$0 \to U' \xrightarrow{\binom{f'}{g'}} M \oplus W \to L \to 0$$

is $\operatorname{Hom}_R(-, \mathcal{P}_\omega(R))$-exact. Because $\mathcal{P}_\omega(R)\text{-id}_R U' \leq n-1$ and $\operatorname{Ext}^i_R(\omega, f')$ is bijective for any $1 \leq i \leq n-1$, we have that the sequence

$$0 \to U'_* \xrightarrow{\binom{f'_*}{g'_*}} (M \oplus W)_* \to L_* \to 0$$

is exact, $\operatorname{Ext}^{1 \leq i \leq n-1}_R(\omega, L) = 0$, and $\operatorname{Ext}^n_R(\omega, M) \cong \operatorname{Ext}^n_R(\omega, L)$. Take a projective resolution

$$Q_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \to \operatorname{Ext}^n_R(\omega, M) \to 0 \tag{6.4}$$

of $\operatorname{Ext}^n_R(\omega, M)$ in $\operatorname{Mod} S$. By assumption, $\text{T-cograde}_\omega \operatorname{Ext}^n_R(\omega, M) \geq n$, so we get the exact sequence

$$0 \to N \to \omega \otimes_S Q_n \xrightarrow{1_\omega \otimes f_n} \cdots \xrightarrow{1_\omega \otimes f_2} \omega \otimes_S Q_1 \xrightarrow{1_\omega \otimes f_1} \omega \otimes_S Q_0 \to 0 \tag{6.5}$$

in $\operatorname{Mod} R$ with all $\omega \otimes_S Q_i$'s in $\mathcal{P}_\omega(R)$ and $N = \operatorname{Ker}(1_\omega \otimes f_n)$. Then $\mathcal{P}_\omega(R)$-$\text{id}_R N \leq n$. Applying the functor $(-)_*$ to the exact sequence (6.5), we get the sequence

$$0 \to N_* \to (\omega \otimes_S Q_n)_* \xrightarrow{(1_\omega \otimes f_n)_*} \cdots \xrightarrow{(1_\omega \otimes f_2)_*} (\omega \otimes_S Q_1)_* \xrightarrow{(1_\omega \otimes f_1)_*} (\omega \otimes_S Q_0)_* \to 0. \tag{6.6}$$

Comparing the sequences (6.4) with (6.6) we get that $\operatorname{Ext}^{1 \leq i \leq n-1}_R(\omega, N) = 0$ and $\operatorname{Ext}^n_R(\omega, N) \cong \operatorname{Ext}^n_R(\omega, M)$.

Because $\mathrm{Ext}_R^i(\omega, L) = 0$ for any $1 \le i \le n-1$, we get an exact sequence

$$I^0(L)_* \to I^1(L)_* \to \cdots \to I^{n-1}(L)_* \to K_* \to \mathrm{Ext}_R^n(\omega, L) \to 0$$

in $\mathrm{Mod}\, S$, where $K = \mathrm{Coker}(I^{n-2}(L) \to I^{n-1}(L))$. Since all Q_i's are projective, there exist homomorphisms $g_0, g_1, \dots, g_n$ such that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
Q_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_2} & Q_1 & \xrightarrow{f_0} & Q_0 & \longrightarrow & \mathrm{Ext}_R^n(\omega, L) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle g_n} & & & & \Big\downarrow{\scriptstyle g_1} & & \Big\downarrow{\scriptstyle g_0} & & \Big\| & & \\
I^0(L)_* & \longrightarrow & \cdots & \longrightarrow & I^{n-1}(L)_* & \longrightarrow & K_* & \longrightarrow & \mathrm{Ext}_R^n(\omega, L) & \longrightarrow & 0
\end{array}$$

(6.7)

Then there exists a homomorphism h such that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & N & \xrightarrow{s} & \omega \otimes_S Q_n & \xrightarrow{1_\omega \otimes f_n} & \cdots & \xrightarrow{1_\omega \otimes f_1} & \omega \otimes_S Q_1 & \xrightarrow{1_\omega \otimes f_0} & \omega \otimes_S Q_0 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h_n} & & & & \Big\downarrow{\scriptstyle h_1} & & \Big\downarrow{\scriptstyle h_0} & & \\
0 & \longrightarrow & L & \longrightarrow & I^0(L) & \longrightarrow & \cdots & \longrightarrow & I^{n-1}(L) & \longrightarrow & K & \longrightarrow & 0
\end{array}$$

(6.8)

where $h_i = \theta_{I^{n-i}(L)} \cdot (1_\omega \otimes g_i)$ for any $1 \le i \le n$ and $h_0 = \theta_K \cdot (1_\omega \otimes g_0)$. Notice that the functor $(-)_*$ takes diagram (6.8) back to diagram (6.7), so $\mathrm{Ext}_R^n(\omega, h)$ is bijective.

Put $W' = \omega \otimes_S Q_n$. Then we get an exact sequence

$$0 \to N \xrightarrow{\binom{h}{s}} L \oplus W' \to N' \to 0$$

and a $\mathrm{Hom}_R(-, \mathcal{P}_\omega(R))$-exact exact sequence

$$0 \to U' \xrightarrow{u} M \oplus W \oplus W' \to L \oplus W' \to 0$$

in $\mathrm{Mod}\, R$, where $u = \begin{pmatrix} f' \\ g' \\ 0 \end{pmatrix}$. Consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \xrightarrow{\alpha} & U & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \lambda} & & \downarrow & & \\
0 & \longrightarrow & U' & \xrightarrow{u} & M \oplus W \oplus W' & \longrightarrow & L \oplus W' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N' & = & N' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

It is easy to see that the first row in the above diagram is $\mathrm{Hom}_R(-,\mathcal{P}_\omega(R))$-exact exact. Because $\mathcal{P}_\omega(R)\text{-id}_R\, U' \leq n-1$ and $\mathcal{P}_\omega(R)\text{-id}_R\, N \leq n$, $\mathcal{P}_\omega(R)\text{-id}_R\, U \leq n$ by the dual version of [15, Lemma 8.2.1].

Put $p = (1_M, 0, 0) : M \oplus W \oplus W' \twoheadrightarrow M$ and $f = p \cdot \lambda$. Then $\mathrm{Ext}_R^i(\omega, f) = \mathrm{Ext}_R^i(\omega, p) \cdot \mathrm{Ext}_R^i(\omega, \lambda)$ for any $i \geq 0$. Because $W \oplus W' \in \mathcal{P}_\omega(R)$, $\mathrm{Ext}_R^i(\omega, p)$ is bijective for any $i \geq 1$. Note that $\mathrm{Ext}_R^i(\omega, f')$ is bijective for any $1 \leq i \leq n-1$ and $\mathrm{Ext}_R^{1\leq i\leq n-1}(\omega, N) = 0 = \mathrm{Ext}_R^{1\leq i\leq n-1}(\omega, L)$. We have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
 & & \mathrm{Ext}_R^i(\omega, U') & \xrightarrow{\mathrm{Ext}_R^i(\omega,\alpha)} & \mathrm{Ext}_R^i(\omega, U) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{Ext}_R^i(\omega,\lambda)} & & \\
0 & \longrightarrow & \mathrm{Ext}_R^i(\omega, U') & \xrightarrow{\mathrm{Ext}_R^i(\omega,u)} & \mathrm{Ext}_R^i(\omega, M \oplus W \oplus W') & \longrightarrow & 0
\end{array}
$$

So $\mathrm{Ext}_R^i(\omega, \lambda)$ and $\mathrm{Ext}_R^i(\omega, f)$ are bijective for $1 \leq i \leq n-1$. On the other hand, because $\mathrm{Ext}_R^n(\omega, h)$ is bijective and $\mathrm{Ext}_R^{n+1}(\omega, U') = 0 = \mathrm{Ext}_R^{n-1}(\omega, L)$, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
 & & \mathrm{Ext}_R^n(\omega, U') & \xrightarrow{\mathrm{Ext}_R^n(\omega,\alpha)} & \mathrm{Ext}_R^n(\omega, U) & \xrightarrow{\mathrm{Ext}_R^n(\omega,\beta)} & \mathrm{Ext}_R^n(\omega, N) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{Ext}_R^n(\omega,\lambda)} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathrm{Ext}_R^n(\omega, U') & \xrightarrow{\mathrm{Ext}_R^n(\omega,u)} & \mathrm{Ext}_R^n(\omega, M \oplus W \oplus W') & \longrightarrow & \mathrm{Ext}_R^n(\omega, L \oplus W') & &
\end{array}
$$

So $\mathrm{Ext}_R^n(\omega, \lambda)$ and $\mathrm{Ext}_R^n(\omega, f)$ are bijective. The proof is finished. □

Dual to Theorem 6.3, we have the following result.

THEOREM 6.4

Let $N \in \operatorname{Mod} S$, *and let* $n \geq 1$. *If* $\text{E-cograde}_\omega \operatorname{Tor}_i^S(\omega, N) \geq i$ *for any* $1 \leq i \leq n$, *then there exist a module* $V \in \operatorname{Mod} S$ *and a homomorphism* $g : N \to V$ *in* $\operatorname{Mod} S$ *satisfying the following properties:*

(1) $\mathcal{I}_\omega(S)\text{-pd}_S V \leq n$, *and*
(2) $\operatorname{Tor}_i^S(\omega, g)$ *is bijective for any* $1 \leq i \leq n$.

In the rest of this section, we give several applications of Theorems 6.3 and 6.4.

Let Λ be an Artin R-algebra over a commutative Artin ring R, and let $\operatorname{mod}\Lambda$ be the class of finitely generated left Λ-modules. It is well known that the ordinary Matlis duality functor $D(-)$ induces a duality between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{op}$. Recall from [23] that Λ is called *right quasi-Auslander n-Gorenstein* provided that $\operatorname{fd}_{\Lambda^{op}} I^i(\Lambda_\Lambda) \leq i+1$ for any $0 \leq i \leq n-1$. As an application of Theorem 6.3, we get the following result.

COROLLARY 6.5

Let Λ *be a right quasi-Auslander n-Gorenstein–Artin algebra, and let* $M \in \operatorname{mod}\Lambda$. *Then there exist a module* $U \in \operatorname{mod}\Lambda$ *and a homomorphism* $f : U \to M$ *in* $\operatorname{mod}\Lambda$ *satisfying the following properties:*

(1) $\operatorname{id}_\Lambda U \leq n$, *and*
(2) $\operatorname{Ext}_\Lambda^i(D(\Lambda), f)$ *is bijective for any* $1 \leq i \leq n$.

Proof

Let $M \in \operatorname{mod}\Lambda$, and let $i, j \geq 0$. Then we have

$$\begin{aligned}
&\operatorname{Ext}_\Lambda^i\big(D(\Lambda), M\big)\\
&\quad\cong \operatorname{Ext}_\Lambda^i\big(D(\Lambda), D\big(D(M)\big)\big)\\
&\quad\cong D\big(\operatorname{Tor}_i^\Lambda\big(D(M), D(\Lambda)\big)\big) \quad \text{(by [9, Chapter VI, Proposition 5.1])}\\
&\quad\cong D\big(D\big(\operatorname{Ext}_{\Lambda^{op}}^i\big(D(M), \Lambda\big)\big)\big) \quad \text{(by [9, Chapter VI, Proposition 5.3])}\\
&\quad\cong \operatorname{Ext}_{\Lambda^{op}}^i\big(D(M), \Lambda\big).
\end{aligned}$$

So for any $i \geq 1$ and $j \geq 0$, we have

$$\begin{aligned}
&\operatorname{Tor}_j^\Lambda\big(D(\Lambda), \operatorname{Ext}_\Lambda^i\big(D(\Lambda), M\big)\big)\\
&\quad\cong \operatorname{Tor}_j^\Lambda\big(D(\Lambda), \operatorname{Ext}_{\Lambda^{op}}^i\big(D(M), \Lambda\big)\big)\\
&\quad\cong D\big(\operatorname{Ext}_\Lambda^j\big(\operatorname{Ext}_{\Lambda^{op}}^i\big(D(M), \Lambda\big), \Lambda\big)\big) \quad \text{(by [9, Chapter VI, Proposition 5.3]).}
\end{aligned}$$

Since Λ is right quasi-Auslander n-Gorenstein, $\operatorname{grade}_\Lambda \operatorname{Ext}_{\Lambda^{op}}^i(D(M), \Lambda) \geq i$ for any $1 \leq i \leq n$ by [3, Theorem 4.7]. It follows from the above argument that $\text{T-cograde}_{D(\Lambda)} \operatorname{Ext}_\Lambda^i(D(\Lambda), M) \geq i$ for any $1 \leq i \leq n$. In addition, note that $D(\Lambda)$ is an injective cogenerator for $\operatorname{Mod}\Lambda$, so $\mathcal{P}_{D(\Lambda)}(\Lambda)\text{-id}_\Lambda X = \operatorname{id}_\Lambda X$ for any $X \in \operatorname{mod}\Lambda$. Now the assertion follows from Theorem 6.3. $\square$

We give the second application of Theorems 6.3 and 6.4 as follows.

COROLLARY 6.6

Let $M \in \operatorname{Mod} R$ *and* $N \in \operatorname{Mod} S$. *Then for any* $n \geq 0$, *we have the following.*

(1) *If* $\text{T-cograde}_\omega \operatorname{Ext}_R^i(\omega, M) \geq i+1$ *for any* $0 \leq i \leq n$, *then* $\text{E-cograde}_\omega M \geq n+1$.

(2) *If* $\text{E-cograde}_\omega \operatorname{Tor}_i^S(\omega, N) \geq i+1$ *for any* $0 \leq i \leq n$, *then* $\text{T-cograde}_C N \geq n+1$.

Proof

(1) We proceed by induction on n. Let $n = 0$ and $\omega \otimes_S M_* = 0$. Since $(\theta_M)_* \cdot \mu_{M_*} = 1_{M_*}$ by Lemma 6.1(1), μ_{M_*} is a split monomorphism and $M_* = 0$.

Now suppose $n \geq 1$. By the induction hypothesis, we have that $\text{E-cograde}_\omega M \geq n$ and $\operatorname{Ext}_R^{0 \leq i \leq n-1}(\omega, M) = 0$. It is left to show $\operatorname{Ext}_R^n(\omega, M) = 0$. By Theorem 6.3, there exist a module $U \in \operatorname{Mod} R$ and a homomorphism $f : U \to M$ in $\operatorname{Mod} R$ such that $\mathcal{P}_\omega(R)\text{-id}_R U \leq n$ and $\operatorname{Ext}_R^i(\omega, f)$ is bijective for any $1 \leq i \leq n$. It follows that $\operatorname{Ext}_R^{1 \leq i \leq n-1}(\omega, U) = 0$. Let

$$0 \to U \xrightarrow{g} W_0 \to W_1 \to \cdots \to W_n \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all W_i's in $\mathcal{P}_\omega(R)$. Applying the functor $(-)_*$ to it, we get an exact sequence

$$0 \to U_* \to W_{0*} \to W_{1*} \to \cdots \to W_{n*} \to \operatorname{Ext}_R^n(\omega, U) \to 0$$

in $\operatorname{Mod} S$. Since $\operatorname{Ext}_R^n(\omega, M) \cong \operatorname{Ext}_R^n(\omega, U)$, we have $\text{T-cograde}_\omega \operatorname{Ext}_R^n(\omega, U) \geq n+1$ by assumption. Then we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccc}
 & & \omega \otimes_R U_* & \longrightarrow & \omega \otimes_S W_{0*} & \longrightarrow & \omega \otimes_S W_{1*} & \longrightarrow & \cdots & \longrightarrow & \omega \otimes_S W_{n*} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_U} & & \downarrow{\scriptstyle \theta_{W_0}} & & \downarrow{\scriptstyle \theta_{W_1}} & & & & \downarrow{\scriptstyle \theta_{W_n}} & & \\
0 & \longrightarrow & U & \longrightarrow & W_0 & \longrightarrow & W_1 & \longrightarrow & \cdots & \longrightarrow & W_n & \longrightarrow & 0
\end{array}$$

Because all θ_{W_i}'s are bijective, θ_U is epic. Note that we have the following commutative diagram:

$$\begin{array}{ccc}
\omega \otimes_S U_* & \xrightarrow{1_\omega \otimes f_*} & \omega \otimes_S M_* \\
\downarrow{\scriptstyle \theta_U} & & \downarrow{\scriptstyle \theta_M} \\
U & \xrightarrow{f} & M
\end{array}$$

Because $\omega \otimes_S M_* = 0$ by assumption, $f \cdot \theta_U = 0$. But θ_U is epic, so $f = 0$. It follows that the bijection $\operatorname{Ext}_R^n(\omega, f)$ is zero and $\operatorname{Ext}_R^n(\omega, M) = 0$.

(2) The proof is dual to that of (1), so we omit it. □

Before giving the third application of Theorem 6.3, we need the following result.

PROPOSITION 6.7

Let

$$V_1 \xrightarrow{g} V_0 \to N \to 0 \tag{6.9}$$

be an exact sequence in $\operatorname{Mod} S$ *satisfying the following conditions.*

(1) *Both* μ_{V_0} *and* μ_{V_1} *are isomorphisms.*
(2) $\operatorname{Ext}^1_R(\omega, \omega \otimes_S V_0) = 0$ *and* $\operatorname{Ext}^1_R(\omega, \omega \otimes_S V_1) = 0 = \operatorname{Ext}^2_R(\omega, \omega \otimes_S V_1)$.

Then there exists an exact sequence

$$0 \to \operatorname{Ext}^1_R(\omega, L) \to N \xrightarrow{\mu_N} (\omega \otimes_S N)_* \to \operatorname{Ext}^2_R(\omega, L) \to 0,$$

where $L = \operatorname{Ker}(1_\omega \otimes g)$.

Proof

By applying the functor $\omega \otimes_S -$ to (6.9), we get an exact sequence

$$0 \to L \to \omega \otimes_S V_1 \xrightarrow{1_\omega \otimes g} \omega \otimes_S V_0 \to \omega \otimes_S N \to 0$$

in $\operatorname{Mod} R$. Let $g = \alpha \cdot \pi$ (where $\pi : V_1 \twoheadrightarrow \operatorname{Im} g$ and $\alpha : \operatorname{Im} g \rightarrowtail V_0$) and $1_\omega \otimes g = \alpha' \cdot \pi'$ (where $\pi' : \omega \otimes_S V_1 \twoheadrightarrow \operatorname{Im}(1_\omega \otimes g)$ and $\alpha' : \operatorname{Im}(1_\omega \otimes g) \rightarrowtail \omega \otimes_S V_0$) be the natural epic-monic decompositions of g and $1_\omega \otimes g$, respectively. Since $\operatorname{Ext}^1_R(\omega, \omega \otimes_S V_0) = 0$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im} g & \xrightarrow{\alpha} & V_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle \mu_{V_0}} & & \Big\downarrow{\scriptstyle \mu_N} & & \\
0 & \longrightarrow & \big(\operatorname{Im}(1_\omega \otimes g)\big)_* & \xrightarrow{\alpha'_*} & (\omega \otimes_S V_0)_* & \longrightarrow & (\omega \otimes_S N)_* & \longrightarrow & \operatorname{Ext}^1_R\big(\omega, \operatorname{Im}(1_\omega \otimes g)\big) \longrightarrow 0
\end{array}$$

where h is an induced homomorphism. Then $\alpha'_* \cdot h = \mu_{V_0} \cdot \alpha$. In addition, since μ_{V_0} is an isomorphism by assumption, by the snake lemma we have $\operatorname{Coker} \mu_N \cong \operatorname{Ext}^1_R(\omega, \operatorname{Im}(1_\omega \otimes g))$ and $\operatorname{Ker} \mu_N \cong \operatorname{Coker} h$.

On the other hand, since $\operatorname{Ext}^1_R(\omega, \omega \otimes_S V_1) = 0 = \operatorname{Ext}^2_R(\omega, \omega \otimes_S V_1)$ by assumption, by applying the functor $(-)_*$ to the exact sequence

$$0 \to L \to \omega \otimes_S V_1 \xrightarrow{\pi'} \operatorname{Im}(1_\omega \otimes g) \to 0,$$

we get the exact sequence

$$0 \to L_* \to (\omega \otimes_S V_1)_* \xrightarrow{\pi'_*} \big(\operatorname{Im}(1_\omega \otimes g)\big)_* \to \operatorname{Ext}^1_R(\omega, L) \to 0$$

and the isomorphism

$$\operatorname{Ext}^1_R\big(\omega, \operatorname{Im}(1_\omega \otimes g)\big) \cong \operatorname{Ext}^2_R(\omega, L).$$

Because

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad g \quad} & V_0 \\ \downarrow{\scriptstyle \mu_{V_1}} & & \downarrow{\scriptstyle \mu_{V_0}} \\ (\omega \otimes_S V_1)_* & \xrightarrow{(1_\omega \otimes g)_*} & (\omega \otimes_S V_0)_* \end{array}$$

is a commutative diagram, $(1_\omega \otimes g)_* \cdot \mu_{V_1} = \mu_{V_0} \cdot g$. Because $1_\omega \otimes g = \alpha' \cdot \pi'$, $(1_\omega \otimes g)_* = \alpha'_* \cdot \pi'_*$. Thus, we have $\alpha'_* \cdot h \cdot \pi = \mu_{V_0} \cdot \alpha \cdot \pi = \mu_{V_0} \cdot g = (1_\omega \otimes g)_* \cdot \mu_{V_1} = \alpha'_* \cdot \pi'_* \cdot \mu_{V_1}$. Because α'_* is monic, $h \cdot \pi = \pi'_* \cdot \mu_{V_1}$. Note that π is epic and that μ_{V_1} is an isomorphism, so $\operatorname{Ker} \mu_N \cong \operatorname{Coker} h \cong \operatorname{Coker} \pi'_* \cong \operatorname{Ext}^1_R(\omega, L)$. Consequently, we obtain the desired exact sequence. □

As a consequence of Proposition 6.7, we have the following result.

COROLLARY 6.8

Let $M \in \operatorname{Mod} R$. Then there exists an exact sequence

$$0 \to \operatorname{Ext}^1_R(\omega, M) \to \operatorname{cTr}_\omega M \xrightarrow{\mu_{\operatorname{cTr}_\omega M}} (\omega \otimes_S \operatorname{cTr}_\omega M)_* \to \operatorname{Ext}^2_R(\omega, M) \to 0.$$

Proof

Let $M \in \operatorname{Mod} R$. Then from the exact sequence (2.1), we get the exact sequence

$$0 \to M_* \to I^0(M)_* \xrightarrow{f^0_*} I^1(M)_* \to \operatorname{cTr}_\omega M \to 0$$

in $\operatorname{Mod} S$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ker}(1_\omega \otimes f^0_*) & \to & \omega \otimes_S I^0(M)_* & \xrightarrow{1_\omega \otimes f^0_*} & \omega \otimes_S I^1(M)_* & \to & \omega \otimes_S \operatorname{cTr}_\omega M & \to 0 \\ & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \theta_{I^0(M)}} & & \downarrow{\scriptstyle \theta_{I^0(M)}} & & & \\ 0 \to & M & \to & I^0(M) & \xrightarrow{f^0} & I^1(M) & & & \end{array}$$

Because $I^0(M), I^1(M) \in \mathcal{B}_\omega(R)$ by [21, Theorem 6.2], both $\theta_{I^0(M)}$ and $\theta_{I^1(M)}$ are isomorphisms. So the induced homomorphism h is also an isomorphism and $M \cong \operatorname{Ker}(1_\omega \otimes f^0_*)$. Note that $I^0(M)_*, I^1(M)_* \in \mathcal{A}_\omega(S)$ by [21, Proposition 4.1]. So both $\mu_{I^0(M)_*}$ and $\mu_{I^1(M)_*}$ are isomorphisms, and then the assertion follows from Proposition 6.7. □

We are now in a position to prove the following result.

THEOREM 6.9

For any $n \geq 1$, the following statements are equivalent:

(1) $\text{s.E-cograde}_\omega \operatorname{Tor}^S_i(\omega, N) \geq i$ *for any* $N \in \operatorname{Mod} S$ *and* $1 \leq i \leq n$,
(2) $\text{s.T-cograde}_\omega \operatorname{Ext}^i_R(\omega, M) \geq i$ *for any* $M \in \operatorname{Mod} R$ *and* $1 \leq i \leq n$.

Proof

(1) $\Rightarrow$ (2) We proceed by induction on n. Let $n=1$. Given a module M in $\operatorname{Mod} R$, by Corollary 6.8 we have an exact sequence

$$0 \to \operatorname{Ext}_R^1(\omega, M) \to \operatorname{cTr}_\omega M \xrightarrow{\mu_{\operatorname{cTr}_\omega M}} (\omega \otimes_S \operatorname{cTr}_\omega M)_* \to \operatorname{Ext}_R^2(\omega, M) \to 0.$$

Let $N = \operatorname{Im} \mu_{\operatorname{cTr}_\omega M}$, and let $\mu_{\operatorname{cTr}_\omega M} = \alpha \cdot \beta$ (where $\beta : \operatorname{cTr}_\omega M \twoheadrightarrow N$ and $\alpha : N \rightarrowtail (\omega \otimes_S \operatorname{cTr}_\omega M)_*$) be the natural epic-monic decomposition of $\mu_{\operatorname{cTr}_\omega M}$. Applying the functor $\omega \otimes_S -$ to the exact sequence

$$0 \to \operatorname{Ext}_R^1(\omega, M) \to \operatorname{cTr}_\omega M \xrightarrow{\beta} N \to 0, \tag{6.10}$$

we get an exact sequence

$$\operatorname{Tor}_1^S(\omega, N) \to \omega \otimes_S \operatorname{Ext}_R^1(\omega, M) \to \omega \otimes_S \operatorname{cTr}_\omega M \xrightarrow{1_\omega \otimes \beta} \omega \otimes_S N \to 0.$$

Since $(1_\omega \otimes \alpha) \cdot (1_\omega \otimes \beta) = 1_\omega \otimes \mu_{\operatorname{cTr}_\omega M}$ and $1_\omega \otimes \mu_{\operatorname{cTr}_\omega M}$ is a split monomorphism by Lemma 6.1(2), $1_\omega \otimes \beta$ is an isomorphism. It follows that $\omega \otimes_S \operatorname{Ext}_R^1(\omega, M)$ is isomorphic to a quotient module of $\operatorname{Tor}_1^S(\omega, N)$ in $\operatorname{Mod} R$. Then by assumption $\text{E-cograde}_\omega(\omega \otimes_S \operatorname{Ext}_R^1(\omega, M)) \geq 1$. Using Corollary 6.6(2), we have that $\omega \otimes_S \operatorname{Ext}_R^1(\omega, M) = 0$.

Let X be a submodule of $\operatorname{Ext}_R^1(\omega, M)$ in $\operatorname{Mod} S$. Then the exact sequence (6.10) induces the exact sequences

$$\begin{gathered} 0 \to \operatorname{Ext}_R^1(\omega, M)/X \to (\operatorname{cTr}_\omega M)/X \xrightarrow{\gamma} N \to 0, \\ 0 \to X \to \operatorname{cTr}_\omega M \xrightarrow{\pi} (\operatorname{cTr}_\omega M)/X \to 0 \end{gathered} \tag{6.11}$$

such that $\beta = \gamma \cdot \pi$. Then $1_\omega \otimes \beta = (1_\omega \otimes \gamma) \cdot (1_\omega \otimes \pi)$. On the other hand, since $\omega \otimes_S \operatorname{Ext}_R^1(\omega, M) = 0$, $\omega \otimes_S (\operatorname{Ext}_R^1(\omega, M)/X) = 0$ and $1_\omega \otimes \gamma$ is bijective. So $1_\omega \otimes \pi$ is also bijective. Hence, from the exact sequence

$$\operatorname{Tor}_1^S(\omega, (\operatorname{cTr}_\omega M)/X) \to \omega \otimes_S X \to \omega \otimes_S \operatorname{cTr}_\omega M \xrightarrow{1_\omega \otimes \pi} \omega \otimes_S (\operatorname{cTr}_\omega M)/X \to 0$$

induced by (6.11), we get that $\omega \otimes_S X$ is isomorphic to a quotient module of $\operatorname{Tor}_1^S(\omega, (\operatorname{cTr}_\omega M)/X)$. Then by assumption $\text{E-cograde}_\omega(\omega \otimes_S X) \geq 1$. It follows from Corollary 6.6(2) that $\text{T-cograde}_\omega X \geq 1$.

Now suppose $n \geq 2$. By the induction hypothesis, it suffices to prove that $\text{s.T-cograde}_\omega \operatorname{Ext}_R^n(\omega, M) \geq n$. Because $\operatorname{Ext}_R^n(\omega, M) \cong \operatorname{Ext}_R^{n-1}(\omega, \operatorname{co}\Omega^1(M))$, $\text{s.T-cograde}_\omega \operatorname{Ext}_R^n(\omega, M) \geq n-1$ by the induction hypothesis.

We suppose that X is a submodule of $\operatorname{Ext}_R^n(\omega, M)$ in $\operatorname{Mod} S$. Because $\text{s.T-cograde}_\omega \operatorname{Ext}_R^i(\omega, M) \geq i$ for any $1 \leq i \leq n-1$, by Theorem 6.3 there exist a module $U \in \operatorname{Mod} R$ and a homomorphism $f : U \to M$ in $\operatorname{Mod} R$ such that $\mathcal{P}_\omega(R)$-$\operatorname{id}_R U \leq n-1$ and such that $\operatorname{Ext}_R^i(\omega, f)$ is bijective for any $1 \leq i \leq n-1$. Let

$$0 \to U \xrightarrow{g} W_0 \to W_1 \to \cdots \to W_{n-1} \to 0$$

be an exact sequence in $\operatorname{Mod} R$ with all W_i's in $\mathcal{P}_\omega(R)$ and $L = \operatorname{Coker} \binom{f}{g}$. Then it is not difficult to verify that $\operatorname{Ext}_R^{1 \leq i \leq n-1}(\omega, L) = 0$ and $\operatorname{Ext}_R^n(\omega, M) \cong \operatorname{Ext}_R^n(\omega, L)$.

So we have an exact sequence

$$0 \to L_* \to I^0(L)_* \to I^1(L)_* \to \cdots \to I^n(L)_* \to Y \to 0$$

such that $\mathrm{Ext}_R^n(\omega, L) \subseteq Y$. Applying the functor $\omega \otimes_S -$ to it, we get the following commutative diagram:

$$\begin{array}{ccccccccc} \omega \otimes_S I^0(L)_* & \longrightarrow & \omega \otimes_S I^1(L)_* & \longrightarrow \cdots \longrightarrow & \omega \otimes_S I^n(L)_* & \longrightarrow & \omega \otimes_S Y & \longrightarrow & 0 \\ \cong \downarrow \theta_{I^0(L)} & & \cong \downarrow \theta_{I^1(L)} & & \cong \downarrow \theta_{I^n(L)} & & & & \\ I^0(L) & \longrightarrow & I^1(L) & \longrightarrow \cdots \longrightarrow & I^n(L) & & & & \end{array}$$

Because the bottom row in this diagram is exact, so is the upper row. It implies that $\mathrm{Tor}^S_{1 \leq i \leq n-1}(\omega, Y) = 0$. Since X is isomorphic to a submodule of $\mathrm{Ext}_R^n(\omega, L)$ ($\cong \mathrm{Ext}_R^n(\omega, M)$) in $\mathrm{Mod}\, S$ and $\text{s.T-cograde}_\omega \mathrm{Ext}_R^n(\omega, L) = \text{s.T-cograde}_\omega \mathrm{Ext}_R^n(\omega, M) \geq n-1$, $\text{T-cograde}_C X \geq n-1$. Since $\mathrm{Tor}^S_{n-1}(\omega, Y) = 0$, we have an exact sequence

$$\mathrm{Tor}^S_n(\omega, Y/X) \to \mathrm{Tor}^S_{n-1}(\omega, X) \to 0.$$

By assumption $\text{s.E-cograde}_\omega \mathrm{Tor}^S_n(\omega, Y/X) \geq n$, so $\text{E-cograde}_\omega \mathrm{Tor}^S_{n-1}(\omega, X) \geq n$. Thus, we have $\text{E-cograde}_\omega \mathrm{Tor}^S_i(\omega, X) \geq i+1$ for any $0 \leq i \leq n-1$. It follows from Corollary 6.6(2) that $\text{T-cograde}_C X \geq n$.

Dually, we get (2) $\Rightarrow$ (1). □

For any $n \geq 1$, recall that an Artin algebra Λ is called *Auslander n-Gorenstein* provided that $\mathrm{fd}_\Lambda I^i({}_\Lambda\Lambda) \leq i$ for any $0 \leq i \leq n-1$. The following result extends [17, Theorem 3.7].

COROLLARY 6.10

Let Λ be an Artin algebra. Then the following statements are equivalent for any $n \geq 1$:

(1) *Λ is Auslander n-Gorenstein.*
$(1)^{op}$ *Λ^{op} is Auslander n-Gorenstein.*
(2) $\text{s.grade}_\Lambda \mathrm{Ext}^i_\Lambda(M, \Lambda) \geq i$ *for any* $M \in \mathrm{mod}\, \Lambda$ *and* $1 \leq i \leq n$.
$(2)^{op}$ $\text{s.grade}_\Lambda \mathrm{Ext}^i_{\Lambda^{op}}(N, \Lambda) \geq i$ *for any* $N \in \mathrm{mod}\, \Lambda^{op}$ *and* $1 \leq i \leq n$.
(3) $\text{s.E-cograde}_{D(\Lambda)} \mathrm{Tor}^\Lambda_i(D(\Lambda), M) \geq i$ *for any* $M \in \mathrm{mod}\, \Lambda$ *and* $1 \leq i \leq n$.
(4) $\text{s.T-cograde}_{D(\Lambda)} \mathrm{Ext}^i_\Lambda(D(\Lambda), M) \geq i$ *for any* $M \in \mathrm{mod}\, \Lambda$ *and* $1 \leq i \leq n$.

Proof

(1) $\Leftrightarrow$ $(1)^{op}$ $\Leftrightarrow$ (2) $\Leftrightarrow$ $(2)^{op}$ follow from [17, Theorem 3.7]. Since the proof of Theorem 6.9 is also valid while modules are restricted to finitely generated modules over Artin algebras, (3) $\Leftrightarrow$ (4) holds true.

(3) $\Rightarrow$ (2) Let $M \in \mathrm{mod}\, \Lambda$, and let $1 \leq i \leq n$. If Y is a submodule of $\mathrm{Ext}^i_\Lambda(M, \Lambda)$ in $\mathrm{mod}\, \Lambda^{op}$, then $D(Y)$ is isomorphic to a quotient module of $D(\mathrm{Ext}^i_\Lambda(M, \Lambda))$ in $\mathrm{mod}\, \Lambda$. Thus, we have $D(\mathrm{Ext}^i_\Lambda(M, \Lambda)) \cong \mathrm{Tor}^\Lambda_i(D(\Lambda), M)$ by [9, Chapter VI,

Proposition 5.3]. So $\mathrm{Ext}^j_{\Lambda^{op}}(Y,\Lambda)\cong\mathrm{Ext}^j_{\Lambda}(D(\Lambda),D(Y))=0$ for any $0\le j\le i-1$ by (3).

(2) ⇒ (3) Let $M\in\mathrm{mod}\,\Lambda$, and let $1\le i\le n$. If X is a quotient module of $\mathrm{Tor}^{\Lambda}_i(D(\Lambda),M)$ in $\mathrm{mod}\,\Lambda$, then we have that $D(X)$ is isomorphic to a submodule of $D(\mathrm{Tor}^{\Lambda}_i(D(\Lambda),M))$ in $\mathrm{mod}\,\Lambda^{op}$. By [9, Chapter VI, Proposition 5.1], we have $D(\mathrm{Tor}^{\Lambda}_i(D(\Lambda),M))\cong\mathrm{Ext}^i_{\Lambda}(M,\Lambda)$. So $\mathrm{Ext}^j_{\Lambda}(D(\Lambda),X)\cong\mathrm{Ext}^j_{\Lambda^{op}}(D(X),\Lambda)=0$ for any $0\le j\le i-1$ by (2). □

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