Weak amenability and simply connected Lie groups

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Abstract Following an approach of Ozawa, we show that several semidirect products are not weakly amenable. As a consequence, we are able to completely characterize the simply connected Lie groups that are weakly amenable.

A locally compact group G is weakly amenable if there is a net $(u_i)_{i\in I}$ of compactly supported Herz–Schur multipliers on G converging to 1 uniformly on compact subsets of G and satisfying $\sup_i \|u_i\|_{B_2} \leq C$ for some $C \geq 1$ (see Section 1 for details). The infimum of those C for which such a net exists is the *weak amenability constant of* G, denoted here by $\Lambda_{\mathrm{WA}}(G)$. Weak amenability was introduced by Cowling and Haagerup [6]. By now, a lot is known about weak amenability, especially for (connected) Lie groups (see, e.g., [4], [5], [8]–[11], [15], [21]). Simple Lie groups are weakly amenable if and only if they have real rank at most one. The nonsimple case was treated in [5], though not in full generality (see Theorem 1 below).

A connected Lie group G has a Levi decomposition $G = RS$ coming from a Levi decomposition of its Lie algebra $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$. Here $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a semisimple Lie algebra. The groups R and S are the connected Lie subgroups of G associated with $\mathfrak{r}$ and $\mathfrak{s}$, respectively. The group R is a closed normal solvable subgroup. The group S is called a semisimple Levi factor of G and is a semisimple Lie subgroup. When S has finite center, the authors of [5] were able to completely characterize weak amenability of G.

THEOREM 1 ([5])

Let G be a connected Lie group, and let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition of its Lie algebra. Let S be the associated semisimple Levi factor, and decompose the Lie algebra of S into simple ideals as $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$. Suppose S has finite center. The following are equivalent.

Kyoto Journal of Mathematics, Vol. 56, No. 3 (2016), 689–700
DOI 10.1215/21562261-3600238,
Received June 4, 2015. Revised July 6, 2015. Accepted July 8, 2015.
2010 Mathematics Subject Classification: 43A22, 43A80, 22E40, 22E15.
Author's work supported by European Research Council (ERC) advanced grant no. OAFPG 247321, the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92), and the Deutsche Forschungsgemeinschaft through the Collaborative Research Centre (SFB 878).

(1) *G is weakly amenable.*
(2) *For every $i=1,\dots,n$, one of the following holds:*

- $\mathfrak{s}_i$ *has real rank zero;*
- $\mathfrak{s}_i$ *has real rank one and* $[\mathfrak{s}_i,\mathfrak{r}]=0$.

In that case, if S_i denotes the connected Lie subgroup of G associated with $\mathfrak{s}_i$, then

$$\Lambda_{\mathrm{WA}}(G)=\prod_{i=1}^{n}\Lambda_{\mathrm{WA}}(S_i).$$

For any natural number $n\geq 1$, the group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{R}^n$ by the unique irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ of dimension n. The group $\mathrm{SL}(2,\mathbb{R})$ also acts on the Heisenberg group H_{2n+1} of dimension $2n+1$ by fixing the center and acting on the vector space $\mathbb{R}^{2n}$ by the unique irreducible representation on $\mathbb{R}^{2n}$.

Apart from some structure theory for Lie groups, the proof of Theorem 1 relies on the following result whose proof occupies [8] and the majority of [5].

THEOREM 2 ([5], [8])
The following groups are not weakly amenable:

- $\mathbb{R}^n\rtimes\mathrm{SL}(2,\mathbb{R})$, *where* $n\geq 2$;
- $H_{2n+1}\rtimes\mathrm{SL}(2,\mathbb{R})$, *where* $n\geq 1$.

In this article, we are able to give a new and much simpler proof of Theorem 2 and, hence, implicitly also of Theorem 1. The new proof relies, among other things, on a result of Ozawa [22] about weakly amenable groups. Ozawa noted that his result gave a new proof of the nonweak amenability of $\mathbb{Z}^2\rtimes\mathrm{SL}(2,\mathbb{Z})$, which immediately implies the nonweak amenability of $\mathbb{R}^2\rtimes\mathrm{SL}(2,\mathbb{R})$.

In the study of weak amenability and related properties for Lie groups, the simply connected Lie groups are often more challenging to handle than, for instance, the Lie groups whose Levi factor has finite center (see, e.g., [9], [12]–[15]). This is partly because such groups are often not matrix groups and, thus, are more difficult to describe explicitly. However, our new method for proving Theorem 2 also applies to simply connected Lie groups (Theorem 3).

Let $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ be the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. The group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ acts on $\mathbb{R}^n$ and H_{2n+1} through the actions of $\mathrm{SL}(2,\mathbb{R})$. We prove that the universal covering groups of the groups in Theorem 2 are not weakly amenable.

THEOREM 3
The following groups are not weakly amenable:

- $\mathbb{R}^n\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{R})$, *where* $n\geq 2$;
- $H_{2n+1}\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{R})$, *where* $n\geq 1$.

As an application of Theorem 3, we completely settle the weak amenability question for simply connected Lie groups.

THEOREM 4
Let G be a connected, simply connected Lie group, and let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition of its Lie algebra. Let S be the associated semisimple Levi factor, and decompose the Lie algebra of S into simple ideals as $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$. The following are equivalent.

(1) *G is weakly amenable.*
(2) *For every $i = 1, \ldots, n$, one of the following holds:*

- *$\mathfrak{s}_i$ has real rank zero;*
- *$\mathfrak{s}_i$ has real rank one and $[\mathfrak{s}_i, \mathfrak{r}] = 0$.*

In that case, if S_i denotes the connected Lie subgroup of G associated with $\mathfrak{s}_i$, then

$$\Lambda_{\mathrm{WA}}(G) = \prod_{i=1}^{n} \Lambda_{\mathrm{WA}}(S_i).$$

Note that the value $\Lambda_{\mathrm{WA}}(S_i)$ is known for any simple Lie group S_i (see [5, p. 433]). We expect Theorem 4 to hold also without the assumption of simple connectedness.

1. Weak amenability and semidirect products

Let G be a locally compact group. A complex, continuous function $u\colon G \to \mathbb{C}$ is a *Herz–Schur multiplier* if there are a Hilbert space $\mathcal{H}$ and two bounded continuous functions $P, Q\colon G \to \mathcal{H}$ such that

$$u(y^{-1}x) = \langle P(x), Q(y) \rangle \quad \text{for every } x, y \in G.$$

The Herz–Schur norm of u is $\|u\|_{B_2} = \inf\{\|P\|_\infty \|Q\|_\infty\}$, where the infimum is taken over all P, Q as above. There are other well-known descriptions of Herz–Schur multipliers (see [2], [16], [24, Theorem 5.1]).

Recall that the group G is weakly amenable if there is a net $(u_i)_{i \in I}$ of compactly supported Herz–Schur multipliers on G converging to 1 uniformly on compact subsets of G and satisfying $\sup_i \|u_i\|_{B_2} \leq C$ for some $C \geq 1$. The infimum of those C for which such a net exists is denoted $\Lambda_{\mathrm{WA}}(G)$, with the understanding that $\Lambda_{\mathrm{WA}}(G) = \infty$ if G is not weakly amenable. We refer to [3, Section 12] for a nice introduction to weak amenability. We list below the behaviour of the weak amenability constant under some relevant group constructions (see, e.g., [6, Section 1] and [17]). These results will be needed in the proof of Theorem 4.

When K is a compact normal subgroup of G,

$$\Lambda_{\mathrm{WA}}(G/K) = \Lambda_{\mathrm{WA}}(G). \tag{1}$$

For a closed subgroup H of G,

$$\Lambda_{\mathrm{WA}}(H) \le \Lambda_{\mathrm{WA}}(G), \tag{2}$$

and if H is moreover co-amenable in G (and G is second countable), then equality holds:

$$\Lambda_{\mathrm{WA}}(H) = \Lambda_{\mathrm{WA}}(G). \tag{3}$$

For any two locally compact groups G and H,

$$\Lambda_{\mathrm{WA}}(G \times H) = \Lambda_{\mathrm{WA}}(G)\Lambda_{\mathrm{WA}}(H). \tag{4}$$

The following theorem is the basis for proving Theorems 2 and 3. It relies on Ozawa's work [22] by using the technique in [23, Corollary 2.12] (see also [3, Corollary 12.3.7]). Ozawa [22] proves that if a weakly amenable group G has an amenable closed normal subgroup N, then there is a state on $L^\infty(N)$ which is both left N-invariant and conjugation G-invariant.

THEOREM 5

Let $H \curvearrowright N$ be an action by automorphisms of a discrete group H on a discrete group N, and let $G = N \rtimes H$ be the corresponding semidirect product group. Let N_0 be a proper subgroup of N. Suppose

(1) *H is not amenable;*
(2) *N is amenable;*
(3) *N_0 is H-invariant;*
(4) *for every $x \in N \setminus N_0$, the stabilizer of x in H is amenable.*

Then G is not weakly amenable.

Proof

We suppose that G is weakly amenable and derive a contradiction. By [22, Theorem A], there is an N-invariant mean μ on $\ell^\infty(N)$ which is moreover H-invariant, where H acts on N by conjugation.

Since N_0 is H-invariant, the action $H \curvearrowright N$ restricts to an action $H \curvearrowright N \setminus N_0$. Let S be a system of representatives for the H-orbits in $N \setminus N_0$. For any $x \in S$, let

$$H_x = \{h \in H \mid h.x = x\}$$

be the stabilizer subgroup of x in H. We make the following identification of H-sets:

$$N = N_0 \sqcup \bigsqcup_{x \in S} H/H_x.$$

The stabilizer subgroup H_x is amenable by assumption, so we may choose a left H_x-invariant mean μ_x on $\ell^\infty(H_x)$. Define $\varphi_x \colon \ell^\infty(H) \to \ell^\infty(H/H_x)$ by averaging by μ_x, that is,

$$\varphi_x(f)(hH_x) = \int_{H_x} f(hy)\, d\mu_x(y), \quad f \in \ell^\infty(H).$$

Then φ_x is unital, positive, and H-equivariant. Collecting these maps, we obtain a unital, positive, H-equivariant map $\ell^\infty(H) \to \ell^\infty(N \setminus N_0)$. Since H is not amenable, the H-invariant mean μ is concentrated on N_0. But this contradicts the fact that μ is also N-invariant. □

2. Some semidirect product groups

For any natural number $n \geq 1$, the group $\mathrm{SL}(2,\mathbb{R})$ has a unique irreducible representation on $\mathbb{R}^n$ (see [19, p. 107]). It is described explicitly in [8, p. 710]. The semidirect product $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$ is defined using this representation. It is clear from the explicit description of the action in [8, p. 710] that $\mathrm{SL}(2,\mathbb{Z})$ leaves the integer lattice $\mathbb{Z}^n$ invariant so that $\mathbb{Z}^n \rtimes \mathrm{SL}(2,\mathbb{Z})$ is a well-defined subgroup of $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$.

Let H_{2n+1} denote the real Heisenberg group of dimension $2n+1$. We realize the Heisenberg group as $\mathbb{R}^{2n} \times \mathbb{R}$ with group multiplication given by

$$(u_1,t_1)(u_2,t_2) = \big(u_1+u_2, t_1+t_2+\langle u_1, Ju_2\rangle\big),$$

where J is the symplectic $2n \times 2n$ matrix defined by

$$J_{ij} = \begin{cases} (-1)^j & \text{if } i+j = 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

For $j = 1, \ldots, 2n$, let

$$\alpha_j = \binom{2n-1}{j-1}^{1/2}.$$

The irreducible representation Z of $\mathrm{SL}(2,\mathbb{R})$ of dimension $2n$ can be realized (in a different way than above) as

$$Z(A)_{ij} = \sum_{l=0}^{2n} \binom{j-1}{l}\binom{2n-j}{2n-i-l} \alpha_i^{-1}\alpha_j a^{2n-i-l} b^l c^{i+l-j} d^{j-l-1},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

We refer to [5, Section 2.1] for more details. In [5], it is shown that the map $\bar{Z}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(H_{2n+1})$ given by

$$\bar{Z}(A)(u,t) = \big(Z(A)u, t\big), \quad A \in \mathrm{SL}(2,\mathbb{R}), (u,t) \in H_{2n+1},$$

defines an action by automorphisms of $\mathrm{SL}(2,\mathbb{R})$ on H_{2n+1}. It is with respect to the action $\bar{Z}$ that we define the semidirect product $H_{2n+1} \rtimes \mathrm{SL}(2,\mathbb{R})$.

Consider the lattice $\Lambda_{2n} = \alpha_1^{-1}\mathbb{Z} \oplus \cdots \oplus \alpha_{2n}^{-1}\mathbb{Z}$ in $\mathbb{R}^{2n}$, and let

$$\Gamma_{2n+1} = \Big\{(u,t) \in H_{2n+1} \;\Big|\; u \in \Lambda_{2n}, t \in \frac{1}{N}\mathbb{Z}\Big\},$$

where $N = \alpha_1^2 \cdots \alpha_{2n}^2$.

LEMMA 6

We have that Γ_{2n+1} *is a discrete subgroup of* H_{2n+1} *which is invariant under the action of* $\mathrm{SL}(2,\mathbb{Z})$.

Proof

Observe that $\alpha_{2n+1-j} = \alpha_j$ for any $j = 1,\dots,2n$. It follows that $J\Lambda_{2n} = \Lambda_{2n}$, and $\langle u_1, Ju_2\rangle \in \frac{1}{N}\mathbb{Z}$ for any $u_1,u_2 \in \Lambda_{2n}$. This shows that Γ_{2n+1} is a subgroup of H_{2n+1}, and clearly Γ_{2n+1} is discrete. It is easily checked that if $A \in \mathrm{SL}(2,\mathbb{Z})$, then $Z(A)\Lambda_{2n} \subseteq \Lambda_{2n}$. It follows that Γ_{2n+1} is invariant under $\mathrm{SL}(2,\mathbb{Z})$. □

Let $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ be the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. The Lie group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is simply connected with a covering homomorphism $\pi\colon \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$. The kernel of π is a discrete normal subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ isomorphic to the group of integers. We let $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ act on $\mathbb{R}^n$ and H_{2n+1} through $\mathrm{SL}(2,\mathbb{R})$, and in this way we obtain the semidirect products

$$\mathbb{R}^n \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R}) \qquad \text{and} \qquad H_{2n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R}).$$

We define the subgroup $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ to be $\widetilde{\mathrm{SL}}(2,\mathbb{Z}) = \pi^{-1}(\mathrm{SL}(2,\mathbb{Z}))$ and obtain the semidirect products

$$\mathbb{Z}^n \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{Z}) \qquad \text{and} \qquad \Gamma_{2n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{Z}).$$

LEMMA 7

A proper, real algebraic subgroup of $\mathrm{SL}(2,\mathbb{R})$ *is amenable.*

Proof

Let H be a proper, real algebraic subgroup of $\mathrm{SL}(2,\mathbb{R})$. By a theorem of Whitney [27, Theorem 3], H has only finitely many components (in the usual Hausdorff topology) (see also [25, Theorem 3.6]). Hence, it suffices to show that the identity component H^0 of H is amenable.

Since H^0 is a connected, proper, closed subgroup of $\mathrm{SL}(2,\mathbb{R})$, its Lie algebra $\mathfrak{h}$ is a proper Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. Hence, the dimension of $\mathfrak{h}$ is at most two, and $\mathfrak{h}$ must be solvable. So H^0 is solvable and, in particular, amenable. □

LEMMA 8

Let $n \geq 2$. *For any* $x \in \mathbb{Z}^n$ *with* $x \neq 0$, *the stabilizer of* x *in* $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ *is amenable.*

Proof

The stabilizer in $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ is precisely the preimage under π of the stabilizer in $\mathrm{SL}(2,\mathbb{Z})$. Since the kernel of π is amenable and amenability is preserved under extensions, it suffices to show that the stabilizer in $\mathrm{SL}(2,\mathbb{Z})$ is amenable.

The stabilizer of x in $\mathrm{SL}(2,\mathbb{R})$ is a real algebraic subgroup. Moreover, since $x \neq 0$, the stabilizer of x is proper, and hence by Lemma 7, the stabilizer of x in $\mathrm{SL}(2,\mathbb{R})$ is amenable. It follows that the stabilizer in the closed subgroup $\mathrm{SL}(2,\mathbb{Z})$ is amenable. □

In the following lemma, we consider the action of $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ on Γ_{2n+1} previously described. Note that the center of Γ_{2n+1} is precisely $\{(u,t)\in\Gamma_{2n+1}\mid u=0\}$.

LEMMA 9

Let $n\geq 1$. For any noncentral $x\in\Gamma_{2n+1}$, the stabilizer of x in $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ is amenable.

Proof

As before, it suffices to prove that the stabilizer of x in $\mathrm{SL}(2,\mathbb{R})$ is amenable. If we write $x=(u,t)\in\Gamma_{2n+1}$, then the stabilizer of x in $\mathrm{SL}(2,\mathbb{R})$ is

$$\{A\in\mathrm{SL}(2,\mathbb{R})\mid Z(A)u=u\}.$$

Clearly, this is a real algebraic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Moreover, since $u\neq 0$, the stabilizer of x is proper. By Lemma 7, the stabilizer of x in $\mathrm{SL}(2,\mathbb{R})$ is amenable. □

Proof of Theorem 3

Case of $\mathbb{R}^n\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The group $\mathbb{Z}^n\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ is a closed subgroup (a lattice, in fact) of $\mathbb{R}^n\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{R})$, so it suffices to prove that $\mathbb{Z}^n\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ is not weakly amenable. This is a direct application of Theorem 5 with $H=\widetilde{\mathrm{SL}}(2,\mathbb{Z})$, $N=\mathbb{Z}^2$, and $N_0=\{0\}$.

Case of $H_{2n+1}\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The group $\Gamma_{2n+1}\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ is a closed subgroup (a lattice, in fact) of $H_{2n+1}\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{R})$, so it suffices to prove that $\Gamma_{2n+1}\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ is not weakly amenable. This is a direct application of Theorem 5 with $H=\widetilde{\mathrm{SL}}(2,\mathbb{Z})$, $N=\Gamma_{2n+1}$, and N_0 equal to the center of Γ_{2n+1}. □

Proof of Theorem 2

Similar to the proof of Theorem 3. One just has to replace $\widetilde{\mathrm{SL}}(2,\mathbb{Z})$ by $\mathrm{SL}(2,\mathbb{Z})$. □

REMARK 10

Note that we have, in fact, proved that the following discrete groups are not weakly amenable:

- $\mathbb{Z}^n\rtimes\mathrm{SL}(2,\mathbb{Z})$, where $n\geq 2$;
- $\mathbb{Z}^n\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{Z})$, where $n\geq 2$;
- $\Gamma_{2n+1}\rtimes\mathrm{SL}(2,\mathbb{Z})$, where $n\geq 1$;
- $\Gamma_{2n+1}\rtimes\widetilde{\mathrm{SL}}(2,\mathbb{Z})$, where $n\geq 1$.

3. Simply connected Lie groups

This section contains the proof of Theorem 4. We first review the structure theory of Lie groups needed in the proof, in particular, the Levi decomposition (see [26, Theorem 3.18.13]).

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We denote the solvable radical of $\mathfrak{g}$ by $\mathrm{rad}(\mathfrak{g})$ or $\mathfrak{r}$. In other words, $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$. There is a semisimple Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$. The semisimple Lie algebra $\mathfrak{s}$ is a direct sum $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ of simple Lie algebras (for some $n \geq 0$). If R and S denote the connected Lie subgroups of G associated with $\mathfrak{r}$ and $\mathfrak{s}$, respectively, then R is a closed, normal subgroup of G and S is maximal semisimple but not necessarily closed. Moreover, $G = RS$ as a set. The group S, which in general is not unique, is called a semisimple Levi factor. If G is simply connected, then S is closed, $R \cap S = \{1\}$, and $G = R \rtimes S$ as Lie groups.

For a connected, simply connected Lie group G, we will prove that the following are equivalent.

(1) G is weakly amenable.
(2) For every $i = 1, \ldots, n$, one of the following holds:

- $\mathfrak{s}_i$ has real rank zero;
- $\mathfrak{s}_i$ has real rank one and $[\mathfrak{s}_i, \mathfrak{r}] = 0$.

The following proposition can be found in [7] (see the proof of [7, Proposition 1.9]) and essentially appears already in [5]. Let $\mathfrak{v}_{n+1} \rtimes \mathfrak{sl}_2$ denote the Lie algebra of $\mathbb{R}^{n+1} \rtimes \mathrm{SL}(2,\mathbb{R})$, and let $\mathfrak{h}_{2n+1} \rtimes \mathfrak{sl}_2$ denote the Lie algebra of $H_{2n+1} \rtimes \mathrm{SL}(2,\mathbb{R})$.

PROPOSITION 11 ([5], [7])

Let $\mathfrak{g}$ be a Lie algebra with solvable radical $\mathfrak{r}$ and a Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$. Write $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ by separating compact factors $\mathfrak{s}_c$ (rank zero) and noncompact factors $\mathfrak{s}_{nc}$ (positive rank). Exactly one of the following holds.

(a) *All noncompact factors of $\mathfrak{s}$ commute with $\mathfrak{r}$: $[\mathfrak{r}, \mathfrak{s}_{nc}] = 0$.*

(b) *$\mathfrak{g}$ has a subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{v}_{n+1} \rtimes \mathfrak{sl}_2$ or $\mathfrak{h}_{2n+1} \rtimes \mathfrak{sl}_2$ for some $n \geq 1$, where $\mathrm{rad}(\mathfrak{h}) \subseteq \mathfrak{r}$ and $\mathfrak{sl}_2 \subseteq \mathfrak{s}_{nc}$.*

LEMMA 12

Let G be $\mathbb{R}^{n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$ or $H_{2n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$, where $n \geq 1$. The semisimple Levi factor of G is unique, and if Z is a central subgroup of G contained in the semisimple Levi factor, then G/Z is not weakly amenable.

Proof

If R is the solvable radical of G, then $[R,R]$ is central in G: the commutator group $[R,R]$ is trivial in the first case and is in the second case equal to the center of H_{2n+1}, which is also central in $H_{2n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$. By [26, Theorem 3.18.13], any two Levi factors of G are conjugate by an element of $[R,R]$, and hence, in our case, they are actually equal.

The center of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is isomorphic to the group of integers. If Z is the trivial group, we are done by Theorem 3. Otherwise, Z has finite index in the center of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, and G/Z is isomorphic up to a finite covering to $\mathbb{R}^{n+1} \rtimes \mathrm{SL}(2,\mathbb{R})$ or $H_{2n+1} \rtimes \mathrm{SL}(2,\mathbb{R})$. Then we are done by Theorem 2 and (1). □

Proof of Theorem 4
When G is simply connected, the Levi decomposition expresses G as a semidirect product $G = R \rtimes S$, where R is the solvable radical and S is closed and semisimple (see [26, Theorem 3.18.13]). Both R and S are simply connected. Decompose the Lie algebra of S into simple ideals $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$. Recall that two simply connected Lie groups with isomorphic Lie algebras are isomorphic. If S_i is a simply connected Lie group with Lie algebra $\mathfrak{s}_i$, then S is isomorphic to the direct product $S_1 \times \cdots \times S_n$. We split S into the compact factors S_c and noncompact factors S_{nc}, $S = S_c \times S_{nc}$.

Assume first that (2) holds. Then S_{nc} is a product of simple factors of rank one, so S_{nc} is weakly amenable (see [6], [15]). Moreover, S_{nc} is a direct factor in G and the quotient G/S_{nc} is $R \rtimes S_c$. As S_c is compact and R is solvable, the group $G/S_{nc} = R \rtimes S_c$ is amenable. It follows from (3) and (4) that G is weakly amenable with

$$\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WA}}(S_{nc}) = \prod_{i=1}^{n} \Lambda_{\mathrm{WA}}(S_i).$$

For the last equality, we also used the obvious fact that $\Lambda_{\mathrm{WA}}(S_c) = 1$, since S_c is compact.

Assume next that (2) does not hold. Let $\mathfrak{v}_{k+1} \rtimes \mathfrak{sl}_2$ denote the Lie algebra of $\mathbb{R}^{k+1} \rtimes \mathrm{SL}(2,\mathbb{R})$, and let $\mathfrak{h}_{2k+1} \rtimes \mathfrak{sl}_2$ denote the Lie algebra of $H_{2k+1} \rtimes \mathrm{SL}(2,\mathbb{R})$.

If some $\mathfrak{s}_i$ has real rank at least two, then the simple Lie group S_i is not weakly amenable (see [9, Theorem 1]), and since S_i is closed in G, it follows that G is not weakly amenable. Otherwise, some $\mathfrak{s}_i$ has real rank one, but $[\mathfrak{s}_i, \mathfrak{r}] \neq 0$. By Proposition 11, the Lie algebra $\mathfrak{g}$ contains a subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{v}_{k+1} \rtimes \mathfrak{sl}_2$ or $\mathfrak{h}_{2k+1} \rtimes \mathfrak{sl}_2$ for some $k \geq 1$, where $\mathrm{rad}(\mathfrak{h}) \subseteq \mathfrak{r}$ and $\mathfrak{sl}_2 \subseteq \mathfrak{s}$. Hence, G contains a Lie subgroup H locally isomorphic to $\mathbb{R}^{k+1} \rtimes \mathrm{SL}(2,\mathbb{R})$ or $H_{2k+1} \rtimes \mathrm{SL}(2,\mathbb{R})$. We claim that H is closed and not weakly amenable.

Let $\mathfrak{h} = \mathfrak{r}_0 \rtimes \mathfrak{s}_0$ be a Levi decomposition of $\mathfrak{h}$, that is, $\mathfrak{r}_0$ is $\mathfrak{v}_{k+1}$ or $\mathfrak{h}_{2k+1}$ and $\mathfrak{s}_0 = \mathfrak{sl}_2$. Let R_0 and S_0 denote the connected Lie subgroups of G associated with $\mathfrak{r}_0$ and $\mathfrak{s}_0$, respectively.

The group S_0 is a semisimple connected Lie subgroup of S and hence it is closed (see [20, p. 615]). Moreover, S_0 is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. The group R_0 is simply connected and closed in R (see [26, Theorem 3.18.12]). Clearly, S_0 normalizes R_0 and $H = R_0 S_0$, and since moreover $R \cap S = \{1\}$, we get that $H = R_0 \rtimes_\beta S_0$, where β denotes the conjugation action of S_0 on R_0. In particular, H is closed in G.

Let $\widetilde{S}_0$ be the universal cover of S_0 (so $\widetilde{S}_0 = \widetilde{\mathrm{SL}}(2,\mathbb{R})$), and consider the semidirect product $\widetilde{H} = R_0 \rtimes_\beta \widetilde{S}_0$, where $\widetilde{S}_0$ acts on R_0 through the covering $\widetilde{S}_0 \to S_0$ and the action of S_0 on R_0. The group $\widetilde{H}$ is simply connected and hence isomorphic to $\mathbb{R}^{k+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$ or $H_{2k+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$. The group H is a quotient of

$\widetilde{H}$ by a central subgroup contained in the Levi factor of $\widetilde{H}$, so by Lemma 12 the group H is not weakly amenable. It follows that G is not weakly amenable. □

One can also obtain the last part of Lemma 12 in a different way, by exploiting that $\mathbb{R}^{n+1}$ has relative property (T) in $\mathbb{R}^{n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$ and that H_{2n+1} has relative property (T) in $H_{2n+1} \rtimes \widetilde{\mathrm{SL}}(2,\mathbb{R})$ (see [7, Proposition 4.3]) combined with the following proposition. Details on relative property (T), also called property (T) of a pair, can be found in the book [1]. We thank the referee for pointing out the following proposition.

PROPOSITION 13

Let $N \rtimes H$ be a semidirect product of locally compact groups (where N is normal) with a continuous homomorphism $Q\colon N \rtimes H \to G$ into a σ-compact, locally compact group G. Assume that N is amenable and has relative property (T) in $N \rtimes H$. If G is weakly amenable, then $Q(N)$ is relatively compact.

Proof

The group H acts by conjugation on N, and N acts on itself by translation. These actions are compatible and define an action of $N \rtimes H$ on N. This also defines an action of $N \rtimes H$ on $Q(N)$ and by continuity on $\overline{Q(N)}$. Note that this action preserves the measure class of the Haar measure on $\overline{Q(N)}$ and so induces an action of $N \rtimes H$ on $L^2(\overline{Q(N)})$.

Since N is amenable, it follows from Ozawa's theorem [22, Theorem A] that there is a mean on $L^\infty(\overline{Q(N)})$ which is translation $\overline{Q(N)}$-invariant and conjugation G-invariant. In particular, the mean is $N \rtimes H$-invariant. By a standard argument, this is equivalent to the existence of almost $N \rtimes H$-invariant unit vectors in $L^2(\overline{Q(N)})$. Since N has relative property (T) in $N \rtimes H$, this implies the existence of an N-invariant unit vector in $L^2(\overline{Q(N)})$, so that $Q(N)$ must be relatively compact. □

We end with an application of Proposition 13 to some algebraic groups over local fields. Let K be a local field of characteristic zero. Then the semidirect products $K^{n+1} \rtimes \mathrm{SL}(2,K)$ and $H_{2n+1}(K) \rtimes \mathrm{SL}(2,K)$ are not weakly amenable when $n \geq 1$ (see [7] for more on these groups). Indeed, these semidirect products have relative property (T) by [7, Proposition 4.3], so Proposition 13 applies. When K is an arbitrary local field, possibly of positive characteristic, it is still true that the semidirect products $K^2 \rtimes \mathrm{SL}(2,K)$ and $K^3 \rtimes \mathrm{SL}(2,K)$ have relative property (T) (see [1, Corollary 1.4.13] and [1, Corollary 1.5.2]). Hence, these semidirect products are also not weakly amenable. Since these two groups are closed subgroups of $\mathrm{SL}(3,K)$ and $\mathrm{Sp}(4,K)$, respectively, it follows that the latter are also not weakly amenable. This has previously been shown by Lafforgue [18] in an unpublished manuscript, in which he also remarked that the weak amenability question for $K^2 \rtimes \mathrm{SL}(2,K)$ and $K^3 \rtimes \mathrm{SL}(2,K)$ was open. We record this as a final theorem.

THEOREM 14

Let K be a local field. The groups $K^2 \rtimes \mathrm{SL}(2,K)$ and $K^3 \rtimes \mathrm{SL}(2,K)$ are not weakly amenable. If, in addition, K has characteristic zero, then the groups $K^{n+1} \rtimes \mathrm{SL}(2,K)$ and $H_{2n+1}(K) \rtimes \mathrm{SL}(2,K)$ are not weakly amenable for $n \geq 1$.

COROLLARY 15 ([18])

Let K be a local field. The groups $\mathrm{SL}(3,K)$ and $\mathrm{Sp}(4,K)$ are not weakly amenable.

Acknowledgment. We are grateful to the referee for pointing out Proposition 13.

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