The equivariant integral cohomology ring of the flag manifold of type C

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Abstract We determine the T-equivariant integral cohomology ring of the flag manifold $\mathrm{Sp}(n)/T$ as a quotient ring of a polynomial ring, where T is a maximal torus of $\mathrm{Sp}(n)$ and acts on $\mathrm{Sp}(n)/T$ by left multiplication.

1. Introduction

Let G be a compact connected Lie group, and let T be a maximal torus of G. Goresky, Kottwitz, and MacPherson [GKM] gave a method to determine the T-equivariant cohomology ring of the flag manifold G/T combinatorially, where T acts on G/T by left multiplication. They regarded the equivariant cohomology ring $H_T^*(G/T) = H^*(ET \times_T G/T)$ as a subring of the equivariant cohomology ring of the T-fixed point set $(G/T)^T$ by using the fact that the restriction map $H_T^*(G/T) \to H_T^*((G/T)^T)$ is injective, and they gave a characterization of the elements of the image of the restriction map with complex coefficients. Harada, Holm, and Henriques [HHH] showed that the same characterization holds for the elements of the image with integer coefficients if G is simple and not of type C. Guillemin and Zara [GZ] introduced a special graph called a GKM graph which indicates the characterization as a diagram. This method is called the GKM theory. Using the GKM theory, Fukukawa, Ishida, and Masuda [FIM] determined the equivariant integral cohomology rings of flag manifolds of classical type except of type C, Fukukawa [F] determined that of the flag manifold of type G_2, and the author [S1], [S2] determined those of the flag manifolds of type F_4 and E_6. In fact, the equivariant cohomology ring of the flag manifold of type C was determined with $\mathbb{Z}[\frac{1}{2}]$-coefficients in [FIM]. In this article we determine the equivariant integral cohomology ring of the flag manifold of type C. Let T be the standard maximal torus of $\mathrm{Sp}(n)$, and let $\{t_i \mid 1 \le i \le n\}$ be the standard basis of $H^2(BT)$. The main result of this article is the following theorem.

THEOREM 1.1

The T-equivariant integral cohomology ring of $\mathrm{Sp}(n)/T$ *is given as*

$$H_T^*\big(\mathrm{Sp}(n)/T\big) \cong H^*(BT)[\tau_i \mid 1 \le i \le n]/\big(c_i(\tau^2) - c_i(t^2) \,\big|\, 1 \le i \le n\big),$$

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where $c_i(x^2)$ denotes the ith elementary symmetric polynomial in $x_1^2, \dots, x_n^2$ for $x = \tau$ or t.

2. GKM theory

Let G be a compact connected Lie group, and let T be a maximal torus of G. We define the *GKM graph* of G/T as follows. The GKM graph of G/T is a simple graph whose edges are equipped with elements of $H^2(BT)$. The vertex set is the Weyl group $W(G)$ and two vertices v, $v' \in W(G)$ are adjacent if and only if there exists a positive root α satisfying $\sigma_\alpha v = v'$, where σ_α denotes the reflection associated with α. Moreover, for the edge vv' satisfying $\sigma_\alpha v = v'$, we assign the positive root $\alpha \in H^2(BT)$, which is called the *label* of vv'.

According to [HHH, Theorem 2.3], the restriction

$$i^*\colon H_T^*(G/T) \to H_T^*\big((G/T)^T\big) \cong \prod_{W(G)} H^*(BT)$$

is injective. We often identify $H_T^*(G/T)$ with the image of i^*. An element f of $\prod_{W(G)} H^*(BT) \cong \mathrm{Map}(W(G), H^*(BT))$ is called a *GKM function* if it satisfies the following condition: for any root $\alpha \in \Phi(G)$ and $v \in W(G)$, $f(v) - f(\sigma_\alpha v) \in (\alpha) \subset H^*(BT)$, where, for elements $x_1, \dots, x_n$ of some ring, $(x_1, \dots, x_n)$ denotes the ideal generated by $x_1, \dots, x_n$. When G is simple and not of type C, Harada, Henriques, and Holm [HHH, Theorem 3.1 and Lemma 5.2] showed that the subring of $\mathrm{Map}(W(G), H^*(BT))$ consisting of all GKM functions coincides with the image of $i^*\colon H_T^*(G/T) \to \mathrm{Map}(W(G), H^*(BT))$. Next we define the GKM functions called the *equivariant Schubert classes*, which give an $H^*(BT)$-module basis of $H_T^*(G/T)$. Recall that for $w \in W(G)$ the length $l(w)$ denotes the number of positive roots which go to negative roots through w, and recall that the Bruhat order on $W(G)$ is the reflexive transitive closure of the following relation: for $w \in W(G)$ and a root α, if $l(\sigma_\alpha w) = l(w) + 1$, then $w \le \sigma_\alpha w$. Let Φ^+ be the set of all positive roots, and let Φ^- be the set of all negative roots. The equivariant Schubert classes $\{S_w\}_{w \in W(G)}$ are defined by the following conditions:

(1) for any $v \in W(G)$, $S_w(v)$ is 0 or homogeneous of degree $2l(w)$,
(2) $S_w(w) = \prod_{\alpha \in \Phi^+ \cap w(\Phi^-)} \alpha$,
(3) if $w \not\le v$, then $S_w(v) = 0$.

The equivariant Schubert classes exist and are unique (cf. [S2, Proposition 3.9 and 3.11]). We obtain the following proposition as an easy consequence of [HHH, Theorem 3.1] and the existence and uniqueness of the equivariant Schubert classes.

PROPOSITION 2.1
The equivariant Schubert classes $\{S_w\}_{w \in W(G)}$ are contained in the image of $i^*\colon H_T^*(G/T) \to \mathrm{Map}(W(G), H^*(BT))$ and form an $H^*(BT)$-basis of $H_T^*(G/T)$.

Let $\mathcal{G}(G/T)$ be the GKM graph of G/T, and let $H^*(\mathcal{G}(G/T))$ be the subring of $\mathrm{Map}(W(G), H^*(BT))$ consisting of all GKM functions. The Weyl group $W(G)$

acts on $H^2(BT)$ naturally, and we can extend the action onto $H^*(BT)$ naturally. Let us introduce the action of the Weyl group $W(G)$ on $H^*(\mathcal{G}(G/T))$: for a GKM function f on $\mathcal{G}(G/T)$, $w \in W(G)$, and a vertex v of $\mathcal{G}(G/T)$, the GKM function $w \cdot f$ is defined by

$$(w \cdot f)(v) = w\bigl(f(w^{-1}v)\bigr).$$

It is easily shown that $w \cdot f$ is also a GKM function. For a short while let us assume that Φ^+ is pairwise relatively prime in $H^*(BT)$; that is, for any distinct α, $\beta \in \Phi^+$, α and β are relatively prime in $H^*(BT)$. For $\alpha \in \Phi(G)$, let us define the *divided difference operator* $\delta_\alpha \colon H^*(\mathcal{G}(G/T)) \to H^*(\mathcal{G}(G/T))$ as follows: for any GKM function $f \in H^*(\mathcal{G}(G/T))$,

$$\delta_\alpha f = \frac{1}{\alpha}(f - \sigma_\alpha \cdot f).$$

One can easily see that $\delta_\alpha f$ is well defined from the formula

$$f(v) - (\sigma_\alpha \cdot f)(v) = (1 - \sigma_\alpha)f(v) + \sigma_\alpha\bigl(f(v) - f(\sigma_\alpha v)\bigr)$$

and that it is actually contained in $H^*(\mathcal{G}(G/T))$ from the formula

$$\delta_\alpha f(v) - \delta_\alpha f(\sigma_\beta v) = \frac{1}{\alpha}\Bigl(f(v) - f(\sigma_\beta v) - \sigma_\alpha\bigl(f(\sigma_\alpha v) - f(\sigma_{\sigma_\alpha \beta}\sigma_\alpha v)\bigr)\Bigr).$$

Note that we need the assumption on Φ^+ for $\delta_\alpha f \in H^*(\mathcal{G}(G/T))$, and note that $\delta_\alpha f$ is contained in $\mathrm{Map}(W(G), H^*(BT))$ for general G. Let w_0 denote the longest element of $W(G)$. Any element $w \in W(G)$ has the form $w = \sigma_{i_1} \cdots \sigma_{i_k} w_0$, where σ_{i_j} denotes the reflection associated to the simple root α_{i_j} and $l(w) = l(w_0) - k$ (cf. [BGG, Corollary 2.6]). By [S2, Lemma 3.10] we have

$$\delta_\alpha S_w = \begin{cases} S_{\sigma_\alpha w} & l(\sigma_\alpha w) < l(w), \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

This equation holds without the assumption on Φ^+. For any $f \in H_T^*(G/T)$, one can see that $\delta_\alpha f$ is also contained in $H_T^*(G/T)$ since the equivariant Schubert classes form an $H^*(BT)$-basis of $H_T^*(G/T)$. Therefore, we regard δ_α as an operator on $H_T^*(G/T)$ for general G.

PROPOSITION 2.2

For any root α and f, $g \in H_T^*(G/T)$, we have $\delta_\alpha(fg) = (\delta_\alpha f)g + (\sigma_\alpha \cdot f)\delta_\alpha g$.

Proof

By definition, for any vertex v, we have

$$\begin{aligned}\delta_\alpha(fg)(v) &= \frac{1}{\alpha}\bigl(fg(v) - \sigma_\alpha\bigl(fg(\sigma_\alpha^{-1}v)\bigr)\bigr) \\ &= \frac{1}{\alpha}\bigl(f(v) - \sigma_\alpha\bigl(f(\sigma_\alpha^{-1}v)\bigr)\bigr)g(v) + \frac{1}{\alpha}\bigl(g(v) - \sigma_\alpha\bigl(g(\sigma_\alpha^{-1}v)\bigr)\bigr)\sigma_\alpha\bigl(f(\sigma_\alpha^{-1}v)\bigr) \\ &= (\delta_\alpha f)(v)g(v) + (\sigma_\alpha \cdot f)(v)\delta_\alpha g(v).\end{aligned}$$

□

By Proposition 2.2, one can see that the action of $W(G)$ on $H^*(\mathcal{G}(G/T))$ is restricted onto $H_T^*(G/T)$ from the formula $\sigma_\alpha \cdot f = f - \alpha\delta_\alpha f$.

3. The equivariant cohomology ring of $\mathrm{Sp}(n)/T$

Let T be the standard maximal torus of $\mathrm{Sp}(n)$, and let $\{t_i \mid 1 \le i \le n\}$ be the standard basis of the dual of the Lie algebra of T. So $\{t_i \mid 1 \le i \le n\}$ is a basis of $H^2(BT)$. Then the root system $\Phi(\mathrm{Sp}(n))$ is given as

$$\Phi\bigl(\mathrm{Sp}(n)\bigr) = \{\pm t_i \pm t_j, \pm 2t_k \mid i \ne j, 1 \le i,j,k \le n\}.$$

By identifying $\pm t_i$ with $\pm i$, the Weyl group $W(\mathrm{Sp}(n))$ is given as

$$W\bigl(\mathrm{Sp}(n)\bigr) \cong \bigl\{\sigma\colon \pm[n] \to \pm[n], \text{ bijection} \bigm| i,j \in \pm[n], \sigma(i) = j \text{ implies } \sigma(-i) = -j\bigr\},$$

where $\pm[n] = \{\pm i \mid 1 \le i \le n\}$. Note that $W(\mathrm{Sp}(n))$ acts on the Lie algebra of $\mathrm{Sp}(n)$ and that the action is restricted onto $\{\pm t_i \mid 1 \le i \le n\}$. The signed permutation σ is uniquely determined by the sequence of its values $\sigma(1), \sigma(2), \dots, \sigma(n)$. Hence, when we refer to some element σ of $W(\mathrm{Sp}(n))$ concretely, we write σ as $\sigma(1)\,\sigma(2)\,\cdots\,\sigma(n)$.

For $1 \le i \le n$ let us define a GKM function τ_i on the GKM graph of $\mathrm{Sp}(n)/T$ as

$$\tau_i(w) = w(t_i) \quad \bigl(w \in W\bigl(\mathrm{Sp}(n)\bigr)\bigr).$$

For $2 \le i \le n$ let σ_i denote the reflection associated with the simple root $t_i - t_{i-1}$, and let σ_1 denote the reflection associated with the simple root $2t_1$. Since

$$S_{\sigma_i} = \sum_{k=i}^{n} (t_k - \tau_k)$$

for any i, the τ_i's are actually contained in $H_T^*(\mathrm{Sp}(n)/T)$.

The longest element w_0 of $W(\mathrm{Sp}(n))$ is $-1\,{-2}\,\cdots\,{-n}$, and actually it maps all positive roots to negative roots. By the characterization of the equivariant Schubert classes, the value of S_{w_0} is given as

$$S_{w_0}(v) = \begin{cases} \prod_{k=1}^{n}(-2t_k)\prod_{i<j}(-t_i - t_j)(-t_i + t_j) & v = w_0, \\ 0 & \text{otherwise,} \end{cases}$$

and we can describe S_{w_0} concretely as a polynomial in the τ_i's over $H^*(BT)$.

PROPOSITION 3.1

The equivariant Schubert class $S_{w_0} \in H_T^*(\mathrm{Sp}(n)/T)$ is given as

$$S_{w_0} = \prod_{k=1}^{n} (\tau_k - t_k) \prod_{i<j} (\tau_i - t_j)(\tau_i + t_j).$$

For the proof of Theorem 1.1 we need some algebraic preliminaries.

DEFINITION 3.2

A sequence $a_1, \dots, a_n$ of elements of a ring R is called *regular* if, for any i, a_i is not a zero divisor in $R/(a_1, \dots, a_{i-1})$ and $R/(a_1, \dots, a_n) \neq 0$.

Propositions 3.3 and 3.4 are obvious by definition.

PROPOSITION 3.3

If $a_1, \dots, a_n$ is a regular sequence, then so is $a_1, \dots, a_{i-1}, a_i + b, a_{i+1}, \dots, a_n$ for $1 \leq i \leq n$ and any $b \in (a_1, \dots, a_{i-1})$.

PROPOSITION 3.4

If $a_1, \dots, a_n$ is a regular sequence, then so is $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$ for $1 \leq i \leq n$.

THEOREM 3.5 ([M, THEOREM 17.4(III)])

Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring, and let $a_1, \dots, a_r$ be elements of $\mathfrak{m}$. Then the sequence $a_1, \dots, a_r$ is regular if and only if the height of the homogeneous ideal $(a_1, \dots, a_r)$ is r.

THEOREM 3.6 (CF. [NS, THEOREM 5.5.1])

Let F be a field, and let $R = F[g_i \mid 1 \leq i \leq m]$ be a nonnegatively graded polynomial ring with degree $|g_i| > 0$ for any $1 \leq i \leq m$. Assume that $a_1, \dots, a_n$ is a regular sequence in R, which consists of homogeneous elements of positive degree. Then the Poincaré series of $R/(a_i \mid 1 \leq i \leq n)$ is given as

$$\frac{\prod_{i=1}^{n}(1 - x^{|a_i|})}{\prod_{i=1}^{m}(1 - x^{|g_i|})}.$$

Proof

For a nonnegatively graded F-module M of finite type, let $P(M, x)$ denote the Poincaré series of M, namely,

$$P(M, x) = \sum_{n=0}^{\infty} (\dim_F M_n) x^n,$$

where M_n denotes the degree n part of M. Then obviously we have

$$P(R, x) = \frac{1}{\prod_{i=1}^{m}(1 - x^{|g_i|})}.$$

Since $a_1, \dots, a_n$ is a regular sequence, the multiplication by a_i induces an injection on a graded F-module $R/(a_1, \dots, a_{i-1})$. Therefore,

$$P\big(R/(a_1, \dots, a_i), x\big) = (1 - x^{|a_i|}) P\big(R/(a_1, \dots, a_{i-1}), x\big).$$

The induction on i completes the proof. $\square$

Proof of Theorem 1.1
Since divided difference operators decrease the degree by two, any divided difference operator maps τ_i's and t_i's to constant functions. Hence, by Proposition 2.2, Proposition 3.1, and (2.1), all equivariant Schubert classes are written as a polynomial in the τ_i's over $H^*(BT)$. Therefore, by Proposition 2.1, $H_T^*(\mathrm{Sp}(n)/T)$ is generated by the τ_i's as an $H^*(BT)$-algebra.

Since $W(\mathrm{Sp}(n))$ is the signed permutation group on n letters, for $1 \le i \le n$ we have $c_i(\tau^2) - c_i(t^2) = 0$ as a GKM function. The natural surjection $H^*(BT)[\tau_i \mid 1 \le i \le n] \to H_T^*(\mathrm{Sp}(n)/T)$ factors through $H^*(BT)[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) - c_i(t^2) \mid 1 \le i \le n)$. We will show that $H^*(BT)[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) - c_i(t^2) \mid 1 \le i \le n)$ is a free $\mathbb{Z}$-module and has the same rank with $H_T^*(\mathrm{Sp}(n)/T)$ at each degree by an argument of regular sequences. Then we will see that the surjective homomorphism

$$H^*(BT)[\tau_i \mid 1 \le i \le n]/\big(c_i(\tau^2) - c_i(t^2) \bigm| 1 \le i \le n\big) \to H_T^*\big(\mathrm{Sp}(n)/T\big)$$

is an isomorphism.

Let us show that the sequence

$$c_1(\tau^2) - c_1(t^2), c_2(\tau^2) - c_2(t^2), \ldots, c_n(\tau^2) - c_n(t^2) \tag{3.1}$$

is a regular sequence in $H^*(BT)[\tau_i \mid 1 \le i \le n] \otimes (\mathbb{Z}/p\mathbb{Z})$ for any prime number p. Then, by Theorem 3.6, we will see that the Poincaré series of $H^*(BT)[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) - c_i(t^2) \mid 1 \le i \le n) \otimes (\mathbb{Z}/p\mathbb{Z})$ does not depend on p. Hence, $H^*(BT)[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) - c_i(t^2) \mid 1 \le i \le n)$ must be free because it is of finite type.

By Propositions 3.3 and 3.4, it is sufficient to show that the sequence

$$t_1, t_2, \ldots, t_n, c_1(\tau^2), c_2(\tau^2), \ldots, c_n(\tau^2) \tag{3.2}$$

is regular in $H^*(BT)[\tau_i \mid 1 \le i \le n] \otimes (\mathbb{Z}/p\mathbb{Z})$. Since the ordinary cohomology $H^*(\mathrm{Sp}(n)/T) \cong \mathbb{Z}[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) \mid 1 \le i \le n)$ (cf. [B, Proposition 30.2]) is finite, one can see that the height of $(c_i(\tau^2) \mid 1 \le i \le n)$ is n by dimensional reasoning. Since a polynomial ring over a field is Cohen–Macaulay, by Theorem 3.5, the sequence $c_1(\tau^2), \ldots, c_n(\tau^2)$ is regular in $\mathbb{Z}/p\mathbb{Z}[\tau_i \mid 1 \le i \le n]$ for any p. Hence, (3.2) is a regular sequence.

By Theorem 3.6, the Poincaré series of the graded $\mathbb{Z}/p\mathbb{Z}$-module $H^*(BT)[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) - c_i(t^2) \mid 1 \le i \le n) \otimes (\mathbb{Z}/p\mathbb{Z})$ is given as

$$\frac{\prod_{i=1}^{n}(1 - x^{2i})}{(1 - x^2)^{2n}}, \tag{3.3}$$

and it does not depend on p and coincides with the Poincaré series of the graded $\mathbb{Z}$-module $H^*(BT)[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) - c_i(t^2) \mid 1 \le i \le n)$. By the Serre spectral sequence of the fibration $\mathrm{Sp}(n)/T \to ET \times_T \mathrm{Sp}(n)/T \to BT$, one can see that the Poincaré series of $H_T^*(\mathrm{Sp}(n)/T)$ is the product of those of the graded $\mathbb{Z}$-modules $H^*(BT)$ and $H^*(\mathrm{Sp}(n)/T) \cong \mathbb{Z}[\tau_i \mid 1 \le i \le n]/(c_i(\tau^2) \mid 1 \le i \le n)$, and it coincides with (3.3). □

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