Numerical adjunction formulas for weighted projective planes and lattice point counting

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Abstract This article gives an explicit formula for the Ehrhart quasipolynomial of certain 2-dimensional polyhedra in terms of invariants of surface quotient singularities. Also, a formula for the dimension of the space of quasihomogeneous polynomials of a given degree is derived. This admits an interpretation as a numerical adjunction formula for singular curves on the weighted projective plane.

1. Introduction

This article deals with the general problem of counting lattice points in a polyhedron with rational vertices and its connection with both the singularity theory of surfaces and adjunction formulas for curves in the weighted projective plane. In addition, we focus on rational polyhedra (whose vertices are rational points) as opposed to lattice polyhedra (whose vertices are integers). Our approach exploits the connection between Dedekind sums (as originated from the work of Hirzebruch and Zagier [18]) and the geometry of cyclic quotient singularities, which has been proposed by several authors (see, e.g., [23], [7], [10], [19], [28], [11], [4], [5]).

According to Ehrhart [17], the number of integer points of a lattice (resp., rational) polygon $\mathcal{P}$ and its dilations $d\mathcal{P} = \{dp \mid p \in \mathcal{P}\}$ is a polynomial (resp., quasipolynomial) in d of degree $\dim \mathcal{P}$ referred to as the *Ehrhart (quasi-)polynomial* of $\mathcal{P}$ (cf. [12]). In this article we focus on the Ehrhart quasipolynomial of polygons of type

$$dD_w = D_{w,d} = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x,y,z \geq 0, w_0x + w_1y + w_2z = d\right\}, \tag{1}$$

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where $w=(w_0,w_1,w_2)$ are pairwise coprime and

$$D_w := \left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; x,y,z\geq 0, w_0x+w_1y+w_2z=1\right\} \tag{2}$$

is a rational polygon. In Theorem 1.1 we give an explicit formula for the Ehrhart quasipolynomial of (1), which in Theorem 1.2 is shown to be an invariant of the quotient singularities of the weighted projective plane $\mathbb{P}^2_w$.

Throughout this article, w_0,w_1,w_2 are assumed to be pairwise coprime integers. Denote by $w=(w_0,w_1,w_2)$, $\bar{w}=w_0w_1w_2$, and $|w|=w_0+w_1+w_2$. Finally, the key ingredients to connect the arithmetical problem referred to above with the geometry of weighted projective planes come from the observation that

$$\mathrm{L}_w(d) := \#(D_{w,d}\cap\mathbb{Z}^3) = h^0\bigl(\mathbb{P}^2_w;\mathcal{O}(d)\bigr), \tag{3}$$

that is, the dimension of the vector space of weighted homogeneous polynomials of degree d, and from a numerical adjunction formula relating $h^0(\mathbb{P}^2_w;\mathcal{O}(d))$ with the genus of a curve in $\mathbb{P}^2_w$.

To explain what we mean by *numerical adjunction formulas*, assume that a quasismooth curve $\mathcal{C}\subset\mathbb{P}^2_w$ of degree d exists. In that case, according to the classical adjunction formula, one has the following equality relating canonical divisors on $\mathcal{C}$ and $\mathbb{P}^2_w$:

$$K_{\mathcal{C}} = (K_{\mathbb{P}^2_w}+\mathcal{C})|_{\mathcal{C}}. \tag{4}$$

Equating degrees on both sides of (4) and using the weighted Bézout's theorem, one has

$$2g(\mathcal{C})-2 = \deg(K_{\mathbb{P}^2_w}+\mathcal{C})|_{\mathcal{C}} = \frac{\deg(\mathcal{C})\deg(K_{\mathbb{P}^2_w}+\mathcal{C})}{\bar{w}} = \frac{d(d-|w|)}{\bar{w}}.$$

Notice that the generic curve of degree $k\bar{w}$ is smooth (see [15, Lemma 5.4]). In that case, one has (cf. Section 4)

$$h^0\bigl(\mathbb{P}^2_w;\mathcal{O}\bigl(k\bar{w}-|w|\bigr)\bigr) = \mathrm{L}_w\bigl(k\bar{w}-|w|\bigr) = g_{w,k\bar{w}}, \tag{5}$$

where $g_{w,t} := \frac{t(t-|w|)}{2\bar{w}}+1$.

However, for a general d the generic curve of degree d in $\mathbb{P}^2_w$ is not necessarily quasismooth (see [15, Lemma 5.4]). The final goal of this article is to revisit (5) in the general (singular) case.

Let us present the main results of this work. The first main statement shows an explicit formula for the Ehrhart quasipolynomial $\mathrm{L}_w(d)$ of degree two of $D_{w,d}$ in terms of d.

THEOREM 1.1
The Ehrhart quasipolynomial $\mathrm{L}_w(d)$ for the polygon D_w in (2) satisfies

$$\mathrm{L}_w(d) = g_{w,d+|w|} - \sum_{P\in\mathrm{Sing}(\mathbb{P}^2_w)} \Delta_P\bigl(d+|w|\bigr).$$

The quadratic polynomial $g_{w,k} = \frac{k(k-|w|)}{2\bar{w}} + 1$ in k is called the *virtual genus* (see [15, Definition 5.1]) and $\Delta_P(k)$ is a periodic function of period $\bar{w}$ which is an invariant associated to the singularity $P \in \operatorname{Sing}(\mathbb{P}^2_w)$ (see Definition 3.8). The proof of Theorem 1.1 relies heavily on computations with Dedekind sums.

The next result aims to show that the previous combinatorial number $\Delta_P(k)$ has a geometric interpretation and can be computed via invariants of curve singularities on a singular surface. To do so, we recall the recently defined invariant $\delta_P(f)$ of a curve $(\{f=0\}, P)$ on a surface with quotient singularity (see [15, Section 4.2]), and we define a new invariant $\kappa_P(f)$ in Section 3.1.

THEOREM 1.2
Let (f, P) be a reduced curve germ at $P \in X$, a surface cyclic quotient singularity. Then

$$\Delta_P(k) = \delta_P(f) - \kappa_P(f)$$

for any reduced germ $f \in \mathcal{O}_{X,P}(k)$.

The module $\mathcal{O}_{X,P}(k)$ of k-invariant germs of X at P can be found in Definition 2.3.

As an immediate consequence of Theorems 1.1 and 1.2 one has a method to compute $\mathrm{L}_w(d)$ by means of appropriate curve germs $(\{f=0\}, P)$ on surface quotient singularities. In an upcoming article (see [14]), we will study the $\Delta_P(k)$-invariant by means of singularity theory and intersection theory on surface quotient singularities to give a closed effective formula for the Ehrhart quasipolynomial $\mathrm{L}_w(d)$. In fact, Theorem 1.1 can also be seen as a version of [7], where an explicit interpretation of the *correction term* is given in Theorem 1.2.

Finally, we generalize the numerical adjunction formula for a general singular curve $\mathcal{C}$ on $\mathbb{P}^2_w$ relating $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2_w}(d-|w|))$, its genus $g(\mathcal{C})$, and the newly defined invariant κ_P.

THEOREM 1.3 (NUMERICAL ADJUNCTION FORMULA)
Consider $\mathcal{C} = \{f=0\} \subset \mathbb{P}^2_w$ an irreducible curve of degree d. Then

$$h^0\big(\mathbb{P}^2_w, \mathcal{O}_{\mathbb{P}^2_w}(d-|w|)\big) = g(\mathcal{C}) + \sum_{P \in \operatorname{Sing}(\mathcal{C})} \kappa_P(f).$$

This article is organized as follows. In Section 2 some basic definitions and preliminary results on surface quotient singularities, logarithmic forms, and Dedekind sums are given. In Section 3, after defining the three local invariants mentioned above, a proof of Theorem 1.2 is given. An introductory example is treated in Section 4, and finally, the main results, Theorems 1.1 and 1.3, are proven in Section 5.

2. Definitions and preliminaries

In this section some needed definitions and results are provided.

2.1. V-manifolds and quotient singularities

We start by giving some basic definitions and properties of V-manifolds, weighted projective spaces, embedded $\mathbf{Q}$-resolutions, and weighted blowups (for a detailed exposition see, e.g., [16], [2], [3], [20], [22]). Let us fix the notation and introduce several tools to calculate a special kind of embedded resolution, called *embedded* $\mathbf{Q}$*-resolutions* (see Definition 2.4), for which the ambient space is allowed to contain abelian quotient singularities. To do this, we study weighted blowups at points.

DEFINITION 2.1

A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(n,\mathbb{C})$.

We are interested in V-surfaces where the quotient spaces B_i/G_i are given by (finite) abelian groups.

Let $\boldsymbol{G}_d \subset \mathbb{C}^*$ be the cyclic group of dth roots of unity generated by ξ_d. Consider a vector of weights $(a,b) \in \mathbb{Z}^2$ and the action

$$(6) \qquad \begin{aligned} \boldsymbol{G}_d \times \mathbb{C}^2 &\xrightarrow{\rho} \mathbb{C}^2, \\ \big(\xi_d,(x,y)\big) &\mapsto (\xi_d^a x, \xi_d^b y). \end{aligned}$$

The set of all orbits $\mathbb{C}^2/\boldsymbol{G}_d$ is called a *cyclic quotient space of type* $(d;a,b)$ and it is denoted by $X(d;a,b)$.

The type $(d;a,b)$ is *normalized* if and only if $\gcd(d,a)=\gcd(d,b)=1$. If this is not the case, then one uses the isomorphism (assuming $\gcd(d,a,b)=1$)

$$\begin{aligned} X(d;a,b) &\longrightarrow X\Big(\frac{d}{(d,a)(d,b)}; \frac{a}{(d,a)}, \frac{b}{(d,b)}\Big), \\ \big[(x,y)\big] &\mapsto \big[(x^{(d,b)}, y^{(d,a)})\big] \end{aligned}$$

to normalize it.

We present different properties of some important sheaves associated to a V-surface (see [8, Section 4], [16]).

PROPOSITION 2.2 ([8])

Let $\mathcal{O}_X$ be the structure sheaf of a V-surface X.

- *If P is not a singular point of X, then $\mathcal{O}_{X,P}$ is isomorphic to the ring of convergent power series $\mathbb{C}\{x,y\}$.*
- *If P is a singular point of X, then $\mathcal{O}_{X,P}$ is isomorphic to the ring of $\boldsymbol{G}_d$-invariant convergent power series $\mathbb{C}\{x,y\}^{\boldsymbol{G}_d}$.*

If no ambiguity seems likely to arise, then we simply write $\mathcal{O}_P$ for the corresponding local ring or just $\mathcal{O}$ in the case $P=0$.

DEFINITION 2.3

Let $\boldsymbol{G}_d$ be an arbitrary finite cyclic group, and let $(a,b)\in\mathbb{Z}^2$ be a vector of weights. Consider the action given in (6). Associated with $X(d;a,b)$ one has the following $\mathcal{O}_{X,P}$-module:

$$\mathcal{O}_{X,P}(k) := \left\{h\in\mathbb{C}\{x,y\} \;\middle|\; h(\xi_d^a x, \xi_d^b y) = \xi_d^k h(x,y)\right\},$$

also known as the module of *k-invariant* germs in $X(d;a,b)$.

REMARK 2.1

Note that

$$\mathbb{C}\{x,y\} = \bigoplus_{k=0}^{d-1} \mathcal{O}_{X,P}(k). \tag{7}$$

REMARK 2.2 ([8])

Let $l,k\in\mathbb{Z}$. Using the notation above one clearly has the following properties:

- $\mathcal{O}_{X,P}(k) = \mathcal{O}_{X,P}(d+k)$,
- $\mathcal{O}_{X,P}(l)\otimes\mathcal{O}_{X,P}(k)\subset\mathcal{O}_{X,P}(l+k)$.

These modules produce the corresponding sheaves $\mathcal{O}_X(k)$ on a V-surface X, which are also called *orbisheaves*.

One of the main examples of V-surfaces is the so-called *weighted projective plane* (e.g., [16]). Let $w:=(w_0,w_1,w_2)\in\mathbb{Z}^3_{>0}$ be a weight vector, that is, a triple of pairwise coprime positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^3\setminus\{0\}$ given by

$$(x_0,x_1,x_2)\longmapsto(t^{w_0}x_0, t^{w_1}x_1, t^{w_2}x_2).$$

The universal geometric quotient of $\frac{\mathbb{C}^3\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^2_w$ and it is called the *weighted projective plane* of type w.

Let us recall the adapted concept of resolution in this category.

DEFINITION 2.4 ([20])

An *embedded* $\mathbf{Q}$*-resolution* of a hypersurface $(H,0)\subset(M,0)$ in an abelian quotient space is a proper analytic map $\pi: X\to(M,0)$ such that

(1) X is a V-manifold with abelian quotient singularities,
(2) π is an isomorphism over $X\setminus\pi^{-1}(\mathrm{Sing}(H))$,
(3) $\pi^{-1}(H)$ is a $\mathbf{Q}$-normal crossing hypersurface on X (see [26, Definition 1.16]).

Embedded $\mathbf{Q}$-resolutions are a natural generalization of the usual embedded resolutions for which some invariants such as δ can be effectively calculated (see

[15]). As a key tool to construct embedded **Q**-resolutions of abelian quotient surface singularities we will recall toric transformations or weighted blowups in this context (see [21] as a general reference), which can be interpreted as blowups of $\mathfrak{m}$-primary ideals.

Let X be an analytic surface with abelian quotient singularities. Let us define the weighted blowup $\pi : \widehat{X} \to X$ at a point $P \in X$ with respect to $w = (p, q)$. Since it will be used throughout the article, we briefly describe the local equations of a weighted blowup at a point P of type $(d; a, b)$ (see [20, Chapter 1] for further details).

The birational morphism $\pi = \pi_{(d;a,b),w} : \widehat{X(d;a,b)}_w \to X(d;a,b)$ can be described as usual by covering $\widehat{X(d;a,b)}_w$ into two charts $\widehat{U}_1 \cup \widehat{U}_2$, where for instance $\widehat{U}_1$ is of type $X(\frac{pd}{e}; 1, \frac{-q+a'pb}{e})$, with $a'a = b'b \equiv 1 \mod (d)$ and $e = \gcd(d, pb - qa)$. The first chart is given by

$$
\begin{aligned}
X\Big(\frac{pd}{e}; 1, \frac{-q+a'pb}{e}\Big) &\longrightarrow \widehat{U}_1, \\
\big[(x^e, y)\big] &\mapsto \big[\big((x^p, x^q y), [1:y]_w\big)\big]_{(d;a,b)}.
\end{aligned} \tag{8}
$$

The second one is given analogously.

The exceptional divisor $E = \pi^{-1}_{(d;a,b),w}(0)$ is identified with $\mathbb{P}^1_w(d;a,b) := \mathbb{P}^1_w / \boldsymbol{G}_d$. The singular points are cyclic and correspond to the origins of the two charts.

2.2. Log-resolution logarithmic forms

All the preliminaries about de Rham cohomology for projective varieties with quotient singularities can be found in [26, Chapter 1], and the ones about the C^∞ log complex of quasiprojective algebraic varieties can be found in [13, Section 1.3]. Here we focus on the nonnormal crossing $\mathbb{Q}$-divisor case in weighted projective planes.

Let D be a $\mathbb{Q}$-divisor in a surface X with quotient singularities. The complement of D will be denoted by X_D. Let us fix a **Q**-resolution $\pi : Y \longrightarrow X$ of the singularities of D so that the reduced $\mathbb{Q}$-divisor $\bar{D} = \pi^*(D)_{\text{red}}$ is a union of smooth $\mathbb{Q}$-divisors on Y with **Q**-normal crossings.

Using the results in [26] we can generalize in [13, Definition 2.7] for a nonnormal crossing $\mathbb{Q}$-divisor in X.

DEFINITION 2.5

The sheaf $\pi_* \Omega_Y(\log\langle \bar{D} \rangle)$ is called the sheaf of *log-resolution logarithmic forms* on X with respect to D.

REMARK 2.3

Note that the space Y in the previous definition is not smooth, and we use the standard definition of logarithmic sheaf for V-manifolds and V-normal crossing divisors $\bar{D}$ due to Steenbrink [26].

In what follows, a log-resolution logarithmic form with respect to a $\mathbb{Q}$-divisor D and a $\mathbf{Q}$-resolution π will be referred to as simply a logarithmic form if D and π are known and no ambiguity is likely to arise.

REMARK 2.4

Let h be an analytic germ on $X(d;a,b)$ where the type is normalized. Notice that $\omega = h\frac{dx\wedge dy}{xy}$ automatically defines a logarithmic form with poles along xy, whereas expressions of the form $\tilde{\omega} = h\frac{dx\wedge dy}{x}$ might not even define a 2-form unless h is such that $\tilde{\omega}$ is invariant under $\boldsymbol{G}_d$.

Also note that $\pi_*\Omega_Y(\log\langle\bar{D}\rangle)$ depends, in principle, on the given resolution π. The following result shows that this is not the case.

PROPOSITION 2.6

The sheaves $\pi_*\Omega^\bullet{}_Y(\log\langle\bar{D}\rangle)$ of logarithmic forms on X with respect to the $\mathbb{Q}$-divisor D do not depend on the chosen $\mathbf{Q}$-resolution.

Proof

Let Y and Y' be two $\mathbf{Q}$-resolutions of (X,D). After resolving $(Y,\bar{D})$ and $(Y',\bar{D}')$ and applying the strong factorization theorem for smooth surfaces, there exists a smooth surface $\tilde{Y}$ obtained as a finite number of blowups of both Y and Y', which is a common resolution of $(Y,\bar{D})$ and $(Y',\bar{D}')$. Note that (see [26, p. 351])

$$\Omega^\bullet_Y\big(\log\langle\bar{D}\rangle\big) = \rho_*\Omega^\bullet_{\tilde{Y}}\big(\log\langle\tilde{D}\rangle\big) \qquad \text{and} \qquad \Omega^\bullet_{Y'}\big(\log\langle\bar{D}'\rangle\big) = \rho'_*\Omega^\bullet_{\tilde{Y}}\big(\log\langle\tilde{D}\rangle\big),$$

where $\bar{D}$, $\bar{D}'$, and $\tilde{D}$ are the corresponding total preimages and ρ and ρ' are the corresponding resolutions. The result follows, since

$$\pi_*\Omega^\bullet_Y\big(\log\langle\bar{D}\rangle\big) = \pi_*\rho_*\Omega^\bullet_{\tilde{Y}}\big(\log\langle\tilde{D}\rangle\big) = \pi'_*\rho'_*\Omega^\bullet_{\tilde{Y}}\big(\log\langle\tilde{D}\rangle\big) = \pi'_*\Omega^\bullet_{Y'}\big(\log\langle\bar{D}'\rangle\big)$$

due to the commutativity of the diagram $\pi\rho = \pi'\rho'$. □

NOTATION 2.7

In the future, we will refer to such sheaves as logarithmic sheaves on D and they will be denoted simply as $\Omega^\bullet_X(\mathrm{LR}\langle D\rangle)$.

2.3. Dedekind sums

Let a,b,c,t be positive integers with $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$. The aim of this part (see [6] for further details) is to give a way to compute the cardinal of the following two sets:

$$\Delta_1 := \big\{(x,y)\in\mathbb{Z}^2_{\geq 0} \;\big|\; ax+by\leq t\big\},$$
$$\Delta_2 := \big\{(x,y,z)\in\mathbb{Z}^3_{\geq 0} \;\big|\; ax+by+cz = t\big\}.$$

Note that $\#\Delta_1$ cannot be computed by means of Pick's theorem unless t is divisible by a and b.

Denote by $L_{\Delta_i}(t)$ the cardinal of Δ_i. Let us consider the following notation.

NOTATION 2.8

Denote by $\xi_a := e^{\frac{2i\pi}{a}}$. Consider

$$
\begin{aligned}
p_{\{a,b,c\}}(t) := \operatorname{poly}_{\{a,b,c\}}(t) + \frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{(1-\xi_a^{kb})(1-\xi_a^{kc})\xi_a^{kt}} \\
+ \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{(1-\xi_b^{ka})(1-\xi_b^{kc})\xi_b^{kt}} + \frac{1}{c}\sum_{k=1}^{c-1} \frac{1}{(1-\xi_c^{ka})(1-\xi_c^{kb})\xi_c^{kt}},
\end{aligned} \tag{9}
$$

with

$$
\operatorname{poly}_{\{a,b,c\}}(t) := \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) + \frac{3(ab+ac+bc)+a^2+b^2+c^2}{12abc}.
$$

REMARK 2.5

Notice that, in particular, one has

$$
\begin{aligned}
p_{\{a,b,1\}}(t) = \operatorname{poly}_{\{a,b,1\}}(t) \\
+ \frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{(1-\xi_a^{kb})(1-\xi_a^{k})\xi_a^{kt}} + \frac{1}{b}\sum_{k=1}^{b-1} \frac{1}{(1-\xi_b^{ka})(1-\xi_b^{k})\xi_b^{kt}},
\end{aligned} \tag{10}
$$

with

$$
\operatorname{poly}_{\{a,b,1\}}(t) = \frac{t^2}{2ab} + \frac{t}{2}\left(\frac{1}{ab} + \frac{1}{a} + \frac{1}{b}\right) + \frac{3(ab+a+b)+a^2+b^2+1}{12ab}.
$$

THEOREM 2.9 ([6])

One has

$$
L_{\Delta_1}(t) = p_{\{a,b,1\}}(t) \qquad \textit{and} \qquad L_{\Delta_2}(t) = p_{\{a,b,c\}}(t).
$$

Now we are going to define the Dedekind sums, giving some properties which will be particularly useful for future results. See [24] and [6] for a more detailed exposition.

DEFINITION 2.10 ([24])

Let a, b be integers, with $\gcd(a,b) = 1$ and $b \geq 1$. The Dedekind sum $s(a,b)$ is defined as

$$
s(a,b) := \sum_{j=1}^{b-1} \left(\left(\frac{ja}{b}\right)\right)\left(\left(\frac{j}{b}\right)\right), \tag{11}
$$

where the symbol $((x))$ denotes

$$
((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer,} \end{cases}
$$

with $[x]$ being the greatest integer not exceeding x.

The following result, referred to as a reciprocity theorem (see [6, Corollary 8.5] or [24, Theorem 2.1] for further details), will be key in what follows.

THEOREM 2.11 (RECIPROCITY THEOREM; SEE [6], [24])

Let a and b be two coprime integers. Then

$$s(a,b)+s(b,a)=-\frac{1}{4}+\frac{1+a^2+b^2}{12ab}.$$

Let us express the sum (11) in terms of ath roots of the unity (see, e.g., [6, Example 8.1] or [24, Chapter 2, (18b)] for further details).

PROPOSITION 2.12 ([6], [24])

Let a,b be integers, with $\gcd(a,b)=1$ and $b\geq 1$. Denote by ξ_b a primitive bth root of unity. The Dedekind sum $s(a,b)$ can be written as

$$s(a,b)=\frac{b-1}{4b}-\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{ka})(1-\xi_b^{k})}.$$

Let us exhibit some useful properties of the Dedekind sum $s(a,b)$.

Since $((-x))=-((x))$ it is clear that

$$s(-a,b)=-s(a,b)$$

and also

$$s(a,-b)=s(a,b).$$

If we define a' by $a'a\equiv 1 \mod b$, then

$$s(a',b)=s(a,b).$$

PROPOSITION 2.13 ([6], [24])

Let a,b,c be integers with $\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=1$. Define a' by $a'a\equiv 1$ $\mod b$, b' by $b'b\equiv 1 \mod c$, and c' by $c'c\equiv 1 \mod a$. Then

$$s(bc',a)+s(ca',b)+s(ab',c)=-\frac{1}{4}+\frac{a^2+b^2+c^2}{12abc}.$$

DEFINITION 2.14 ([6])

Let $a_1,\dots,a_m$, let $n\in\mathbb{Z}$, and let $b\in\mathbb{Z}_{>0}$. Then the Fourier–Dedekind sum is defined as

$$s_n(a_1,\dots,a_m;b):=\frac{1}{b}\sum_{k=1}^{b-1}\frac{\xi_b^{kn}}{(1-\xi_b^{ka_1})(1-\xi_b^{ka_2})\cdots(1-\xi_b^{ka_m})}.$$

Let us see some interesting properties of these sums.

REMARK 2.6 ([6])

Let $a,b,c\in\mathbb{Z}$.

(1) For all $n\in\mathbb{Z}$, $s_n(a,b;1)=0$.
(2) For all $n\in\mathbb{Z}$, $s_n(a,b;c)=s_n(b,a;c)$.

(3) One has $s_0(a,1;b) = -s(a,b) + \frac{b-1}{4b}$.
(4) If a' denotes the inverse of a modulo c, then $s_0(a,b;c) = -s(a'b,c) + \frac{c-1}{4c}$.

With this notation we can express (9) and (10) as

$$p_{\{a,b,1\}}(t) = \mathrm{poly}_{\{a,1,b\}}(t) + s_{-t}(a,1;b) + s_{-t}(1,b;a), \tag{12}$$

$$p_{\{a,b,c\}}(t) = \mathrm{poly}_{\{a,b,c\}}(t) + s_{-t}(a,b;c) + s_{-t}(b,c;a) + s_{-t}(a,c;b). \tag{13}$$

As a consequence of Zagier reciprocity in dimension 3 (see [6, Theorem 8.4]), one has the following result.

COROLLARY 2.15 (RADEMACHER'S RECIPROCITY LAW; SEE [6])
Substituting $t=0$ in the previous expression one gets

$$1 - \mathrm{poly}_{\{a,b,c\}}(0) = s_0(a,b;c) + s_0(c,b;a) + s_0(a,c;b) = -\frac{1}{4} + \frac{a^2+b^2+c^2}{12abc}.$$

3. Local algebraic invariants on quotient singularities

In this section we study two local invariants of a curve in a V-surface, the δ-invariant $\delta_P(\mathcal{C})$ and the dimension $\kappa_P(\mathcal{C})$ (see Definitions 3.2 and 3.5), and a local invariant of the surface, $\Delta_P(k)$ (see Definition 3.8), as well as the relation among them (see Theorem 1.2), so as to understand the right-hand side of the formula in Theorem 1.1.

In [15] and [22] we started extending the concept of the Milnor fiber and Milnor number of a curve singularity allowing the ambient space to be a quotient surface singularity. A generalization of the local δ-invariant is also defined and described in terms of a $\mathbf{Q}$-resolution of the curve singularity. All these tools allow for an explicit description of the genus formula of a curve defined on a weighted projective plane in terms of its degree and the local type of its singularities.

DEFINITION 3.1 ([15])
Let $\mathcal{C} = \{f=0\} \subset X(d;a,b)$ be a curve germ, where f is quasi-invariant. The *Milnor fiber* f_t^w of $(\mathcal{C},[0])$ is defined as

$$f_t^w := \{f^d = t\}/\boldsymbol{G}_d.$$

The *Milnor number* μ^w of $(\mathcal{C},P)$ is defined as

$$\mu^w := 1 - \chi^{\mathrm{orb}}(f_t^w).$$

We recall that $\chi^{\mathrm{orb}}(O) := \frac{1}{|G|}\sum_\Delta (-1)^{\dim\Delta}|G_\Delta|$ for an orbifold O with a finite CW-complex structure given by the cells Δ and the finite group G acting on it, where G_Δ denotes the stabilizer of Δ.

Note that alternative generalizations of Milnor numbers can be found, for instance, in [1], [9], [27], and [25]. The one proposed here seems more natural for quotient singularities, but more importantly, it allows for the existence of an explicit formula relating the Milnor number, δ-invariant, and genus of a curve on a singular surface.

We define the local invariant δ for curve singularities on $X(d;a,b)$.

DEFINITION 3.2 ([15])

Let $\mathcal{C}$ be a reduced curve germ at $[0] \in X(d;a,b)$. Then we define δ (or $\delta_0(\mathcal{C})$) as the number verifying

$$\chi^{\mathrm{orb}}(f_t^w) = r^w - 2\delta,$$

where r^w is the number of irreducible branches of $\mathcal{C}$ at $[0]$.

REMARK 3.1

Note that the δ-invariant of a reduced curve (i.e., a reduced $\mathbb{Q}$-divisor) on a surface with quotient singularity is not an integer number in general, but rather a rational number (see [15, Example 4.6]). However, in the case when $\mathcal{C}$ is in fact Cartier, $\delta_0(\mathcal{C})$ is an integer number that has an interpretation as the dimension of the quotient $\bar{R}/R$ where R is the coordinate ring of $\mathcal{C}$ and $\bar{R}$ is its normalization (see [15, Theorem 4.14]). An alternative definition of $\delta_0(\mathcal{C})$ on normal surfaces in terms of a resolution can be found in [7].

Also note that r^w can also be seen as the number of irreducible k-invariant factors of the defining equation f. For instance, the germ defined by $f = (x^2 - y^4)$ in $\mathcal{C}^2$ is not irreducible since $(x^2 - y^4) = (x - y^2)(x + y^2)$. However, $f = 0$ also defines a set of zeroes in $X(2;1,1)$, which is irreducible (and hence $r^w = 1$), since $(x - y^2)$ and $(x + y^2)$ are not k-invariant for any k (recall Definition 2.3).

A recurrent formula for δ based on a $\mathbf{Q}$-resolution of the singularity is provided in Theorem 3.3.

Assume $(f,0) \subset X(d;a,b)$, and consider a (p,q)-blowup π at the origin. Denote by $\nu_0(f)$ the (p,q)-multiplicity of f at 0, and let $e := \gcd(d, pb - qa)$. As an interpretation of $\nu = \nu_0(f)$, we recall that $\pi^*(C) = \hat{C} + \frac{\nu}{e}E$, where $\pi^*(C)$ is the total transform of C, $\hat{C}$ is its strict transform, and E is the exceptional divisor.

We will use the following notation:

$$\delta_{0,\pi}(f) = \frac{\nu_0(f)}{2dpq}\big(\nu_0(f) - p - q + e\big). \tag{14}$$

THEOREM 3.3 ([15])

Let $(C,[0])$ be a curve germ on an abelian quotient surface singularity. Then

$$\delta(C) = \sum_{Q \prec [0]} \delta_{Q,\pi_{(p,q)}}(f), \tag{15}$$

where Q runs over all the infinitely near points of a $\mathbf{Q}$-resolution of $(C,[0])$ and $\pi_{(p,q)}$ is a (p,q)-blowup of Q, the origin of $X(d;a,b)$.

3.1. Logarithmic modules

For a given $k \geq 0$, one has the module $\mathcal{O}_P(k)$ of k-invariant germs (see Definition 2.3)

$$\mathcal{O}_P(k) := \left\{h \in \mathbb{C}\{x,y\} \,\middle|\, h(\xi_d^a x, \xi_d^b y) = \xi_d^k h(x,y)\right\}.$$

Let $\{f=0\}$ be a germ in $X(d;a,b)$. Note that if $f \in \mathcal{O}_P(k)$, then one has the following $\mathcal{O}_P$-module:

$$\mathcal{O}_P(k-a-b) = \left\{h \in \mathbb{C}\{x,y\} \,\middle|\, h\frac{dx \wedge dy}{f} \text{ is } \boldsymbol{G}_d\text{-invariant}\right\}.$$

DEFINITION 3.4

Let $D = \{f=0\}$ be a germ in $P \in X(d;a,b)$, where $f \in \mathcal{O}_P(k)$.

(1) Let $\mathcal{M}_D^{\mathrm{LR}}$ denote the submodule of $\mathcal{O}_P(k-a-b)$ consisting of all $h \in \mathcal{O}_P(k-a-b)$ such that the 2-form (recall Notation 2.7)

$$\omega = h\frac{dx \wedge dy}{f} \in \Omega_X^2\big(\mathrm{LR}\langle D\rangle\big).$$

(2) Let $\mathcal{M}_D^{\mathrm{nul}}$ denote the submodule of $\mathcal{M}_D^{\mathrm{LR}}$ consisting of all $h \in \mathcal{M}_D^{\mathrm{LR}}$ such that the 2-form

$$\omega = h\frac{dx \wedge dy}{f} \in \Omega_X^2\big(\mathrm{LR}\langle D\rangle\big)$$

admits a holomorphic extension outside the strict transform $\widehat{f}$.

(3) Any $\mathcal{O}_P$-module $\mathcal{M} \subseteq \mathcal{M}_D^{\mathrm{LR}}$ will be called a *logarithmic module*.

DEFINITION 3.5

Let $D = \{f=0\}$ be a germ in $P \in X(d;a,b)$. Let us define the following dimension:

$$\kappa_P(D) = \kappa_P(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_P(s)}{\mathcal{M}_D^{\mathrm{nul}}},$$

for $s = \deg f - a - b$.

REMARK 3.2

From the discussion in Section 2.2 note that $\kappa_P(f)$ turns out to be a finite number independent of the chosen $\mathbf{Q}$-resolution. Intuitively, the number $\kappa_P(f)$ provides the minimal number of conditions required for a generic germ $h \in \mathcal{O}_P(s)$ so that $h \in \mathcal{M}_D^{\mathrm{nul}}$.

REMARK 3.3

It is known (see [13, Chapter 2]) that if f is a holomorphic germ in $(\mathbb{C}^2, 0)$, then

$$\kappa_0(f) = \delta_0(f).$$

3.2. The δ-invariant in the general case of local germs

Let us start with the following constructive result which allows one to see any singularity on the quotient $X(d;a,b)$ as the strict transform of some $\{g=0\}\subset\mathbb{C}^2$ after performing a certain weighted blowup.

REMARK 3.4

The Weierstrass division theorem states that, given $f,g\in\mathbb{C}\{x,y\}$ with f being y-general of order k, there exist $q\in\mathbb{C}\{x,y\}$ and $r\in\mathbb{C}\{x\}[y]$ of degree in y less than or equal to $k-1$, both uniquely determined by f and g, such that $g=qf+r$. The uniqueness and the linearity of the action ensure that the division can be performed equivariantly for the action of $\boldsymbol{G}_d$ on $\mathbb{C}\{x,y\}$ (see (7)), that is, if $f,g\in\mathcal{O}(l)$, then so are q and r. In other words, the Weierstrass preparation theorem still holds for zero sets in $\mathbb{C}\{x,y\}^{\boldsymbol{G}_d}$.

Let $\{f=0\}\subset(X(d;a,b),0)$ be a reduced analytic germ. Assume that $(d;a,b)$ is a normalized type. After a suitable change of coordinates of the form $X(d;a,b)\to X(d;a,b)$, $[(x,y)]\mapsto[(x+\lambda y^k,y)]$ where $bk\equiv a \mod d$, one can assume $x\nmid f$. Moreover, by Remark 3.4, f can be written in the form

$$f(x,y)=y^r+\sum_{i>0,j<r}a_{ij}x^iy^j\in\mathbb{C}\{x\}[y]\cap\mathcal{O}(k). \tag{16}$$

For technical reasons, in the following results the space $X(d;a,b)$ will be considered to be of type $X(p;-1,q)$. Note that this is always possible.

LEMMA 3.6

Let $f\in\mathcal{O}(k)$ define an analytic germ on $X(p;-1,q)$, $\gcd(p,q)=1$, such that $x\nmid f$. Then there exist $g\in\mathbb{C}\{x,y\}$ with $x\nmid g$ such that $g(x^p,x^qy)=x^{qr}f(x,y)$. Moreover, f is reduced (resp., irreducible) if and only if g is.

Proof

By the discussion after Remark 3.4 one can assume $f\in\mathbb{C}\{x\}[y]$ as in (16). We have $-i+qj\equiv qr\equiv k \mod p$ for all i,j so $p\mid(i+q(r-j))$ and $i+q(r-j)>0$. Consider

$$g(x,y)=y^r+\sum_{i>0,j<r}a_{ij}x^{\frac{i+q(r-j)}{p}}y^j\in\mathbb{C}\{x\}[y],$$

$$g(x^p,x^qy)=x^{qr}y^r+\sum_{i>0,j<r}a_{ij}x^{i+qr}y^j=x^{qr}\Big(y^r+\sum_{i>0,j<r}a_{ij}x^iy^j\Big).$$

Note that the strict transform passes only through the origin of the first chart. $\square$

The following Proposition 3.7 will be useful to give a generalization of Remark 3.3. Before we state the result we need some notation. Given $r,p,q\in\mathbb{Z}_{>0}$ we define the following combinatorial number which generalizes $\binom{d}{2}$:

$$\delta_r^{(p,q)} := \frac{r(qr-p-q+1)}{2p}. \tag{17}$$

Note that $\binom{d}{2} = \delta_d^{(1,1)}$.

PROPOSITION 3.7

Let $p,q,a,r \in \mathbb{Z}_{>0}$ with $\gcd(p,q)=1$ and $r_1 = r + pa$. Consider the following cardinal:

$$A_r^{(p,q)} := \#\{(i,j) \in \mathbb{Z}^2 \mid pi + qj \leq qr; i,j \geq 1\}.$$

(1) *If $r = pa$, then one has $\delta_r^{(p,q)} = A_r^{(p,q)}$.*

(2) *The following equalities hold:*

$$\delta_{r_1}^{(p,q)} - \delta_r^{(p,q)} = \delta_{r_1-r}^{(p,q)} + aqr, \tag{18}$$

$$A_{r_1}^{(p,q)} - A_r^{(p,q)} = A_{r_1-r}^{(p,q)} + aqr. \tag{19}$$

(3) *The difference $A_r^{(p,q)} - \delta_r^{(p,q)}$ only depends on r modulo p.*

Proof

(1) To prove this fact it is enough to apply Pick's theorem (see, e.g., [6, Section 2.6]), noticing that the number of points on the diagonal without counting the ones on the axes is $a-1$. Finally, one gets

$$A_{pa}^{(p,q)} = \frac{a(pqa-p-q+1)}{2} = \delta_{pa}^{(p,q)}.$$

(2) Proving (18) is a simple and direct computation. To prove (19), let us describe $A_{r_1}^{(p,q)}$, $A_r^{(p,q)}$, and $A_{r_1-r}^{(p,q)}$:

$$\begin{aligned}
A_{r_1}^{(p,q)} &= \#\{(i,j) \in \mathbb{Z}^2 \mid pi + qj \leq qr + pqa; i,j \geq 1\}, \\
A_r^{(p,q)} &= \#\{(i,j) \in \mathbb{Z}^2 \mid pi + qj \leq qr; i,j \geq 1\} \\
&= \#\{(i,j) \in \mathbb{Z}^2 \mid pi + qj \leq qr + apq - apq; i,j \geq 1\} \\
&= \#\{(i,j) \in \mathbb{Z}^2 \mid p(i+aq) + qj \leq qr_1; i \geq 1, j \geq 1\} \\
&= \#\{(i,j) \in \mathbb{Z}^2 \mid pi + qj \leq qr_1; i \geq aq+1, j \geq 1\}, \\
A_{r_1-r}^{(p,q)} &= \#\{(i,j) \in \mathbb{Z}^2 \mid pi + qj \leq qr_1 - qr; i,j \geq 1\} \\
&= \#\{(i,j) \in \mathbb{Z}^2 \mid pi + q(j+r) \leq qr_1; i,j \geq 1\} \\
&= \#\{(i,j) \in \mathbb{Z}^2 \mid p + qj \leq qr_1; i \geq 1, j \geq r+1\}.
\end{aligned}$$

By using the decomposition shown in Figure 1, the claim follows.

(3) Subtracting (18) and (19) shows that the result holds. $\square$

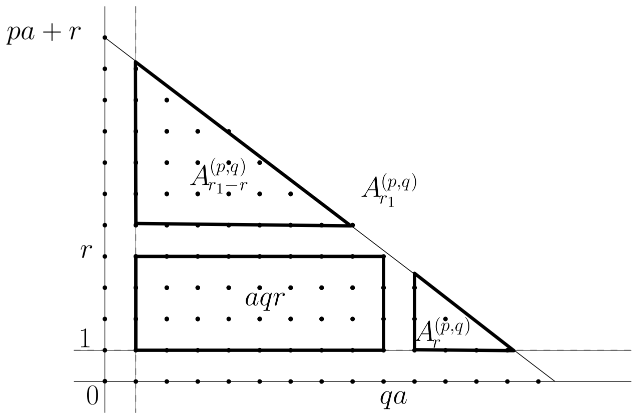

Figure 1

DEFINITION 3.8

Let $k \geq 0$, and let $P \in X(p;-1,q) = X$. The $\Delta_P(k)$*-invariant* of X is defined as

$$\Delta_P(k) := A_r^{(p,q)} - \delta_r^{(p,q)},$$

where $r = q^{-1}k \mod p$.

As a result of Proposition 3.7 one has the following result.

THEOREM 3.9

Let $f_1, f_2 \in \mathcal{O}(k)$ be two germs at $[0] \in X(d;a,b)$. Then,

$$\kappa_0(f_1) - \kappa_0(f_2) = \delta_0(f_1) - \delta_0(f_2).$$

Proof

By Remark 3.4 and the discussion after it, we can assume that

$$f_\ell(x,y) = y^{r_\ell} + \sum_{i>0\leq j<r_\ell} a_{ij}x^iy^j \in \mathbb{C}\{x\}[y]$$

in $X(p;-1,q)$ $(p=d,\ q \equiv -ba^{-1} \mod d)$. Consider $g_1 \in \mathbb{C}\{x,y\}$ the reduced germ obtained after applying Lemma 3.6 to f_1. Denote by $\pi_{(p,q)}$ the blowup at the origin. Note that $\nu_{p,q}(g_1) = qr_1$, and thus, $\delta_{r_1}^{(p,q)} = \delta_{\pi_{(p,q)}}(g_1)$ (see (17) and (14)).

Consider the form $\omega := \phi\frac{dx\wedge dy}{g_1}$, $\phi \in \mathbb{C}\{x,y\}$, and let us calculate the local equations for the pullback of ω after blowing-up the origin on $\mathbb{C}^2$,

$$\phi\frac{dx\wedge dy}{g_1} \overset{\pi_{(p,q)}}{\longleftarrow} x^{\nu_\phi+p+q-1-qr_1}h\frac{dx\wedge dy}{f_1}. \tag{20}$$

Using the definitions of $\mathcal{M}_{g_1}^{\mathrm{nul}}$ and $\mathcal{M}_{f_1}^{\mathrm{nul}}$ (see Definition 3.4), we find that

$$\phi \in \mathcal{M}_{g_1}^{\mathrm{nul}} \Leftrightarrow h \in \mathcal{M}_{f_1}^{\mathrm{nul}} \qquad \text{and} \qquad \nu_\phi + p + q - 1 - qr_1 \geq 0.$$

Therefore, $\phi(x,y) \mapsto \phi(x^p, x^q y)$ induces an isomorphism

$$\mathcal{M}_{g_1}^{\mathrm{nul}} \cong \mathcal{A}_{r_1}^{(p,q)} \cap \mathcal{M}_{f_1}^{\mathrm{nul}},$$

where $\mathcal{A}_{r_1}^{(p,q)} := \{h \in \mathbb{C}\{x,y\} \mid \mathrm{ord}_h + p + q - 1 - qr_1 \geq 0\}$ and ord_h is the (p,q)-order of h. Since $\dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\mathcal{A}_{r_1}^{(p,q)}} = A_{r_1}^{(p,q)}$, one obtains

$$\kappa_0(g_1) = A_{r_1}^{(p,q)} + \kappa_0(f_1). \tag{21}$$

On the other hand (see Remark 3.3 and Theorem 3.3),

$$\kappa_0(g_1) = \delta_0(g_1) = \delta_{\pi_{(p,q)}}(f) + \delta_0(f_1) = \delta_{r_1}^{(p,q)} + \delta_0(f_1). \tag{22}$$

Therefore, from (21) and (22),

$$\kappa_0(f_1) = \delta_0(f_1) + \delta_{r_1}^{(p,q)} - A_{r_1}^{(p,q)}. \tag{23}$$

Following a similar procedure we get

$$\kappa_0(f_2) = \delta_0(f_2) + \delta_{r_2}^{(p,q)} - A_{r_2}^{(p,q)}. \tag{24}$$

Notice that $k \equiv qr_1 \equiv qr_2 \mod p$, which implies $r_1 \equiv r_2 \mod p$ since p and q are coprime. Therefore by Proposition 3.7,

$$A_{r_1}^{(p,q)} - \delta_{r_1}^{(p,q)} = A_{r_2}^{(p,q)} - \delta_{r_2}^{(p,q)},$$

and finally from (23) and (24) it can be concluded that

$$\kappa_0(f_1) - \kappa_0(f_2) = \delta_0(f_1) - \delta_0(f_2). \qquad \square$$

Proof of Theorem 1.2
The result follows directly from (23). $\square$

REMARK 3.5
If $(f,[0])$ is a function germ on $X = X(d;a,b)$, then from Proposition 3.7(1) and Theorem 1.2, one has

$$\kappa_P(f) = \delta_P(f).$$

In particular, if P is a smooth point of X, then this generalizes Remark 3.3.

4. An introductory example

Let us start this section with one basic illustrative example. Let us compute the number of solutions $(a,b,c) \in \mathbb{Z}_{\geq 0}^3$ of the equation

$$aw_0 + bw_1 + cw_2 = k\bar{w}$$

with $w_0, w_1, w_2 \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{\geq 0}$ fixed, or equivalently, the number of monomials in $\mathcal{O}_{\mathbb{P}_w^2}$ of quasihomogeneous degree $k\bar{\omega}$. This number will be denoted by $\mathrm{L}_w(k\bar{w})$ (recall (3)). Notice that this is equivalent to computing the number of nonnegative integer solutions (a,b,c) to $aw_0 + bw_1 = (kw_{01} - c)w_2$ (with $w_{ij} := w_i w_j$), which can be achieved by considering the following sets:

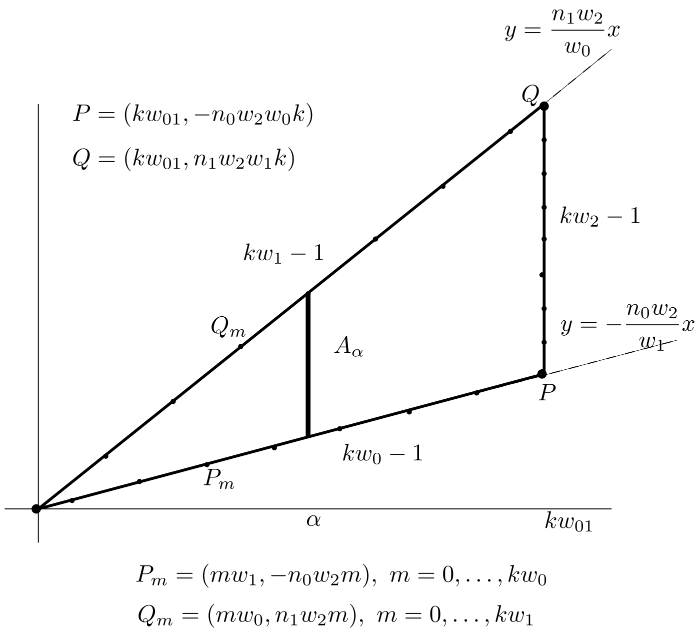

Figure 2

$$\tilde{A} := \left\{(a,b) \in \mathbb{Z}^2_{>0} \;\middle|\; aw_0 + bw_1 = \alpha w_2, \alpha = 0, \ldots, kw_{01}\right\},$$

$$\tilde{B} := \left\{(a,0) \in \mathbb{Z}^2_{>0} \;\middle|\; aw_0 = \alpha w_2, \alpha = 0, \ldots, kw_{01}\right\}$$
$$\cup \left\{(0,b) \in \mathbb{Z}^2_{>0} \;\middle|\; bw_1 = \alpha w_2, \alpha = 0, \ldots, kw_{01}\right\}.$$

If we denote by $A = \#\tilde{A}$ and $B = \#\tilde{B}$, then one has $\mathrm{L}_w(k\bar{w}) = A + B + 1$. To compute A take two integers n_0, n_1 such that $n_0 w_0 + n_1 w_1 = 1$ with $n_1 > 0$ and $n_0 \leq 0$. (This can always be done since the weights are pairwise coprime.) There exists a positive integer λ satisfying $a = n_0 \alpha w_2 + \lambda w_1$ and $-\frac{n_0 \alpha w_2}{w_1} < \lambda < \frac{n_1 \alpha w_2}{w_0}$. This justifies the following definition (see Figure 2):

$$A_\alpha := \#\left\{\lambda \in \mathbb{Z}_{>0} \;\middle|\; -\frac{n_0 \alpha w_2}{w_1} < \lambda < \frac{n_1 \alpha w_2}{w_0}\right\}.$$

Note that by virtue of Pick's theorem the area of the triangle is equal to the number of natural points in its interior I plus one half the number of points in the boundary minus one. The area of the triangle equals $\frac{k^2 \bar{\omega}}{2}$. Thus,

$$\frac{k^2 \bar{\omega}}{2} = I + \frac{k|w|}{2} - 1,$$

which implies

$$A = I + (kw_2 - 1) = \left(\frac{k^2 \bar{w}}{2} - \frac{k|w|}{2} + 1\right) + (kw_2 - 1) = \frac{1}{2}\left(k^2 \bar{w} - k|w|\right) + kw_2.$$

It is easy to check that $B = kw_0 + kw_1$. Then we have

$$\mathrm{L}_w(k\bar{w}) = \frac{1}{2} k\left(k\bar{w} + |w|\right) + 1.$$

It is known that the genus of a smooth curve on $\mathbb{P}^2_w$ of degree d transversal with respect to the axes is

$$g_{w,d} = \frac{d(d-|w|)}{2\bar{w}} + 1.$$

We want to find d such that $\mathrm{L}_w(k\bar{w}) = g_{w,d}$. To do that it is enough to solve the equation

$$\frac{1}{2}k\big(k\bar{w} + |w|\big) + 1 = \frac{d(d-|w|)}{2\bar{w}} + 1.$$

One finally gets that

$$\mathrm{L}_w(k\bar{w}) = g_{w,|w|+k\bar{w}}.$$

The rest of this article deals with the extension of this example when d is not necessarily a multiple of $\bar{w}$.

5. Proof of the main results

Proof of Theorem 1.1
We will prove the equivalent formula

$$\mathrm{L}_w\big(d-|w|\big) = g_{w,d} - \sum_{P\in\mathrm{Sing}(\mathbb{P}^2_w)} \Delta_P(d).$$

From the definitions note that

$$\mathrm{L}_w\big(d-|w|\big) = p_{\{w_0,w_1,w_2\}}\big(d-|w|\big) =: p_{\{w\}}\big(d-|w|\big).$$

Fix a point $P \in \mathrm{Sing}(\mathbb{P}^2_w)$, and describe for simplicity the local singularity as $X(w_i; w_{i+1}, w_{i+2}) = X(w_i; -1, q_i)$, where $q_i := -w_{i+1}^{-1} w_{i+2} \mod w_i$, for $i = 0, 1, 2$. (The indices are considered modulo 3.) Define $r_i := w_{i+2}^{-1} d \mod w_i$. Then

$$A_{r_i}^{(w_i,q_i)} = p_{\{w_i,q_i,1\}}(q_i r_i - w_i - q_i).$$

On the one hand from (13) and a direct computation one obtains

$$\mathrm{L}_w\big(d-|w|\big) - g_{w,d} = -1 + \mathrm{poly}_{\{w\}}(0) + \sum_{i=0}^{2} s_{|w|-d}(w_i, w_{i+1}; w_{i+2}).$$

By Definition 2.14 and Corollary 2.15 one obtains

$$\mathrm{L}_w\big(d-|w|\big) - g_{w,d} = \sum_{i=0}^{2} \big(s_{|w|-d}(w_{i+1}, w_{i+2}; w_i) - s_0(w_{i+1}, w_{i+2}; w_i)\big). \tag{25}$$

On the other hand, from (12) and straightforward computations, one obtains

$$\begin{aligned} \delta_{r_i}^{(w_i,q_i)} - A_{r_i}^{(w_i,q_i)} = {} & \frac{w_i + q_i}{2w_i q_i} - \mathrm{poly}_{\{w_i,q_i,1\}}(0) \\ & - \big(s_{w_i+q_i-q_i r_i}(w_i, 1; q_i) + s_{w_i+q_i-q_i r_i}(q_i, 1; w_i)\big), \end{aligned} \tag{26}$$

with

$$s_{w_i+q_i-q_ir_i}(q_i,1;w_i)=\frac{1}{w_i}\sum_{k=1}^{w_i-1}\frac{1}{(1-\xi_{w_i}^{kq_i})(1-\xi_{w_i}^{k})\xi_{w_i}^{k(q_ir_i-w_i-q_i)}}$$

and

$$\begin{aligned}s_{w_i+q_i-q_ir_i}(w_i,1;q_i)&=\frac{1}{q_i}\sum_{k=1}^{q_i-1}\frac{1}{(1-\xi_{q_i}^{kw_i})(1-\xi_{q_i}^{k})\xi_{q_i}^{k(q_ir_i-w_i-q_i)}}\\&=-\frac{1}{q_i}\sum_{k=1}^{q_i-1}\frac{1}{(1-\xi_{q_i}^{-kw_i})(1-\xi_{q_i}^{k})},\end{aligned}$$

which implies by Proposition 2.12 that

$$s_{w_i+q_i-q_ir_i}(w_i,1;q_i)=s_{w_i}(w_i,1;q_i)=s(-w_i,q_i)-\frac{q_i-1}{4q_i}. \tag{27}$$

Since by hypothesis $q_i=-w_{i+1}^{-1}w_{i+2} \mod w_i$ and $r_i=w_{i+2}^{-1}d \mod w_i$, one obtains

$$\begin{aligned}&s_{w_i+q_i-q_ir_i}(q_i,1;w_i)\\&\quad=\frac{1}{w_i}\sum_{\ell=1}^{w_i-1}\frac{1}{(1-\xi_{w_i}^{\ell q_i})(1-\xi_{w_i}^{\ell})\xi_{w_i}^{\ell(q_ir_i-w_i-q_i)}}\\&\quad=\frac{1}{w_i}\sum_{\ell=1}^{w_i-1}\frac{1}{(1-\xi_{w_i}^{\ell(-w_{i+1}^{-1}w_{i+2})})(1-\xi_{w_i}^{\ell})\xi_{w_i}^{\ell(-w_{i+1}^{-1}d+w_{i+1}^{-1}w_{i+2})}}\\&\overset{\ell=-w_{i+1}\bar\ell}{=}\frac{1}{w_i}\sum_{\bar\ell=1}^{w_i-1}\frac{1}{(1-\xi_{w_i}^{\bar\ell w_{i+2}})(1-\xi_{w_i}^{-\bar\ell w_{i+1}})\xi_{w_i}^{\bar\ell(d-w_{i+2})}}\\&\quad=-\frac{1}{w_i}\sum_{\bar\ell=1}^{w_i-1}\frac{1}{(1-\xi_{w_i}^{\bar\ell w_{i+2}})(1-\xi_{w_i}^{\bar\ell w_{i+1}})\xi_{w_i}^{\bar\ell(d-|w|)}}\\&\quad=-s_{|w|-d}(w_{i+1},w_{i+2};w_i).\end{aligned} \tag{28}$$

Thus,

$$s_{|w|-d}(w_{i+1},w_{i+2};w_i)=-s_{w_i+q_i-q_ir_i}(q_i,1;w_i), \tag{29}$$

for $i=0,1,2$.

By using (25), (26), and (29), it only remains to show that

$$-s_0(w_{i+1},w_{i+2};w_i)=\frac{w_i+q_i}{2w_iq_i}-\operatorname{poly}_{\{w_i,1,q_i\}}(0)-s_{w_i+q_i-q_ir_i}(w_i,1;q_i). \tag{30}$$

For the left-hand side we use Remark 2.6(4) and obtain

$$s_0(w_{i+1},w_{i+2};w_i)=-s(-q_i,w_i)+\frac{w_i-1}{4w_i}=s(q_i,w_i)+\frac{w_i-1}{4w_i}.$$

For the right-hand side, using Corollary 2.15 and (27) we have

$$\begin{aligned}&\operatorname{poly}_{\{w_i,1,q_i\}}(0)+s_{w_i+q_i-q_ir_i}(w_i,1;q_i)\\&\quad=1-s_0(q_i,1;w_i)-s_0(w_i,1;q_i)+s_{w_i}(w_i,1;q_i),\end{aligned}$$

which by Remark 2.6(3) and (27) becomes

$$\left(1+s(q_i,w_i)-\frac{w_i-1}{4w_i}+s(w_i,q_i)-\frac{q_i-1}{4q_i}\right)+s(-w_i,q_i)-\frac{q_i-1}{4q_i}.$$

Combining these equalities into (30) one obtains the result. $\square$

Proof of Theorem 1.3

It is enough to apply [15, Theorem 5.7] and Theorem 1.1 and recall the characterization of $\kappa_P(f)$ in the proof of Theorem 3.9 (see (23)):

$$\begin{aligned} g(\mathcal{C}) &= g_{w,d} - \sum_{P\in\operatorname{Sing}(\mathcal{C})} \delta_P(f) \\ &= \underbrace{g_{w,d} + \sum_{i=0}^{2}(\delta_{r_i}^{(w_i,q_i)} - A_{r_i}^{(w_i,q_i)})}_{\mathrm{L}_w(d-|w|)} \\ &\quad - \underbrace{\Big(\sum_{P\in\operatorname{Sing}(\mathcal{C})} \delta_P(f) + \sum_{i=0}^{2}(\delta_{r_i}^{(w_i,q_i)} - A_{r_i}^{(w_i,q_i)})\Big)}_{\sum_{P\in\operatorname{Sing}(\mathcal{C})}\kappa_P(f)}. \end{aligned}$$

$\square$

REMARK 5.1

The second equality in the previous identity always holds, and therefore if one considers $\mathcal{C}\subset\mathbb{P}^2_w$ a reduced curve of degree d, then (recall Theorem 1.2)

$$\mathrm{L}_w\big(d-|w|\big) = g_{w,d} - \sum_{P\in\operatorname{Sing}(\mathcal{C})}\delta_P(f) + \sum_{P\in\operatorname{Sing}(\mathcal{C})}\kappa_P(f).$$

Let us see an example of the previous result.

EXAMPLE 5.1

Consider the polygon $D_w := \{(x,y,z)\in\mathbb{R}^3 \mid w_0x+w_1y+w_2z=1\}$, for $w=(w_0,w_1,w_2)=(2,3,7)$. As an example, we want to obtain the Ehrhart quasipolynomial $\mathrm{L}_w(d)$ for D_w. Note that according to Theorem 1.1

$$\mathrm{L}_w(d)=\frac{1}{84}d^2+\frac{1}{7}d+a_0(d),$$

where $a_0(d)$ is a rational periodic number of period $\bar{w}=42$. Moreover, $a_0(d)=1-(\sum_{i=0}^{2}\Delta_i(d+12))$, where Δ_i has period w_i and depends only on the singular point $P_i=\{x_j=x_k=0\}$ ($\{i,j,k\}=\{0,1,2\}$) in the weighted projective plane $\mathbb{P}^2_w$.

In order to describe $\Delta_i(d)$ we will introduce some notation. Given a list of rational numbers $q_0,\dots,q_{r-1}$ we denote by $[q_0,\dots,q_{r-1}]$ the periodic function $f:\mathbb{Z}\to\mathbb{Q}$ whose period is r and such that $f(i)=[q_0,\dots,q_{r-1}]_i=q_i$. By using

Table 1. Local invariants at $X(7;2,3)$.

d	0	1	2	3	4	5	6
Δ_P	0	2/7	3/7	3/7	2/7	0	4/7
δ_P	0	9/7	3/7	3/7	9/7	1	4/7
κ_P	0	1	0	0	1	1	0
Branches	0	2	1	1	2	2	1
Equation	1	$x(x^3+y^2)$	x	y	$x(x+y^3)$	xy	x^3+y^2

this notation it is easy to check that

$$\Delta_0(d) = \left[0, \frac{1}{4}\right]_d \qquad \text{and} \qquad \Delta_1(d) = \left[0, \frac{1}{3}, \frac{1}{3}\right]_d.$$

Finally, in order to obtain $\Delta_2(d)$, one needs to compute both δ- and K-invariants for the singular point $P_2 \in X = X(7;2,3)$.

The typical way to obtain the first row is by applying Theorem 1.3 to a generic germ f_d in $\mathcal{O}_X(d)$. This is how the second and third rows in the previous table were obtained. The last two rows indicate the local equations and the number of branches of such a generic germ $f_d \in \mathcal{O}_X(d)$.

Let us detail the computations for the third column in Table 1 (case $d=1$). One can write the generic germ f_1 in $\mathcal{O}_X(1)$ as $x(x^3+y^2)$. On the one hand, a $(2,3)$-blowup serves as a $\mathbf{Q}$-resolution of X, and thus, by using Theorem 3.3,

$$\delta_P(f_1) = \frac{8(8-2-3+7)}{2\cdot 7\cdot 2\cdot 3} + \frac{3-1}{2\cdot 3} = \frac{9}{7}.$$

For a computation of κ_P one needs to study the quotient $\mathcal{O}_X(3)/\mathcal{M}^{\mathrm{nul}}_{f_1}$. Notice that in the present case $\mathcal{O}_X = \mathbb{C}\{x,y\}^{\boldsymbol{G}_7} = \mathbb{C}\{x^7, y^7, x^2y\}$ and $\mathcal{O}_X(3)$ is the $\mathcal{O}_X$-module generated by y and x^5. In order to study $\mathcal{M}^{\mathrm{nul}}_{f_1}$, consider a generic form

$$(ay+bx^5)\frac{dx\wedge dy}{f_1} \in \Omega^2_X\big(\mathrm{LR}\langle f_1\rangle\big),$$

where $a,b \in \mathcal{O}_X$, and its pullback by a resolution of the singularity $X(7;2,3)$. One obtains the following:

$$\begin{aligned}(ay+bx^5)\frac{dx\wedge dy}{x(x^3-y^2)} \quad &\overset{\substack{x=u_1\bar v_1^2\\ y=\bar v_1^3,\ v_1=\bar v_1^7}}{\longleftarrow} \quad 3\bar v_1^3(\tilde a + \tilde b u_1^5 \bar v_1^7)\frac{\bar v_1^4\, du_1\wedge d\bar v_1}{\bar v_1^8 u_1(u_1^3-1)} \\ &= \quad \frac{3}{7}(\tilde a + \tilde b u_1^5 v_1)\frac{du_1\wedge dv_1}{v_1u_1(u_1^3-1)}.\end{aligned} \tag{31}$$

Therefore, $(ay+bx^5) \notin \mathcal{M}^{\mathrm{nul}}_{f_1}$ if and only if the function $a\in\mathcal{O}_X$ is a unit. Hence, by Definition 3.5,

$$\kappa_P(f_1) = \dim_{\mathbb{C}} \frac{\mathcal{O}_X(3)}{\mathcal{M}^{\mathrm{nul}}_{f_1}} = \dim_{\mathbb{C}}\langle y\rangle_{\mathbb{C}} = 1.$$

Finally,

$$\Delta_P(1) = \delta_P(f_1) - \kappa_P(f_1) = \frac{2}{7}.$$

The rest of values in Table 1 can be computed analogously. Hence, one obtains

$$\Delta_2(d) := \left[0, \frac{2}{7}, \frac{3}{7}, \frac{3}{7}, \frac{2}{7}, 0, \frac{4}{7}\right]_d,$$

and thus,

$$\mathrm{L}_w(d) = \frac{1}{84}d^2 + \frac{1}{7}d + \left(1 - \left[0, \frac{1}{4}\right]_d - \left[0, \frac{1}{3}, \frac{1}{3}\right]_d - \left[0, \frac{4}{7}, 0, \frac{2}{7}, \frac{3}{7}, \frac{3}{7}, \frac{2}{7}\right]_d\right).$$

For instance, to obtain $\mathrm{L}_w(54)$, note that $[0, \frac{1}{4}]_{54} = [0, \frac{1}{4}]_0 = 0$, $[0, \frac{1}{3}, \frac{1}{3}]_{54} = [0, \frac{1}{3}, \frac{1}{3}]_0 = 0$, and $[0, \frac{4}{7}, 0, \frac{2}{7}, \frac{3}{7}, \frac{3}{7}, \frac{2}{7}]_{54} = [0, \frac{4}{7}, 0, \frac{2}{7}, \frac{3}{7}, \frac{3}{7}, \frac{2}{7}]_5 = \frac{3}{7}$. Thus,

$$\mathrm{L}_w(54) = \frac{1}{84}54^2 + \frac{1}{7}54 + \left(1 - \frac{3}{7}\right) = 43.$$

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