Classification of automorphisms on a deformation family of hyper-Kähler four-folds by p-elementary lattices

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Abstract We give a classification of all nonsymplectic automorphisms of prime order p acting on irreducible holomorphic symplectic four-folds deformation equivalent to the Hilbert scheme of two points on a K3 surface, for $p = 2, 3$, and $7 \le p \le 19$. Our classification relates some invariants of the fixed locus to the isometry classes of two natural lattices associated to the action of the automorphism on the second cohomology group with integer coefficients. In several cases we provide explicit examples. As an application, we find new examples of *nonnatural* nonsymplectic automorphisms of order 3.

1. Introduction

Irreducible holomorphic symplectic (IHS) manifolds (or equivalently hyper-Kähler manifolds), together with Calabi–Yau manifolds, are the natural higher-dimensional generalizations of K3 surfaces. In particular, many properties of automorphisms on K3 surfaces generalize to IHS manifolds (see [4, Section 4]). The interest in automorphisms of IHS manifolds has grown markedly in the last few years (see [5], [12], [28], [35], [9], [10]), especially the study of automorphisms of prime order on IHS four-folds deformation equivalent to the Hilbert scheme of two points on a K3 surface, which we call for short $\mathrm{IHS} - \mathrm{K3}^{[2]}$. The case of symplectic automorphisms (i.e., those automorphisms leaving invariant the holomorphic two-form) was studied by the second author [12] for $p = 2$ and then completely settled by Mongardi [29] for all primes. They describe the fixed locus, which is never empty and consists of isolated fixed points, abelian surfaces, and K3 surfaces. The case of nonsymplectic involutions was considered first by Beauville [5] and recently by Ohashi–Wandel [36] who study in detail families of $\mathrm{IHS} - \mathrm{K3}^{[2]}$ with 19 parameters and nonsymplectic involution. In particular, they describe some *nonnatural* involutions: these cannot be deformed to an involution on the Hilbert scheme of two points on a K3 surface induced by an automorphism on the K3 surface.

In this article we classify the nonsymplectic automorphisms of prime order $p \ge 3$ acting on $\mathrm{IHS} - \mathrm{K3}^{[2]}$. As an application of our results, we construct the

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first known examples of nonnatural nonsymplectic automorphisms of order 3 on $\mathrm{IHS}-\mathrm{K3}^{[2]}$. This comes from the study of nonsymplectic automorphisms of order 3 on a special 20-dimensional and a special 14-dimensional family of Fano varieties of lines on cubic four-folds (Corollary 7.6).

Let X be an $\mathrm{IHS}-\mathrm{K3}^{[2]}$. In the study of nonsymplectic automorphisms on X, two natural lattices play an important role: the invariant sublattice T of $H^2(X,\mathbb{Z}) = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$ and its orthogonal complement S. The lattice T is contained in the Néron–Severi group of X, while the lattice S contains the transcendental lattice. These two lattices play an important role when studying moduli spaces. In the case of K3 surfaces, they also determine completely the topology of the fixed locus (see [1], [2]). In this article, using lattice theory and a formula relating topological invariants of the fixed locus with lattice invariants (see [10]), we classify the lattices S and T when the order is $p = 2, 3$, and $7 \leq p \leq 19$. Our first main result (Theorem 3.8) classifies all possible lattices T and S for $p \neq 2, 5$. Our second main result (Theorem 5.5, Theorem 7.1) proves that all cases (except one) can be realized by an automorphism. For $p = 11, 13, 17, 19$ all the examples that we find are natural; for $p = 3$ some examples are constructed using the Fano variety of lines of a cubic four-fold. In particular, in a 12-dimensional family we find an example of a nonsymplectic automorphism of order 3 of *different kind*: it has the same invariant lattice T and orthogonal complement S as a natural automorphism, but its fixed locus is different (see Remark 7.7). This is very surprising: it shows that in the case of $\mathrm{IHS}-\mathrm{K3}^{[2]}$ the lattice invariants do not uniquely determine the fixed locus, contrary to the case of K3 surfaces. In several cases we construct coarse moduli spaces of $\mathrm{IHS}-\mathrm{K3}^{[2]}$ with a nonsymplectic automorphism of order p. This construction uses the classification of nonsymplectic automorphisms of order p on K3 surfaces (Theorem 5.5).

In the last section of the article we discuss the case $p = 2$. The situation is more complicated because the lattice T can have different embeddings in the lattice $U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$. This has an important influence on the construction of the moduli spaces. Our main result is Proposition 8.5, where we show that every embedding of T can be realized as the invariant lattice of a nonsymplectic involution on an $\mathrm{IHS}-\mathrm{K3}^{[2]}$. Many examples can be constructed explicitly by using natural involutions, but in several cases concrete realizations of the automorphisms are still unknown.

2. Preliminary results on lattice theory

A *lattice* L is a free $\mathbb{Z}$-module of finite rank equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ with integer values. Its *dual lattice* is $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. It can be also described as

$$L^\vee \cong \{ x \in L \otimes \mathbb{Q} \mid \langle x, v \rangle \in \mathbb{Z} \ \forall v \in L \}.$$

Clearly, L is a sublattice of $L^\vee$ of the same rank, so the *discriminant group* $A_L := L^\vee / L$ is a finite abelian group whose order is denoted $\mathrm{discr}(L)$ and called the *discriminant of* L. We denote by $\ell(A_L)$ the *length* of A_L, that is, the minimal number of generators of A_L. In a basis $\{e_i\}_i$ of L, for the Gram matrix $M := (\langle e_i, e_j\rangle)_{i,j}$ one has $\mathrm{discr}(\Lambda) = |\det(M)|$.

A lattice L is called *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. In this case the bilinear form induces a quadratic form $q_L : A_L \longrightarrow \mathbb{Q}/2\mathbb{Z}$. Denoting by $(s_{(+)}, s_{(-)})$ the signature of $L \otimes \mathbb{R}$, we have that the triple of invariants $(s_{(+)}, s_{(-)}, q_L)$ characterizes the *genus* of the even lattice L (see [15, Chapter 15, Section 7], [32, Corollary 1.9.4]).

A lattice L is called *unimodular* if $A_L = \{0\}$. A sublattice $M \subset L$ is called *primitive* if L/M is a free $\mathbb{Z}$-module. If L is unimodular and $M \subset L$ is a primitive sublattice, then M and its orthogonal $M^\perp$ in L have isomorphic discriminant groups and $q_M = -q_{M^\perp}$ (see [32]).

Let p be a prime number. A lattice L is called *p-elementary* if $A_L \cong (\frac{\mathbb{Z}}{p\mathbb{Z}})^{\oplus a}$ for some nonnegative integer a (also called the *length* $\ell(A_L)$ of A). We write $\frac{\mathbb{Z}}{p\mathbb{Z}}(\alpha)$, $\alpha \in \mathbb{Q}/2\mathbb{Z}$, to denote that the quadratic form q_L takes value α on the generator of the $\frac{\mathbb{Z}}{p\mathbb{Z}}$ component of the discriminant group. (To be precise we assume that α is a rational number contained in the interval $[0, 2)$, and it is the least representative of the corresponding equivalence class in $\mathbb{Q}/2\mathbb{Z}$.) Recall the following classification result.

THEOREM 2.1 ([37, SECTION 1])

(1) *An even, hyperbolic, p-elementary lattice of rank r with $p \neq 2$ and $r > 2$ is uniquely determined by the integer a.*

(2) *For $p \neq 2$, a hyperbolic p-elementary lattice with invariants r, a exists if and only if the following conditions are satisfied: $a \leq r$, $r \equiv 0 \pmod 2$, and*

$$\begin{cases} \text{if } a \equiv 0 \pmod 2, & r \equiv 2 \pmod 4, \\ \text{if } a \equiv 1 \pmod 2, & p \equiv (-1)^{r/2-1} \pmod 4. \end{cases}$$

Moreover, if $r \not\equiv 2 \pmod 8$, then $r > a > 0$.

We formulate also the following generalization of Theorem 2.1. The proof is essentially contained in [15, Chapter 15, Section 8.2]. We give it here again for convenience.

THEOREM 2.2

Let S be an even, indefinite, p-elementary lattice of rank $r \geq 3$, $p \geq 3$. Then S is uniquely determined by its signature and its discriminant form.

Proof

By a result of Eichler (see [15, Chapter 15, Theorem 14]), since $r \geq 3$, the genus and the spinor genus of S coincide, so by [15, Chapter 15, Theorem 13] the genus contains only one isomorphism class. Then by [32, Corollary 1.9.4] the genus of an even lattice is uniquely determined by the signature and the discriminant form. $\square$

REMARK 2.3

For 2-elementary lattices the situation is different: an even indefinite 2-elementary lattice is determined up to isometry by its signature, length, and a third invariant $\delta \in \{0,1\}$. We refer to [16, Theorem 1.5.2], [33, Theorem 4.3.1, Theorem 4.3.2], and [37, Section 1] for the relations between these invariants.

The following results on the unicity of the isometry class of a lattice of a given genus and on the splitting of lattices will be needed in the remainder of the paper.

THEOREM 2.4 ([30, THEOREM 2.2])

Let L be an even lattice of invariants $(s_{(+)}, s_{(-)}, q_L)$. Assume that $s_{(+)} > 0$, $s_{(-)} > 0$, and $\ell(A_L) \leq \operatorname{rank} L - 2$. Then up to isometry, L is the only lattice with those invariants.

THEOREM 2.5 ([15, CHAPTER 15, THEOREM 21])

If L is an indefinite lattice of rank n and discriminant d with more than one isometry class in its genus, then $4^{[\frac{n}{2}]}d$ is divisible by $k^{\binom{n}{2}}$ for some nonsquare natural number $k \equiv 0, 1 \mod (4)$.

We denote by U the unique even unimodular hyperbolic lattice of rank two and by A_k, D_h, E_l the even, negative definite lattices associated to the Dynkin diagrams of the corresponding type ($k \geq 1$, $h \geq 4$, $l = 6, 7, 8$). We denote by $L(t)$ the lattice whose bilinear form is the one on L multiplied by $t \in \mathbb{N}^*$. The following p-elementary lattices will be used in the remainder of the paper (see [2]).

- For $p \equiv 3 \mod (4)$, the lattice

$$K_p := \begin{pmatrix} -(p+1)/2 & 1 \\ 1 & -2 \end{pmatrix}$$

is negative definite and p-elementary with $a = 1$. Note that $K_3 = A_2$.

- For $p \equiv 1 \mod (4)$ the lattice

$$H_p := \begin{pmatrix} (p-1)/2 & 1 \\ 1 & -2 \end{pmatrix}$$

is hyperbolic and p-elementary with $a = 1$.

- The lattice

$$L_{17} := \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & -4 \end{pmatrix}$$

is negative definite and 17-elementary with $a = 1$.

- The lattice $E_6^\vee(3)$ is even, negative definite, and 3-elementary with $a = 5$. To get a simple form of its discriminant group one can proceed as follows. By [1, Table 2] the lattice $U(3) \oplus E_6^\vee(3)$ admits a primitive embedding in the K3 lattice (which is unimodular) with orthogonal complement isometric to $U \oplus U(3) \oplus A_2^{\oplus 5}$.

It follows that the discriminant form of $E_6^\vee(3)$ is the opposite of those of $A_2^{\oplus 5}$, that is, $\mathbb{Z}/3\mathbb{Z}(2/3)^{\oplus 5}$.

THEOREM 2.6 ([32, THEOREM 1.13.5])

Let L be an even indefinite lattice of signature $(s_{(+)}, s_{(-)})$, and assume that $s_{(+)} > 0$ and $s_{(-)} > 0$.

(1) *If $s_{(+)} + s_{(-)} \geq 3 + \ell(A_L)$, then $L \cong U \oplus W$ for a certain even lattice W.*
(2) *If $s_{(-)} \geq 8$ and $s_{(+)} + s_{(-)} \geq 9 + \ell(A_L)$, then $L \cong E_8 \oplus W'$ for a certain even lattice W'.*

The following result is an application of Nikulin's [32] results on primitive embeddings.

PROPOSITION 2.7

Let S be an even p-elementary lattice, $p \neq 2$, with invariants $(s_{(+)}, s_{(-)}, q_S, a)$, and let $L := U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$. If S admits a primitive embedding in L, then the orthogonal complement T of S in L has discriminant group $(\mathbb{Z}/p\mathbb{Z})^{\oplus a} \oplus \mathbb{Z}/2\mathbb{Z}$ and discriminant form $(-q_S) \oplus q_L$. If moreover $s_{(+)} < 3$, $s_{(-)} < 20$, and $a \leq 21 - \operatorname{rank}(S)$, then T is uniquely determined and there is at most one embedding of S in L up to an isomorphism of L.

Proof

The lattice L has signature $(3, 20)$ and discriminant form $q_L = \frac{\mathbb{Z}}{2\mathbb{Z}}(\frac{3}{2})$; hence, by Theorem 2.4 it is unique in its genus. By Nikulin [32, Proposition 1.15.1], a primitive embedding of S in L is equivalent to a quintuple $(H_S, H_L, \gamma, T, \gamma_T)$ satisfying the following conditions.

- H_S is a subgroup of $A_S = (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$, H_L is a subgroup of $A_L = \mathbb{Z}/2\mathbb{Z}$, and $\gamma\colon H_S \to H_L$ is an isomorphism of groups such that, for any $x \in H_S$, $q_L(\gamma(x)) = q_S(x)$. Here the only possibility is $H_S = \{0\}$, $H_L = \{0\}$, and $\gamma = \mathrm{id}$.
- T is a lattice of invariants $(3 - s_{(+)}, 20 - s_{(-)}, q_T)$ with $q_T = ((-q_S) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, where Γ is the graph of γ in $A_S \oplus A_L$, $\Gamma^\perp$ is the orthogonal complement of Γ in $A_S \oplus A_L$ with respect to the bilinear form induced on $A_S \oplus A_L$, with values in $\mathbb{Q}/\mathbb{Z}$, and γ_T is an automorphism of A_T that preserves q_T. Moreover, T is the orthogonal complement of S in L. Here we get $\Gamma = \{0\}$; hence, $\Gamma^\perp = A_S \oplus A_L = A_T$ and $q_T = (-q_S) \oplus q_L$ is the only possibility.

Since $p \neq 2$, one has $\ell(A_T) = a$. If T is indefinite (i.e., $s_{(+)} < 3$, $s_{(-)} < 20$) and $a \leq \operatorname{rank}(T) - 2 = 21 - \operatorname{rank}(S)$, then by Theorem 2.4 the lattice T is uniquely determined up to isometry. Moreover, under these assumptions the natural homomorphism $O(T) \to O(A_T)$ is surjective (see [16, Proposition 1.4.7]) so different choices of the isometry γ_T produce isomorphic embeddings of S in L (see [16, Lemma 1.4.5]). □

3. Automorphisms on deformations of the Hilbert scheme of two points on a K3 surface

3.1. Irreducible holomorphic symplectic manifolds

A complex, compact, Kähler, smooth manifold X is called *irreducible holomorphic symplectic* (IHS) if X is simply connected and $H^0(X, \Omega_X^2)$ is spanned by an everywhere nondegenerate closed two-form, denoted by ω_X. In dimension 4, one of the most famous examples of IHS manifolds is the Hilbert scheme $\Sigma^{[2]}$ of two points on a K3 surface Σ.

The second cohomology group has a Hodge decomposition

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X),$$

and we set $H^{1,1}(X)_{\mathbb{R}} := H^{1,1}(X) \cap H^2(X,\mathbb{R})$. The second cohomology group $H^2(X,\mathbb{Z})$ is torsion-free and equipped with the Beauville–Bogomolov [4] bilinear symmetric nondegenerate two-form of signature $(3, b_2(X)-3)$ with the property that, after scalar extension, $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$. The *Néron–Severi* group of X is defined by

$$\mathrm{NS}(X) := H^{1,1}(X)_{\mathbb{R}} \cap H^2(X,\mathbb{Z}).$$

We set $\rho(X) := \mathrm{rank}(\mathrm{NS}(X))$ as the *Picard number* of X and $\mathrm{Transc}(X) := \mathrm{NS}(X)^{\perp}$ as the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X,\mathbb{Z})$ for the quadratic form, called the *transcendental lattice*. Note that $\mathrm{NS}(X)$ and $\mathrm{Transc}(X)$ are primitively embedded in $H^2(X,\mathbb{Z})$. By [22, Theorem 3.11] X is projective if and only if $\mathrm{NS}(X)$ is a hyperbolic lattice.

Let $G \subset \mathrm{Aut}(X)$ be a finite group of automorphisms of prime order p, and fix a generator $\sigma \in G$. If $\sigma^*\omega_X = \omega_X$, then G is called *symplectic*. Otherwise, there exists a primitive pth root of unity ξ such that $\sigma^*\omega_X = \xi\omega_X$ and G is called *nonsymplectic*. Observe that nonsymplectic actions exist only when X is projective (see [3, Section 4]). Following the notation of [10] we denote by $T := T_G(X)$ the invariant sublattice of $H^2(X,\mathbb{Z})$ and by $S := S_G(X)$ its orthogonal complement (see [10, Lemma 6.1]).

Since $h^0(X, TX) = 0$, the variety X admits a universal deformation $p\colon \mathcal{X} \to \mathrm{Def}(X)$, where p is a smooth proper holomorphic morphism, $\mathrm{Def}(X)$ is a germ of analytic space, and $p^{-1}(0) \cong X$. (The isomorphism is part of the data.) Although $h^2(X, TX)$ is not zero in general, $\mathrm{Def}(X)$ is smooth of dimension $h^1(X, TX)$ (see [23, Section 4] and references therein). If $\mathrm{Def}(X)$ is taken small enough, then all nearby fibers $\mathcal{X}_t := p^{-1}(t)$, $t \in \mathrm{Def}(X)$ are also IHS manifolds and the universal deformation $p\colon \mathcal{X} \to \mathrm{Def}(X)$ is in fact universal also for these fibers $\mathcal{X}_t$ (see Huybrechts [22, Section 1.12]). Two IHS manifolds X and X' are called *deformation equivalent* if there exists a smooth proper holomorphic morphism $p\colon \mathcal{X} \to S$ with connected base S, whose fibers are Kähler manifolds, and with two points $t, t' \in S$ such that $\mathcal{X}_t \cong X$ and $\mathcal{X}_{t'} \cong X'$. We say that an IHS manifold X is an $\mathrm{IHS}-\mathrm{K3}^{[2]}$ if it is deformation equivalent to the Hilbert scheme of two points on a K3 surface.

3.2. Invariant sublattice and fixed locus

We recall the main results of Boissière–Nieper-Wißkirchen–Sarti [10], [8]. In this section we denote by G a finite group of prime order p acting on an $\mathrm{IHS}-\mathrm{K3}^{[2]}$ that we call X.

PROPOSITION 3.1 ([10, DEFINITIONS 4.5 AND 4.9, LEMMA 5.5])

Assume that the order p of G satisfies $3 \le p \le 23$. Then:

- $\operatorname{rank} S = (p-1)m_G(X)$ *for some positive integer* $m_G(X)$*;*
- $\frac{H^2(X,\mathbb{Z})}{S \oplus T} \cong (\frac{\mathbb{Z}}{p\mathbb{Z}})^{\oplus a_G(X)}$ *for some nonnegative integer* $a_G(X)$*;*
- $A_T \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus a_G(X)}$*,* $A_S \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus a_G(X)}$*; and*
- *if G acts nonsymplectically, then S has signature $(2,(p-1)m_G(X)-2)$ and T has signature $(1, 22-(p-1)m_G(X))$.*

REMARK 3.2

We denote by $H^*(X^G,\mathbb{F}_p)$ the cohomology of the fixed locus X^G with coefficients in $\mathbb{F}_p$, and we set $m := m_G(X)$, $a := a_G(X)$, and $h^*(X^G,\mathbb{F}_p) = \sum_{i \ge 0} h^i(X^G,\mathbb{F}_p)$, where $h^i(X^G,\mathbb{F}_p) := \dim H^i(X^G,\mathbb{F}_p)$.

THEOREM 3.3 ([10, THEOREM 6.15])

Assume that the order p of G satisfies $3 \le p \le 19$, $p \ne 5$. Then

$$\begin{aligned} h^*(X^G,\mathbb{F}_p) &= 324 - 2a(25-a) - (p-2)m(25-2a) \\ &\quad + \frac{1}{2} m\big((p-2)^2 m - p\big) \end{aligned} \tag{1}$$

with $2 \le (p-1)m < 23$ and $0 \le a \le \min\{(p-1)m, 23-(p-1)m\}$.

REMARK 3.4

- For technical reasons (see [10, proof of Theorem 5.15]), the case of $p = 5$ is excluded in Theorem 3.3.
- It follows from [10, Proposition 5.17] that the fixed locus X^G is never empty if $p \ne 2$. So one cannot produce new examples of Enriques varieties (see [9], [35]) by using finite quotients of $\mathrm{IHS}-\mathrm{K3}^{[2]}$ other than quotients by (special) involutions.
- If the group G acts on the K3 surface Σ, then it induces a natural action on $\Sigma^{[2]}$. One can similarly define the integers $a_G(\Sigma)$ and $m_G(\Sigma)$, and it is easy to check that $a_G(\Sigma) = a_G(\Sigma^{[2]})$ and $m_G(\Sigma) = m_G(\Sigma^{[2]})$ (see [10, Remark 5.16(2)]). For any $\sigma \in G$ considered as an automorphism of Σ, we denote by $\sigma^{[2]}$ the automorphism induced on $\Sigma^{[2]}$. These automorphisms are called *natural* in [7], but we will use this term in a more general sense in Definition 4.1. To be precise this "old" definition of *natural* means the following: let X be an $\mathrm{IHS}-\mathrm{K3}^{[2]}$, and let f be an automorphism acting on it; then f is called *natural* if there exists a K3 surface Σ with an automorphism σ such that the couple (X, f) is isomorphic to $(\Sigma^{[2]}, \sigma^{[2]})$.

The topological Lefschetz fixed-point formula gives complementary information on the fixed locus X^G. Denote by $\chi(X^G) := \sum_i (-1)^i \dim H^i(X^G, \mathbb{R})$ the Euler characteristic of X^G. If σ is a generator of G, then one has

$$\chi(X^G) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(\sigma^*_{|H^i(X,\mathbb{R})}).$$

Since X has real dimension 8 and trivial odd cohomology, using Poincaré duality we rewrite the formula as

$$\chi(X^G) = 2 + 2\operatorname{tr}(\sigma^*_{|H^2(X,\mathbb{R})}) + \operatorname{tr}(\sigma^*_{|H^4(X,\mathbb{R})}).$$

Setting $r := \operatorname{rank} T$ one sees easily that $\operatorname{tr}(\sigma^*_{|H^2(X,\mathbb{R})}) = r - m$.

LEMMA 3.5

One has $\operatorname{tr}(\sigma^*|_{H^4(X,\mathbb{R})}) = \frac{(m-r)(m-r-1)}{2}$.

Proof

Denote by ξ a primitive pth root of unity, and denote by V_{ξ^i} the one-dimensional representation of G with character ξ^i, for $i = 1, \ldots, p-1$. Then, as a representation of G,

$$H^2(X,\mathbb{R}) \cong \mathbb{R}^{\oplus r} \oplus \bigoplus_{i=1}^{p-1} V_{\xi^i}^{\oplus m},$$

where $\mathbb{R} = V_{\xi^0}$ stands for the trivial representation. Since $H^4(X,\mathbb{R}) \cong \operatorname{Sym}^2 H^2(X,\mathbb{R})$ (cf. [39, Theorem 1.3]) one gets

$$\begin{aligned}\operatorname{Sym}^2 H^2(X,\mathbb{R}) \cong \mathbb{R}^{\oplus \frac{r(r+1)}{2}} &\oplus \bigoplus_{i=1}^{p-1} V_{\xi^i}^{\oplus rm} \oplus \bigoplus_{i=1}^{p-1} \operatorname{Sym}^2(V_{\xi^i}^{\oplus m}) \\ &\oplus \bigoplus_{i=1}^{p-1} \bigoplus_{j=i+1}^{p-1} (V_{\xi^i}^{\oplus m} \otimes V_{\xi^j}^{\oplus m}).\end{aligned}$$

Since $\operatorname{Sym}^2(V_{\xi^i}^{\oplus m}) \cong V_{\xi^{2i}}^{\oplus \frac{m(m+1)}{2}}$ and $V_{\xi^i}^{\oplus m} \otimes V_{\xi^j}^{\oplus m} \cong V_{\xi^{i+j}}^{\oplus m^2}$ one gets

$$\begin{aligned}\operatorname{tr}(\sigma^*_{|H^4(X,\mathbb{R})}) &= \frac{r(r+1)}{2} + \Big(\sum_{i=1}^{p-1} \xi^i\Big) rm + \Big(\sum_{i=1}^{p-1} \xi^{2i}\Big) \frac{m(m+1)}{2} \\ &\quad + \Big(\sum_{i=1}^{p-1} \sum_{j=i+1}^{p-1} \xi^{i+j}\Big) m^2 \\ &= \frac{r(r+1)}{2} - rm - \frac{m(m+1)}{2} + m^2 \\ &= \frac{(m-r)(m-r-1)}{2}.\end{aligned}$$

□

Using the fact that $r = 23 - (p-1)m$ we obtain the following.

COROLLARY 3.6

The Euler characteristic of the fixed locus satisfies

$$\chi(X^G) = 324 - \frac{51}{2}mp + \frac{1}{2}m^2p^2. \tag{2}$$

We deduce one further relation between the parameters a and m.

COROLLARY 3.7

One has $a \le m$.

Proof

By the universal coefficient theorem we have that $H^i(X^G,\mathbb{Z}) \otimes \mathbb{F}_p$ injects in $H^i(X^G,\mathbb{F}_p)$. Since $h^i(X^G,\mathbb{R})$ equals the rank of the free part of $H^i(X^G,\mathbb{Z})$ it follows that $h^i(X^G,\mathbb{R}) \le h^i(X^G,\mathbb{F}_p)$ for all i. So we get $h^*(X^G,\mathbb{F}_p) - \chi(X^G) \ge 0$. Combining (1) and (2) we get

$$h^*(X^G,\mathbb{F}_p) - \chi(X^G) = 2(a-m)(a-25+mp-m).$$

By Theorem 3.3 we have $a - 25 + mp - m < 0$. Hence, $a \le m$. □

3.3. Computation of the invariant lattice

Let X be an IHS $-$ K3$^{[2]}$ with a nonsymplectic action of a group $G = \langle\sigma\rangle$ of prime order p with $3 \le p \le 19$, $p \ne 5$. Recall that $H^2(X,\mathbb{Z})$ is isometric to the lattice $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2\rangle$, T is the invariant sublattice of $H^2(X,\mathbb{Z})$, and S is its orthogonal complement.

For each value of p, combining Proposition 3.1, formulas (1) and (2), and Corollary 3.7, we get all the possible values of $m := m_G(X)$ and $a := a_G(X)$, and we compute the values of $\chi := \chi(X^G)$ and $h^* := h^*(X^G,\mathbb{F}_p)$. By Proposition 3.1, the lattice S has signature $(2,(p-1)m-2)$ and is p-elementary with discriminant group $(\mathbb{Z}/p\mathbb{Z})^{\oplus a}$, and the lattice T has signature $(1, 22-(p-1)m)$ and discriminant group $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$. Considering T and S as sublattices of L we call a triple (p,m,a) *admissible* if such sublattices of L with these invariants do exist. In this case, we compute their isometry class. We prove the following result.

THEOREM 3.8

For every admissible value of (p,m,a), there exists a unique even p-elementary lattice S of signature $(2,(p-1)m-2)$ with $A_S = (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$. This lattice admits a primitive embedding in L, this embedding is unique if $(p,m,a) \notin \{(3,10,2), (3,8,6), (11,2,2)\}$, and its orthogonal complement T in L is uniquely determined by the signature $(1,22-(p-1)m)$ and the discriminant group $A_T = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$.

Proof

The proof follows from a case-by-case analysis. We use Theorems 2.6 and 2.1 on the existence of hyperbolic p-elementary lattices to exclude the nonadmissible values and determine the isometry class of S. The uniqueness of T and of the embedding of S in L are a direct consequence of Proposition 2.7. Only a few special cases require a more specific argument. We first handle one case in detail to explain the method; then we treat the special cases.

The case $(p,m,a)=(3,9,1)$. Here S has signature $(2,16)$ and is 3-elementary with $a=1$. Since $\mathrm{rank}(S)=18\geq 3+\ell(A_S)=4$ by Theorem 2.6 we can write $S=U\oplus W$, where W is an even hyperbolic 3-elementary lattice of signature $(1,15)$ and $a(W)=1$. We use Theorem 2.1 to compute W and to prove its uniqueness. This also gives the uniqueness of S, so one computes directly that the only possibility (up to isometry) is $W=U\oplus E_6\oplus E_8$. Finally, $1=a\leq 21-\mathrm{rank}(S)$, so, by Proposition 2.7, T is uniquely determined and the embedding of S in L is unique.

We discuss now the special cases.

The case $(p,m,a)=(3,11,1)$. One has $T=\langle 6\rangle$. It is easy to check that the homomorphism $O(T)\to O(A_T)$ is surjective, so the argument of the proof of Proposition 2.7 applies and S admits a unique embedding in L.

The case $(p,m,a)=(3,1,1)$. In this case one computes directly that $S=A_2(-1)$. Then one concludes with Proposition 2.7.

The case $(p,m,a)\in\{(3,10,2),(3,8,6),(11,2,2)\}$. Here we have $a>\mathrm{rank}(T)-2$. Using Theorem 2.5 one sees that T is unique in its genus and one deduces its isometry class.

The case $(p,m,a)=(3,9,5)$. We prove that this case cannot occur. We compute as before that S is isometric to $U^{\oplus 2}\oplus E_8\oplus E_6^\vee(3)$, so its discriminant group is $A_S=\frac{\mathbb{Z}}{3\mathbb{Z}}(\frac{2}{3})^{\oplus 5}$. By Proposition 2.7, if S admits a primitive embedding in L, then its orthogonal T has signature $(1,4)$ and discriminant form $\frac{\mathbb{Z}}{3\mathbb{Z}}(\frac{4}{3})^{\oplus 5}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}}(\frac{3}{2})$. Assume that such a lattice does exist. Consider its 3-adic completion $T_3:=T\otimes_{\mathbb{Z}}\mathbb{Z}_3$. One has $A_{T_3}=(A_T)_3$ so $q_{T_3}=\frac{\mathbb{Z}}{3\mathbb{Z}}(\frac{4}{3})^{\oplus 5}$. The rank of T_3 is then equal to $\ell(A_{T_3})$. By Nikulin [32, Theorem 1.9.1], there exists a unique 3-adic lattice K of rank $\ell(A_{T_3})$ and discriminant form q_{T_3}. Then necessarily the determinants of the lattices T_3 and K differ by an invertible square:

$$\det T=\det T_3\equiv\det K\mod(\mathbb{Z}_3^*)^2.$$

One has $\det T=(-1)^4|A_T|=2\cdot 3^5$. Using [32, Proposition 1.8.1] one finds that $K=\langle 3\theta\rangle^{\oplus 5}$ with $\theta=\frac{1}{4}\in\mathbb{Z}_3^*$ so $\det K=(\frac{3}{4})^5$. The relation $\det T\equiv\det K \mod(\mathbb{Z}_3^*)^2$ gives here $2^{11}\in(\mathbb{Z}_3^*)^2$. This is not true (it would imply that 2 is a square modulo 3), so such a lattice T does not exist. □

REMARK 3.9

The isometry classes of the lattices S and T for all admissible values of (p,m,a) are summarized in the Appendix in Tables 1–7 corresponding to $p=3,7,11,13,17,19$, respectively. The excluded values of (p,m,a) are not written in the tables.

PROPOSITION 3.10

Under the same assumptions as in Theorem 3.8, the lattice T admits a unique primitive embedding in the lattice L whose orthogonal complement is S.

Proof

The proof is essentially the same as in Proposition 2.7. Observe that in this case the orthogonal complement is given and it is isometric to S. So a primitive embedding of T into L corresponds to a quadruple $(H_T, H_L, \gamma, \gamma_S)$. The only possibility is $H_T = H_L = \frac{\mathbb{Z}}{2\mathbb{Z}}$, so the only choice is $\gamma = \mathrm{id}$. Finally, observe that all the lattices S in the tables except the case $(p,m,a) = (3,1,1)$ have $\mathrm{rank}(S) \geq \ell(S) + 2$, so by [16, Proposition 1.4.7] the morphism $O(S) \to O(A_S)$ is surjective. In the case $(p,m,a) = (3,1,1)$ one shows the surjectivity by hand. Hence, different choices of the isomorphism γ_S produce isomorphic embeddings of T in L. $\square$

4. Deformation of automorphisms on IHS manifolds

Let X be an IHS manifold, and let $f \in \mathrm{Aut}(X)$ be a biholomorphic automorphism of X. We denote by $p\colon \mathcal{X} \to \mathrm{Def}(X)$, $p^{-1}(0) = X$, the universal deformation of X. By a theorem of Horikawa [21, Theorem 8.1], there exists an open neighborhood Δ of the origin of $\mathrm{Def}(X)$, a family of deformations $p'\colon \mathcal{X}' \to \Delta$, $p'^{-1}(0) = X$, and a holomorphic map $\Phi\colon \mathcal{X}_{|\Delta} \to \mathcal{X}'$ over Δ such that $\Phi_0 = f$. By the universality of p, there exists a unique holomorphic map $\gamma\colon \Delta \to \mathrm{Def}(X)$ with $\gamma(0) = 0$ such that $\mathcal{X}' = \Delta \times_{\mathrm{Def}(X)} \mathcal{X}$, so we obtain by composition a holomorphic map $F\colon \mathcal{X}_{|\Delta} \to \mathcal{X}$ such that $F_0 = f$ with a commutative diagram

$$\begin{tikzcd} \mathcal{X}_\Delta \arrow[r, "F"] \arrow[d, "p_\Delta"'] & \mathcal{X} \arrow[d, "p"] \\ \Delta \arrow[r, "\gamma"] & \mathrm{Def}(X) \end{tikzcd} \tag{3}$$

Denote $D := \Delta^\gamma = \{t \in \Delta \mid \gamma(t) = t\}$. By restricting to D one obtains a family of holomorphic maps

$$\begin{tikzcd} \mathcal{X}_D \arrow[r, "F"] \arrow[d, "p_D"'] & \mathcal{X}_D \arrow[d, "p_D"] \\ D \arrow[r, equal] & D \end{tikzcd}$$

with $F_0 = f$ and such that the holomorphic map $F_t\colon \mathcal{X}_t \to \mathcal{X}_t$ is an automorphism for all $t \in D$ (by shrinking D if necessary). The pair $(p_D\colon \mathcal{X}_D \to D, F)$ is thus a deformation space of the pair (X, f) (see also [28, Definition 1.1]). From the diagram (3) we get a commutative diagram of vector bundles over X with exact rows

$$\begin{tikzcd} 0 \arrow[r] & TX \arrow[r] \arrow[d, "df"] & T\mathcal{X}_\Delta|_X \arrow[r] \arrow[d, "dF"] & T_0\Delta \arrow[r] \arrow[d, "d_0\gamma"] & 0 \\ 0 \arrow[r] & TX \arrow[r] & T\mathcal{X}|_X \arrow[r] & T_0\operatorname{Def}(X) \arrow[r] & 0 \end{tikzcd}$$

that induces in cohomology an exact sequence

$$\begin{tikzcd} T_0\Delta \arrow[r, "\rho"] \arrow[d, "d_0\gamma"'] & H^1(X,TX) \arrow[d, "df"] \\ T_0\operatorname{Def}(X) \arrow[r, "\rho"] & H^1(X,TX) \end{tikzcd}$$

where ρ is the Kodaira–Spencer map. Since the deformation is universal, ρ is an isomorphism (note that $T_0\Delta = T_0\operatorname{Def}(X)$), and df is an isomorphism since f is an automorphism of X. It follows that $d_0\gamma$ is invertible, so the fixed locus $D = \Delta^\gamma$ is smooth and its dimension equals the dimension of the invariant space $H^1(X,TX)^{df}$.

If the automorphism f acts symplectically on X, then the isomorphism $TX \cong \Omega_X$ induced by ω_X is f-equivariant (f induces natural actions denoted df on tangent vectors and denoted f^* on differential forms), so $\dim D = \dim(H^{1,1}(X)^{f^*})$. If the automorphism f acts nonsymplectically on X, that is, $f^*\omega_X = \xi\omega_X$ for some $\xi \in \mathbb{C}^*$, $\xi \neq 1$, then the isomorphism $TX \cong \Omega_X$ is not f-equivariant and one computes that $H^1(X,TX)^{f^*}$ is isomorphic to the eigenspace of $H^{1,1}(X)$ corresponding to the eigenvalue ξ of f^*. Assume that f is an automorphism of prime order p, $G = \langle f \rangle$, so that ξ is a primitive pth root of unity. Since the action of f^* on $H^2(X,\mathbb{C})$ comes from an action on the lattice $H^2(X,\mathbb{Z})$, the eigenspaces of f^* corresponding to the eigenvalues ξ^i, $i = 1,\dots,p-1$, have the same dimension, which is $m_G(X)$. Since $H^{2,0}(X)$ is an eigenspace for the eigenvalue ξ and $H^{0,2}(X)$ is one for $\bar{\xi}$, it follows that $\dim D = m_G(X) - 1$ if $p \geq 3$ and $\dim D = m_G(X) - 2$ if $p = 2$.

If X is an IHS manifold and $f \in \operatorname{Aut}(X)$, $G = \langle f \rangle$, then it follows from Ehresmann's theorem that the G-module structure of $H^*(X,\mathbb{Z})$ is invariant under deformation of the pair (X,f), so the lattices $T_G(X)$ and $S_G(X)$ and the values $h^*(X^G)$ and $\chi(X^G)$ (by (1) and (2)) are also invariant.

DEFINITION 4.1

Let X be an $\mathrm{IHS}-\mathrm{K3}^{[2]}$. An automorphism σ of X is called *natural* if there exists a K3 surface Σ and an automorphism φ of Σ such that (X,σ) is deformation equivalent to $(\Sigma^{[2]},\varphi^{[2]})$, where $\varphi^{[2]}$ denotes the induced automorphism on $\Sigma^{[2]}$ by φ.

5. Moduli spaces of lattice polarized IHS manifolds

5.1. The global Torelli theorem

We recall some well-known facts from [22] and [26]. If X is an irreducible holomorphic symplectic manifold, then a *marking* for X is a choice of an isometry $\eta : L \longrightarrow H^2(X,\mathbb{Z})$. Two marked pairs (X_1,η_1) and (X_2,η_2) are isomorphic if there is an isomorphism $f : X_1 \longrightarrow X_2$ such that $\eta_1 = f^* \circ \eta_2$. There exists a coarse moduli space $\mathfrak{M}_L$ that parameterizes isomorphism classes of marked pairs, which is a non-Hausdorff smooth complex manifold (see [22]). If X is an $\mathrm{IHS}-\mathrm{K3}^{[2]}$, then this has dimension 21. Denote by

$$\Omega_L := \big\{\omega \in \mathbb{P}(L\otimes\mathbb{C}) \,\big|\, q(\omega)=0, q(\omega+\bar\omega)>0\big\}$$

the *period domain*, which is an open (in the usual topology) subset of the nonsingular quadric defined by $q(\omega)=0$. The period map

$$P : \mathfrak{M}_L \longrightarrow \Omega_L, \qquad (X,\eta)\mapsto \eta^{-1}\big(H^{2,0}(X)\big)$$

is a local isomorphism by the local Torelli theorem [3, Théorème 5]. For $\omega\in\Omega_L$ we put

$$L^{1,1}(\omega) := \big\{\lambda\in L \,\big|\, (\lambda,\omega)=0\big\},$$

where $(\cdot,\cdot)$ is the bilinear form associated to the quadratic form q. Then $L^{1,1}(\omega)$ is a sublattice of L. Let $\lambda\in L$, $\lambda\neq 0$, and consider the hyperplane

$$H_\lambda = \big\{\omega\in\Omega_L \,\big|\, (\omega,\lambda)=0\big\}.$$

Then $L^{1,1}(\omega)=\{0\}$ if ω does not belong to the countable union of hyperplanes $\bigcup_{\lambda\in L\setminus\{0\}} H_\lambda$. In particular, given a marked pair (X,η) we get $\eta^{-1}(\mathrm{NS}(X)) = L^{1,1}(P(X,\eta))$. The set $\{\alpha\in H^{1,1}(X,\mathbb{R}) \mid q(\alpha)>0\}$ has two connected components; we denote the connected component containing the Kähler cone $\mathfrak{K}_X$ as the *positive cone* $\mathfrak{C}_X$.

Following the terminology of [26] recall that two points x,y of a topological space M are called *inseparable* if every pair of open neighborhoods $x\in U$ and $y\in V$ has nonempty intersection. A point $x\in M$ is called a *Hausdorff point* if x and y are separable for every $y\in M$, $y\neq x$.

THEOREM 5.1 (GLOBAL TORELLI THEOREM [39], [26, THEOREM 2.2])
Let $\mathfrak{M}_L^0$ be a connected component of $\mathfrak{M}_L$.

(1) *The period map P restricts to a surjective holomorphic map*

$$P_0 : \mathfrak{M}_L^0 \longrightarrow \Omega_L.$$

(2) *For each $\omega\in\Omega_L$, the fiber $P_0^{-1}(\omega)$ consists of pairwise inseparable points.*

(3) *Let (X_1,η_1) and (X_2,η_2) be two inseparable points of $\mathfrak{M}_L^0$. Then X_1 and X_2 are bimeromorphic.*

(4) *The point $(X,\eta)\in\mathfrak{M}_L^0$ is Hausdorff if and only if $\mathfrak{C}_X=\mathfrak{K}_X$.*

(5) *The fiber* $P_0^{-1}(\omega)$, $\omega\in\Omega_L$ *consists of a single Hausdorff point if* $L^{1,1}(\omega)$ *is trivial or if* $L^{1,1}(\omega)$ *is both of rank one and generated by a class* α *satisfying* $q(\alpha)>0$.

REMARK 5.2

In assertion (5) if $L^{1,1}(\omega)$ is trivial, then X is nonprojective. If it is of rank one and generated by a class α with $q(\alpha)>0$, then X is projective, $\pm\alpha$ is an ample class, and the Néron–Severi group of X has signature $(1,0)$, so its transcendental lattice has signature $(2,20)$ (see [22, Theorem 3.11]).

5.2. Lattice polarizations

In this section and the next one, we extend some constructions and results of [19, Section 6], [18, Section 10], [17], and [2, Section 9]. See also [13] for related results.

Let M be an even nondegenerate lattice of rank $\rho\geq 1$ and signature $(1,\rho-1)$. An M*-polarized* $\mathrm{IHS}-\mathrm{K3}^{[2]}$ is a pair (X,j), where X is a projective $\mathrm{IHS}-\mathrm{K3}^{[2]}$ and j is a primitive embedding of lattices $j: M\hookrightarrow \mathrm{NS}(X)$. Two M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$s (X_1,j_1) and (X_2,j_2) are called *equivalent* if there exists an isomorphism $f: X_1\to X_2$ such that $j_1=f^*\circ j_2$. As in [18, Section 10] and [17] one can construct a moduli space of *marked* M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$s as follows. We fix a primitive embedding of M in L and we identify M with its image in L. A *marking* of (X,j) is an isomorphism of lattices $\eta: L\to H^2(X,\mathbb{Z})$ such that $\eta_{|M}=j$. As observed in [17, p. 11], if the embedding of M in L is unique up to an isometry of L, then every M-polarization admits a compatible marking. Two M-polarized marked $\mathrm{IHS}-\mathrm{K3}^{[2]}$s (X_1,j_1,η_1) and (X_2,j_2,η_2) are called *equivalent* if there exists an isomorphism $f: X_1\to X_2$ such that $\eta_1=f^*\circ\eta_2$. (This clearly implies that $j_1=f^*\circ j_2$.) Let $N:=M^\perp\cap L$ be the orthogonal complement of M in L, and set

$$\Omega_M:=\left\{\omega\in\mathbb{P}(N\otimes\mathbb{C})\;\middle|\; q(\omega)=0, q(\omega+\bar{\omega})>0\right\}.$$

Since N has signature $(2,21-\rho)$ the period domain Ω_M is a disjoint union of two connected components of dimension $21-\rho$. For each M-polarized marked $\mathrm{IHS}-\mathrm{K3}^{[2]}$ (X,j,η), since $\eta(M)\subset\mathrm{NS}(X)$ we have $\eta^{-1}(H^{2,0}(X))\in\Omega_M$. On the other hand, by the surjectivity of the period map (see [22, Theorem 8.1]) restricted to any connected component $\mathfrak{M}_L^0$ of $\mathfrak{M}_L$ we can associate to each point $\omega\in\Omega_M$ an M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$ as follows: there exists a marked pair $(X,\eta)\in\mathfrak{M}_L^0$ such that $\eta^{-1}(H^{2,0}(X))=\omega\in\mathbb{P}(N\otimes\mathbb{C})$ so $M=N^\perp\subset\omega^\perp\cap L$; hence, $\eta(M)\subset H^{2,0}(X)^\perp\cap H^{1,1}_{\mathbb{Z}}(X)=\mathrm{NS}(X)$, and we take $(X,\eta_{|M},\eta)$.

By the local Torelli theorem for IHS manifolds, an M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$ (X,j) has a local deformation space $\mathrm{Def}_M(X)$ that is contractible and smooth of dimension $21-\rho$ such that the period map $P\colon \mathrm{Def}_M(X)\to\Omega_M$ is a local isomorphism (see [17]). By gluing all these local deformation spaces one obtains a moduli space K_M of marked M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$s that is a nonseparated analytic space, with a period map $P\colon K_M \to \Omega_M$. The following proposition generalizes [36, Lemma 2.9].

PROPOSITION 5.3
If $\operatorname{rank} M \leq 20$, *then there exists a dense subset* Ω_M^0 *of* Ω_M *such that if* (X,η) *is a marked* $\mathrm{IHS}-\mathrm{K3}^{[2]}$ *whose period is in* Ω_M^0, *then* $\mathrm{NS}(X)$ *is isomorphic to* M.

Proof
For each $\lambda \in N \setminus \{0\}$ consider the hyperplane $H_\lambda := \{\omega \in \Omega_M \mid (\omega,\lambda) = 0\}$, and let $\mathcal{H} := \bigcup_\lambda H_\lambda$. Each subset $\Omega_M \setminus H_\lambda$ is open and dense in Ω_M; hence, by Baire's theorem the subset $\Omega_M^0 := \Omega_M \setminus \mathcal{H}$ is dense in Ω_M since $\mathcal{H}$ is a countable union of complex closed subspaces. If $\omega = P(X,\eta) \in \Omega_M^0$, then $\mathrm{NS}(X) = \eta(L^{1,1}(\omega)) = \eta(M)$. □

This means that for a *general* point of Ω_M the associated marked M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$ has Néron–Severi group isometric to M and transcendental lattice isometric to N. If $\operatorname{rank} M = 21$, then Ω_M consists of two periods that correspond to an $\mathrm{IHS}-\mathrm{K3}^{[2]}$ whose Néron–Severi group is isometric to M. We specialize this construction of the period domain in the case of projective $\mathrm{IHS}-\mathrm{K3}^{[2]}$'s with nonsymplectic automorphism.

REMARK 5.4
Observe that in the construction we fix an embedding of M in L. Different embeddings give a priori different constructions of Ω_M.

5.3. Eigenperiods of projective $\mathrm{IHS}-\mathrm{K3}^{[2]}$ with a nonsymplectic automorphism

Let (X,j) be an M-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$, and let $G = \langle\sigma\rangle$ be a cyclic group of prime order $p \geq 2$ acting nonsymplectically on X. It is easy to see that the invariant sublattice $T = T_G(X)$ is contained in $\mathrm{NS}(X)$. Assume that the action of G on $j(M)$ is the identity and that there exists a group homomorphism $\rho : G \longrightarrow O(L)$ such that

$$M = L^\rho := \big\{x \in L \,\big|\, \rho(g)(x) = x,\ \forall g \in G\big\}.$$

We define a (ρ,M)-*polarization* of (X,j) as a marking $\eta\colon L \to H^2(X,\mathbb{Z})$ such that $\eta_{|M} = j$ and $\sigma^* = \eta \circ \rho(\sigma) \circ \eta^{-1}$.

Two (ρ,M)-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$'s (X_1,j_1) and (X_2,j_2) are isomorphic if there are markings $\eta_1 : L \to H^2(X_1,\mathbb{Z})$ and $\eta_2 : L \longrightarrow H^2(X_2,\mathbb{Z})$ such that $\eta_{i|M} = j_i$ and an isomorphism $f\colon X_1 \to X_2$ such that $\eta_1 = f^* \circ \eta_2$.

Recall that by construction $\mathbb{C}\omega_X$ is the line in $L \otimes \mathbb{C}$ defined by $\mathbb{C}\omega_X = \eta^{-1}(H^{2,0}(X))$. Let $\xi \in \mathbb{C}^*$ such that $\rho(\sigma)(\omega_X) = \xi\omega_X$. Observe that $\xi \neq 1$ since the action is nonsymplectic and it is a primitive pth root of unity since p is prime. The period ω_X belongs to the eigenspace of $N \otimes \mathbb{C}$ relative to the eigenvalue ξ, where $N = M^\perp \cap L$. We denote it by $N(\xi)$. (If $p = 2$, then we have $\xi = -1$ and we denote $N(\xi) = N_{\mathbb{R}}(\xi) \otimes \mathbb{C}$, where $N_{\mathbb{R}}(\xi)$ is the real eigenspace relative to $\xi = -1$.)

Assume that $\xi \neq -1$. Then the period belongs to the space

$$\Omega_M^{\rho,\xi} := \{x \in \mathbb{P}(N(\xi)) \mid q(x+\bar{x}) > 0\}$$

of dimension $\dim N(\xi) - 1$, which is a complex ball if $\dim N(\xi) \geq 2$. By using the fact that $\xi \neq -1$ it is easy to check that every point $x \in \Omega_M^{\rho,\xi}$ satisfies automatically the condition $q(x) = 0$.

If $\xi = -1$, then we set $\Omega_M^{\rho,\xi} := \{x \in \mathbb{P}(N(\xi)) \mid q(x) = 0, q(x+\bar{x}) > 0\}$. It has dimension $\dim N(\xi) - 2$; clearly, $\Omega_M^{\rho,\xi} \subset \Omega_M$.

Assume now that $M = T = \overline{T} \oplus \langle -2 \rangle$ where $\overline{T}$ is an even nondegenerate lattice of signature $(1, 21 - (p-1)m)$. Assume moreover that $\overline{T}$ has a primitive embedding in the K3 lattice Λ. We fix such an embedding, and we call again $\overline{T}$ the image. This induces in a natural way a primitive embedding of T in L. We then identify T with its image. Let $N = S = T^\perp \cap L$, and assume that $S \subset \Lambda$. For $\delta \in S$ with $q(\delta) = -2$, denote $H_\delta = \delta^\perp \cap S$ and $\Delta := \bigcup_{\delta \in S, q(\delta)=-2} H_\delta$. Then we have the following (with the same notation as above).

THEOREM 5.5
Let X be a (ρ,T)-polarized $\mathrm{IHS}-\mathrm{K3}^{[2]}$ *such that $H^{2,0}(X)$ is contained in the eigenspace of $H^2(X,\mathbb{C})$ relative to ξ. Then $\omega_X \in \Omega_T^{\rho,\xi}$, and conversely, if $\dim N(\xi) \geq 2$, then every point of $\Omega_T^{\rho,\xi} \setminus \Delta$ is the period point of some (ρ,T)-polarized* $\mathrm{IHS}-\mathrm{K3}^{[2]}$.

The proof is an application of a result of Namikawa that we recall for convenience.

THEOREM 5.6 ([31, THEOREM 3.10])
Let Σ be a K3 surface, and let G be a finite subgroup of the group of isometries of $H^2(\Sigma,\mathbb{Z})$. Denote by ω the period of Σ in $H^2(\Sigma,\mathbb{C})$, and set $S_G(\Sigma) := (H^2(\Sigma,\mathbb{Z})^G)^\perp \cap \{\mathbb{C}\omega\}^\perp$. Then there exists an element ϖ in the Weyl group $W(\Sigma)$ of Σ such that $\varpi G \varpi^{-1} \subset \mathrm{Aut}(\Sigma)$ if and only if

(i) *$\mathbb{C}\omega$ is G-invariant;*
(ii) *$S_G(\Sigma)$ contains no element of length -2;*
(iii) *if $\omega \in H^2(\Sigma,\mathbb{C})^G$, then $S_G(\Sigma)$ is nondegenerate and negative definite if $S_G(\Sigma) \neq 0$;*
(iii′) *if $\omega \notin H^2(\Sigma,\mathbb{C})^G$, then $H^2(\Sigma,\mathbb{C})^G$ contains an element α with $(\alpha,\alpha) > 0$.*

Proof of Theorem 5.5
We have already proven that $\omega_X \in \Omega_T^{\rho,\xi}$. Conversely, if $\dim N(\xi) \geq 2$, then the locus $\Omega_T^{\rho,\xi} \setminus \Delta$ is nonempty (see the proof of Proposition 5.3). By our assumptions on the lattice T, for any $\omega \in \Omega_T^{\rho,\xi} \setminus \Delta$ the map $\rho(\sigma)$ acts as the identity on the $\langle -2 \rangle$ class of the decomposition $\Lambda \oplus \langle -2 \rangle = L$ (where Λ is the K3 lattice with a fixed embedding to L) so it leaves invariant the lattice Λ. We thus have a corestriction map to $\Lambda = L/\langle -2 \rangle$:

$$\bar{\rho}\colon G \longrightarrow O(\Lambda).$$

This has the property that $\overline{T}=\Lambda^{\bar\rho(\sigma)}$, and by our assumption on S, the orthogonal complement of $\overline{T}$ in Λ is S. Now as in [18, Section 11] we consider the period domain of $(\bar\rho,\overline{T})$-polarized K3 surfaces. For $\xi\neq -1$ this is

$$D^{\rho,\xi}_{\overline{T}} := \{x\in \mathbb{P}(S(\xi)) \mid (x,\bar x)>0\},$$

and for $\xi=-1$ this is

$$D^{\rho,\xi}_{\overline{T}} := \{x\in \mathbb{P}(S(\xi)) \mid (x,x)=0,\ (x,\bar x)>0\}.$$

Observe that $D^{\rho,\xi}_{\overline{T}}\setminus\Delta$ is not empty because it coincides with $\Omega^{\rho,\xi}_{T}\setminus\Delta$ (which is not empty by assumption). The point $\omega\in\Omega^{\rho,\xi}_{T}\setminus\Delta$ is then also a point in $D^{\rho,\xi}_{\overline{T}}\setminus\Delta$; hence, there exists a $\overline{T}$-polarized K3 surface Σ (see [2, Section 9] and [18, Section 11]). It has a $\overline{T}$-polarization $\eta:\Lambda\longrightarrow H^2(\Sigma,\mathbb{Z})$ with $\eta_{|\overline{T}}(\overline{T})\subset \mathrm{NS}(\Sigma)$ and $\eta(\omega)=H^{2,0}(\Sigma)$. The isometry $\eta\circ\bar\rho(\sigma)\circ\eta^{-1}$ acts on $H^2(\Sigma,\mathbb{Z})$, it is the identity on $j(\overline{T})$ (where $j:=\eta_{|\overline{T}}$), and it preserves $H^{2,0}(\Sigma)$. Let us check that the conditions of Theorem 5.6 are satisfied. Since the line generated by ω is preserved by $\bar\rho(\sigma)$ we have condition (i). By assumption, $S\cap\{\omega\}^{\perp}\cap\Lambda$ does not contain classes of length -2; this gives condition (ii). By construction, $\overline{T}$ is the $\bar\rho(\sigma)$-invariant sublattice of Λ and it is hyperbolic, so it contains a class α with $(\alpha,\alpha)>0$; this gives condition (iii′). So there exists $a\in\mathrm{Aut}(\Sigma)$ with $a^*=\eta\circ\bar\rho(\sigma)\circ\eta^{-1}$ (up to conjugation with an element of the Weyl group). Take $Y:=\Sigma^{[2]}$ with the marking $L\to H^2(Y,\mathbb{Z})=H^2(\Sigma,\mathbb{Z})\oplus\langle -2\rangle$ in such a way that its restriction to Λ is equal to η. We still denote this marking by η. By construction, Y admits an automorphism $a^{[2]}$ such that $(a^{[2]})^*=\eta\circ\rho(\sigma)\circ\eta^{-1}$, so Y is (ρ,T)-polarized with period $\omega_Y=\omega$. □

COROLLARY 5.7

Let X and X' be two IHS $-$ K3$^{[2]}$*'s admitting nonsymplectic automorphisms σ and σ', respectively, of the same prime order p. Assume as above that $T_\sigma(X)=\overline{T}\oplus\langle -2\rangle = T_{\sigma'}(X')$ and $S_\sigma(X)=S_{\sigma'}(X')$ have rank at least p. Then there exists $h\in O(L)$ such that $h\circ\rho(\sigma)\circ h^{-1}=\rho'(\sigma')$. (That is, once the order is fixed, the action of σ on L is uniquely determined by T_σ and S_σ.)*

Proof

We use the same notation as above. As in the proof of Theorem 5.5 we can associate to X and X' two K3 surfaces Σ and Σ' with respective automorphisms $\bar\sigma$ and $\bar\sigma'$ such that $\overline{T}_{\bar\sigma}(\Sigma)=\overline{T}_{\bar\sigma'}(\Sigma')$ and $S_{\bar\sigma}(\Sigma)=S_{\bar\sigma'}(\Sigma')$. Then by [2, Proposition 9.3], $\bar\sigma^*$ and $\bar\sigma'^*$ are conjugated via an element of $O(\Lambda)$. More precisely, if $\bar\rho\colon\langle\sigma\rangle\longrightarrow O(\Lambda)$ and $\bar\rho'\colon\langle\sigma'\rangle\longrightarrow O(\Lambda)$ are the corestrictions of the analogous morphisms with image in $O(L)$, then there exists $\bar h\in O(\Lambda)$ such that $\bar h\circ\bar\rho(\sigma)\circ\bar h^{-1}=\bar\rho'(\sigma')$. We define an isometry h of $L=\Lambda\oplus\langle -2\rangle$ by $h_{|\Lambda}=\bar h$ and $h_{|\langle -2\rangle}=\mathrm{id}$. Recall that

$$\rho(\sigma)_{|\langle -2\rangle}=\mathrm{id},\qquad \rho'(\sigma')_{|\langle -2\rangle}=\mathrm{id}.$$

This implies that $h\circ\rho(\sigma)\circ h^{-1}=\rho'(\sigma')$, hence the statement. □

6. Examples of automorphisms

6.1. Natural automorphisms on Hilbert schemes of points

Let Σ be a K3 surface, and let φ be a nonsymplectic automorphism of prime order $p \geq 3$ acting on Σ. By [2, Theorem 0.1] the fixed locus Σ^φ is a disjoint union of curves and points of the form

$$\Sigma^\varphi = C_g \cup R_1 \cup \cdots \cup R_k \cup \{p_1, \ldots, p_N\},$$

where C_g is a smooth curve of genus $g \geq 0$, R_i, $i = 1, \ldots, k$, are smooth rational curves, and p_j, $j = 1, \ldots, N$ are isolated fixed points. The isolated fixed points are of different types depending on the local action of φ at them. The possible local actions are

$$A_{p,t} = \begin{pmatrix} \xi_p^{t+1} & 0 \\ 0 & \xi_p^{p-t} \end{pmatrix}, \quad t = 0, \ldots, p-2;$$

here we use the same notation as in [2] and we denote by n_t the number of isolated fixed points corresponding to the local action $A_{p,t}$.

By using the results of [7, Section 4.2] one computes the fixed locus of $\varphi^{[2]}$ on $\Sigma^{[2]}$. It consists of the following:

- $N(N-1)/2 + 2(N - n_{\frac{p-1}{2}})$ isolated fixed points. For this contribution, one has $\chi = h^* = N(N-1)/2 + 2(N - n_{\frac{p-1}{2}})$.
- $(n_{\frac{p-1}{2}} + Nk + k)$ smooth rational curves: $n_{\frac{p-1}{2}}$ for each fixed point at which the local action has two equal eigenvalues, Nk for each couple of a fixed point and a rational curve, and k for each rational curve. This last case comes from the fact that, taking the schemes of length 2 of $\Sigma^{[2]}$ over a point of a rational curve, we get a curve isomorphic to $\mathbb{P}^1$ contained in the exceptional set. (We also get a surface isomorphic to $\mathbb{P}^2 = (\mathbb{P}^1)^{[2]}$; this contribution is taken into account below.) For this contribution, one has $\chi = h^* = 2(n_{\frac{p-1}{2}} + Nk + k)$.
- $N+1$ curves isomorphic to C_g, one for each couple consisting of a fixed point p_j and the curve C_g, and one for the curve C_g. The explanation for this last contribution is the same as above in the case of the rational curves. Here $\chi = (N+1)(2-2g)$, and $h^* = (N+1)(2+2g)$.
- $k(k-1)/2$ surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, one for each couple of distinct rational curves. Here $\chi = h^* = 4(k(k-1)/2)$.
- k surfaces isomorphic to $\mathbb{P}^1 \times C_g$, one for each rational curve. Here $\chi = (4-4g)k$, and $h^* = (4+4g)k$.
- k surfaces isomorphic to $\mathbb{P}^2 = (\mathbb{P}^1)^{[2]}$, one for each rational curve. Here $\chi = h^* = 3k$.
- one surface $(C_g)^{[2]}$, which is the Hilbert scheme of two points on C_g. Here $\chi = 3 + 2g^2 - 5g$, and $h^* = 3 + 2g^2 + 3g$.

As a consequence, the fixed locus of $\varphi^{[2]}$ on $\Sigma^{[2]}$ has invariants

$$\begin{aligned} \chi = {} & 2(n_{\frac{p-1}{2}} + Nk + k) + N(N-1)/2 \\ & + 2(N - n_{\frac{p-1}{2}}) + 4k(k-1)/2 + 3k + (N+1)(2-2g) \end{aligned}$$

$$
\begin{aligned}
&+k(4-4g)+3-5g+2g^2\\
&=(1/2)(2g-2-N-2k)(2g-5-N-2k),\\
h^* &= N^2/2+7N/2+2Nk+7k+2Ng+5+5g+2k^2+4kg+2g^2.
\end{aligned}
$$

Observe that $\varphi^{[2]}$ acts nonsymplectically on $\Sigma^{[2]}$. In fact, there is an injective morphism $\iota : H^2(\Sigma,\mathbb{C}) \longrightarrow H^2(\Sigma^{[2]},\mathbb{C})$ such that

$$H^2(\Sigma^{[2]},\mathbb{C}) = \iota\big(H^2(\Sigma,\mathbb{C})\big) \oplus \mathbb{C}[E],$$

where E denotes the exceptional set and ι respects the Hodge decomposition (see [3, Proposition 6]). If $\alpha \in H^{2,0}(\Sigma)$, then $\iota(\alpha) \in H^{2,0}(\Sigma^{[2]})$ and, by the definition of ι, one has $\varphi^{[2]}(\iota(\alpha)) = \iota(\varphi(\alpha))$ (see [11]). This implies that $S_\varphi(\Sigma) = S_{\varphi^{[2]}}(\Sigma^{[2]})$ and $\langle -2\rangle \oplus T_\varphi(\Sigma) = T_{\varphi^{[2]}}(\Sigma^{[2]})$. In Tables 1–7 we mark with a ♣ the cases that are realized with natural automorphisms.

REMARK 6.1

Theorem 3.3 for the order 5 automorphisms holds only for natural automorphisms, so in Table 2 the list of admissible triples $(5,m,a)$ is only in this special case. For the moment, there are no other known examples of nonsymplectic automorphisms of order 5 (see also Remark 6.3).

6.2. The Fano variety of lines on a cubic four-fold

Let V be a smooth cubic hypersurface in $\mathbb{P}^5$. The *Fano variety of lines* on V is defined as

$$F(V) := \big\{ l \in \mathrm{Gr}(1,5) \,\big|\, l \subset V \big\}.$$

By [6] the variety $F(V)$ is an $\mathrm{IHS}-\mathrm{K3}^{[2]}$. One can construct examples of nonsymplectic automorphisms of prime order by starting with an automorphism of a cubic hypersurface in $\mathbb{P}^5$ and then looking at the induced automorphism on $F(V)$. By a classical result of Matsumura and Monsky [27] the automorphism group of a cubic hypersurface in $\mathbb{P}^5$ is finite, the automorphisms are induced by linear automorphisms of $\mathbb{P}^5$, and a generic cubic hypersurface in $\mathbb{P}^5$ has no automorphism. All automorphisms of prime order of smooth cubic four-folds are classified in [20, Theorem 3.8]. We are interested in those that induce a nonsymplectic automorphism on $F(V)$.

Denote by $Z \subset F(V) \times V$ the universal family, and denote by p and q the projection to $F(V)$ and V, respectively. By [6, Proposition 4] the Abel–Jacobi map

$$A := p_* q^* : H^4(V,\mathbb{Z}) \longrightarrow H^2\big(F(V),\mathbb{Z}\big)$$

is an isomorphism of Hodge structures with $A(H^{3,1}(V)) \cong H^{2,0}(F(V))$. If σ is an automorphism of V, then by the equivariance of A one has $\sigma^*(\omega_{F(V)}) = A(\sigma^*(A^{-1}(\omega_{F(V)})))$. Let $f := f(x_0,\ldots,x_5)$ be the cubic polynomial whose zero set in $\mathbb{P}^5$ is V. By Griffith's residue theorem (see [40, Proposition 18.2]) the cohomology group $H^{3,1}(V)$ is one-dimensional and generated by the residue $\mathrm{Res}(\frac{\omega_{\mathbb{P}^5}}{f^2})$, where

$$\omega_{\mathbb{P}^5} = \sum_i (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_5.$$

Assume that σ is an automorphism of $\mathbb{P}^5$ such that $\sigma(V) = V$. The form $\frac{\omega_{\mathbb{P}^5}}{f^2}$ is a closed meromorphic five-form on $\mathbb{P}^5$, holomorphic on $\mathbb{P}^5 \setminus V$, with poles of order 2 along V, that we consider as a differential form on $U := \mathbb{P}^5 \setminus V$. Taking its cohomology class we get an element in $H^5(U,\mathbb{C})$. There is a natural σ-equivariant isomorphism $H^5(U,\mathbb{C}) \cong H^5(\mathbb{P}^5, \Omega^\bullet_{\mathbb{P}^5}(\log V))$. For a form with logarithmic pole $\alpha \wedge \frac{df}{f} \in \Omega^\bullet_{\mathbb{P}^5}(\log V)$, with $\alpha \in \Omega^{\bullet -1}_{\mathbb{P}^5}$, one has by definition $\mathrm{Res}(\alpha \wedge \frac{df}{f}) = 2i\pi\alpha_{|V}$ (see [40, Section 18.1.1]). Since σ has finite order we can assume that it acts diagonally; that is, $\sigma(x_0,\dots,x_5) = (\alpha_0 x_0, \dots, \alpha_5 x_5)$, and we denote $\det(\sigma) := \prod_i \alpha_i$. Then $\sigma^*\omega_{\mathbb{P}^5} = \det(\sigma)\omega_{\mathbb{P}^5}$. Assume furthermore that f is a projective invariant for σ, that is, $\sigma^* f = \lambda_\sigma f$ with $\lambda_\sigma \in \mathbb{C}^*$. Then $\sigma^*(\frac{df}{f}) = \frac{df}{f}$ so the map $\mathrm{Res}\colon H^5(U,\mathbb{C}) \to H^4(V,\mathbb{C})$ is σ-equivariant. It follows that

$$\sigma^* \mathrm{Res}\Big(\frac{\omega_{\mathbb{P}^5}}{f^2}\Big) = \frac{\det(\sigma)}{\lambda_\sigma^2} \mathrm{Res}\Big(\frac{\omega_{\mathbb{P}^5}}{f^2}\Big).$$

This proves the following result.

LEMMA 6.2

Let σ be a diagonal automorphism of V. If the homogeneous polynomial of degree three $f \in \mathbb{C}[x_0,\dots,x_5]$ which defines V is a projective invariant for the action of σ with $\sigma^* f = \lambda_\sigma f$, then the action of σ on $F(V)$ is nonsymplectic if and only if $\frac{\det(\sigma)}{\lambda_\sigma^2} \neq 1$.

Looking inside the classification of [20, Theorem 3.8] we see that examples of nonsymplectic automorphisms occur only for $p = 2, 3$. In this section we consider only the case $p = 3$ and we find four families of examples, for which we compute now the fixed locus and the lattices T and S. Put $\xi := \exp(2\pi i/3)$.

REMARK 6.3

There is a small mistake in the classification of [20]: the cubics of the family denoted $\mathcal{F}_5^2$, which would have a nonsymplectic automorphism of order 5, are in fact all singular (as confirmed to us by the authors of [20]). So this family should not be in the list.

EXAMPLE 6.4 (CASE $T = \langle 6 \rangle$)

Consider the automorphism of order 3 of $\mathbb{P}^5$ given by

$$\sigma_1(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_0 : x_1 : x_2 : x_3 : x_4 : \xi x_5).$$

The family of invariant cubics is

$$V_1\colon L_3(x_0,\dots,x_4) + x_5^3 = 0,$$

where L_3 is a homogeneous polynomial of degree 3. The fixed locus of σ_1 on V is the cubic three-fold $\mathcal{C} := \{x_5 = 0, L_3(x_0, \ldots, x_4) = 0\}$. The fixed points on $F(V_1)$ correspond to σ_1-invariant lines on V. If a line $L \subset V$ is invariant, then either it is pointwise fixed or it contains two fixed points. If L is pointwise fixed, then it is contained in $\mathcal{C}$; if it is only invariant but not pointwise fixed, then L intersects $\mathcal{C}$ in two points. Hence it intersects also the hyperplane $\{x_5 = 0\}$ in two points so it is contained in it. Hence, $L \subset V \cap \{x_5 = 0\}$; this means that L is contained in $\mathcal{C}$. We call again σ_1 the induced automorphism on $F(V_1)$; this is nonsymplectic by Lemma 6.2. We have shown that its fixed locus is the Fano surface $F(\mathcal{C})$ of $\mathcal{C}$. This is a well-known surface of general type with Hodge numbers $h^{1,0} = h^{0,1} = 5$, $h^{0,2} = h^{2,0} = 10$, and $h^{1,1} = 25$. (These are computed, e.g., in [14].) Hence, one computes $\chi(F(V_1)^{\sigma_1}) = 27$ and $h^*(F(V_1)^{\sigma_1}, \mathbb{F}_3) = 67$. By using (1) and (2) we get $m = 11$ and $a = 1$, so looking in the table we have $S_{\sigma_1}(F(V_1)) = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$ and $T_{\sigma_1}(F(V_1)) = \langle 6 \rangle$. Finally, observe that the dimension of the family $F(V_1)$ is 10, which is also $\operatorname{rank}(S_{\sigma_1}(F(V_1)))/2 - 1$.

EXAMPLE 6.5 (CASE $T = U \oplus A_2^{\oplus 5} \oplus \langle -2 \rangle$)

Consider the automorphism of order 3 of $\mathbb{P}^5$ given by

$$\sigma_2(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_0 : x_1 : x_2 : x_3 : \xi x_4 : \xi x_5).$$

The family of invariant cubics is

$$V_2 \colon L_3(x_0, x_1, x_2) + M_3(x_4, x_5) = 0,$$

where L_3 and M_3 are homogeneous polynomials of degree 3. The fixed locus of σ_2 on V_2 is $\{x_0 = x_1 = x_2 = x_3 = 0, M_3(x_4, x_5) = 0\}$, which are three distinct points p_1, p_2, p_3 and the cubic surface K of $\mathbb{P}^3$ given by $\{x_4 = x_5 = 0, L_3(x_0, x_1, x_2) = 0\}$. An invariant line through two points must also contain the third point, and in fact, it is the line $\{x_0 = x_1 = x_2 = x_3 = 0\}$, which is not contained in V_2. An invariant line through p_i and a point of K is contained in V_2. So on $F(V_2)$ we have three fixed surfaces isomorphic to the rational cubic K; this has $h^2(K, \mathbb{Z}) = h^{1,1}(K) = 7$. Moreover, each fixed line on K determines a fixed point on $F(V_2)$ so we have 27 isolated fixed points. By Lemma 6.2 the induced automorphism on $F(V_2)$, which we call again σ_2, is nonsymplectic. Using the fact that the odd cohomology of the fixed locus is zero we have

$$\chi(X^{\sigma_2}) = h^*(X^{\sigma_2}, \mathbb{F}_3) = 3(2 + 7) + 27 = 54.$$

Then one computes that $m = a = 5$ by using (2) and (1). Looking in the table we have $S_{\sigma_2}(F(V_2)) = U \oplus U(3) \oplus A_2^{\oplus 3}$ and $T_{\sigma_2}(F(V_2)) = U \oplus A_2^{\oplus 5} \oplus \langle -2 \rangle$. Finally, the dimension of the family is 4 which is equal to $\operatorname{rank}(S_{\sigma_2}(F(V_2)))/2 - 1$. Observe that this automorphism does not have the same fixed locus as the natural automorphism on a Hilbert scheme with the same lattices S and T.

EXAMPLE 6.6 (CASE $T = \langle 6\rangle \oplus E_6^\vee(3)$)

Consider the automorphism of order 3 of $\mathbb{P}^5$ given by

$$\sigma_3(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_0 : x_1 : x_2 : \xi x_3 : \xi x_4 : \xi^2 x_5).$$

The family of invariant cubics is

$$\begin{aligned} V_3\colon\, & L_3(x_0, x_1, x_2) + M_3(x_3, x_4) + x_5^3 \\ & + x_5\big(x_3 L_1(x_0, x_1, x_2) + x_4 M_1(x_0, x_1, x_2)\big) = 0, \end{aligned}$$

where L_3 and M_3 are homogeneous polynomials of degree 3 and L_1 and M_1 are linear forms. The fixed locus of σ_3 on V_3 is the union of the elliptic curve $E :$ $\{x_3 = x_4 = x_5 = 0, L_3(x_0, x_1, x_2) = 0\}$ and the three isolated fixed points q_1, q_2, q_3 given by $\{x_0 = x_1 = x_2 = x_5 = 0, M_3(x_3, x_4) = 0\}$. Any invariant line containing two fixed points on E is contained in the plane $\{x_3 = x_4 = x_5 = 0\}$ so it cannot be contained on V_3. On the other hand a line through two of the points q_j must also contain the third point, and it is the line $\{x_3 = x_4 = x_5 = 0\}$, which is not contained in V_3. Finally, all invariant lines through a point q_j, $j = 1, 2, 3$, and a point of E are contained in V_3 so $F(V_3)^{\sigma_3}$ contains three elliptic curves isomorphic to E. By Lemma 6.2 the induced automorphism on $F(V_3)$ is nonsymplectic. One then computes $\chi(X^{\sigma_3}) = 0$ and $h^*(X^{\sigma_3}, \mathbb{F}_3) = 12$, so by using (1) and (2) we get $m = 8$ and $a = 6$. By looking in Table 1 one finds that $S_{\sigma_3}(F(V_3)) =$ $U^{\oplus 2} \oplus A_2^{\oplus 6}$ and $T_{\sigma_3}(F(V_3)) = \langle 6\rangle \oplus E_6^\vee(3)$. Finally, the family of varieties $F(V_3)$ is 7-dimensional, which is equal to $\mathrm{rank}(S_{\sigma_3}(F(V_3)))/2 - 1$.

EXAMPLE 6.7 (CASE $T = U(3) \oplus E_6^\vee(3) \oplus \langle -2\rangle$)

Consider the automorphism of order 3 of $\mathbb{P}^5$ given by

$$\sigma_4(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = (x_0 : x_1 : \xi x_2 : \xi x_3 : \xi^2 x_4 : \xi^2 x_5).$$

The family of (projective) invariant cubics is

$$\begin{aligned} V_4\colon\, & x_2 L_2(x_0, x_1) + x_3 M_2(x_0, x_1) + x_4^2 L_1(x_0, x_1) + x_4 x_5 M_1(x_0, x_1) \\ & + x_5^2 N_1(x_0, x_1) + x_4 N_2(x_2, x_3) + x_5 P_2(x_2, x_3) = 0, \end{aligned}$$

where L_2, M_2, N_2, and P_2 are homogeneous polynomial of degree 2, and L_1, M_1, and N_1 are linear factors. The fixed locus of σ_4 on V_4 is the union of the three projective lines L_1, L_2, L_3 of equations $\{x_0 = x_1 = x_2 = x_3 = 0\}$, $\{x_0 =$ $x_1 = x_4 = x_5 = 0\}$, and $\{x_2 = x_3 = x_4 = x_5 = 0\}$. Each line determines a fixed point on $F(V_4)$. On the other hand an invariant line L intersects two of the lines L_i at one point each. Take a line L intersecting, for example, the lines L_1 and L_2 at the points $(0 : 0 : 0 : 0 : p_4 : p_5)$ and $(0 : 0 : q_2 : q_3 : 0 : 0)$, respectively. Then these points satisfy the equation $p_4 N_2(q_2, q_3) + p_5 P_2(q_2, q_3) = 0$. This is a rational curve on the surface $L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$; hence, we have a fixed rational curve on $F(V_4)$. In the same way, taking the lines L_1, L_3 and L_2, L_3 we get two rational fixed curves on $F(V_4)$. By Lemma 6.2, σ_4 acts nonsymplectically on $F(V_4)$. One computes $\chi(F(V_4)^{\sigma_4}) = 9$ and $h^*(F(V_4)^{\sigma_4}, \mathbb{F}_3) = 9$; checking in Table 1 one finds that $S_{\sigma_4}(F(V_4)) = U \oplus U(3) \oplus A_2^{\oplus 5}$ and $T_{\sigma_4}(F(V_4)) = U(3) \oplus$

$E_6^\vee(3) \oplus \langle -2 \rangle$. Finally, the family of varieties $F(V_4)$ is 6-dimensional, which is equal to $\operatorname{rank}(S_{\sigma_4}(F(V_4)))/2 - 1$. Observe that this automorphism has the same fixed locus as the natural automorphism on a Hilbert scheme with the same lattices S and T.

In Tables 1–7 we mark with a $\diamondsuit$ the cases that are realized with automorphisms on the Fano variety of lines of a cubic four-fold.

REMARK 6.8

Let V be a 6-dimensional vector space. The wedge product $\wedge : \bigwedge^3 V \times \bigwedge^3 V \longrightarrow \bigwedge^6 V$ induces a symplectic form ω on $\bigwedge^3 V$ by choosing an isomorphism $\bigwedge^6 V \cong \mathbb{C}$. Consider a Lagrangian subspace $A \subset \bigwedge^3 V$, and define $Y_A := \{v \in \mathbb{P}(V)/(v \wedge \bigwedge^2 V) \cap A \neq 0\}$. For general A, Y_A is a hypersurface of degree 6 of the type described by Eisenbud–Popescu–Walter (EPW). Such a hypersurface is not smooth, but for a general A it has a smooth double cover X_A which is an $\mathrm{IHS} - \mathrm{K3}^{[2]}$ (see [34]). These are called *double EPW sextics*. One can construct automorphisms of X_A induced by automorphisms of prime order of V as done in [12] and in [29]. A direct computation shows that, using double EPW sextics, one can only construct nonsymplectic automorphisms of prime order 2 and these examples are already explained in [12] and in [29].

7. Existence of automorphisms

Starting from our list of all admissible values of (p, m, a) we want to realize each case by an automorphism, using Theorem 5.5 and the examples of Section 6. The following result shows that it is always possible, except in one case.

THEOREM 7.1

For every admissible value of $(p, m, a) \neq (13, 1, 0)$ there exists an $\mathrm{IHS} - \mathrm{K3}^{[2]}$ *with a nonsymplectic automorphism σ of order p whose invariant lattice T_σ and orthogonal lattice S_σ are those characterized in Theorem 3.8. The fixed locus is a disjoint union of isolated fixed points, smooth curves, and smooth surfaces whose invariants h^* and χ are given in (1) and (2).*

Proof

We refer to Tables 1–7 in the Appendix for the isometry class of the lattices and the values of χ and h^*, which depend only on (p, m, a).

Existence. Observe that by Proposition 3.10 the lattice T has a unique primitive embedding in L. So except for $(p, m, a) = (3, 11, 1)$ and $(p, m, a) = (3, 8, 6)$ it is not a restriction to assume that $T = \overline{T} \oplus \langle -2 \rangle$ with $\overline{T}$ primitively embedded in Λ. Moreover, by Theorem 2.2 the lattice S is uniquely determined (the case $S = A_2(-1)$ is a direct computation), and it is in fact the orthogonal complement of $\overline{T}$ in Λ. Assume that $T = \overline{T} \oplus \langle -2 \rangle$. If $\operatorname{rank} S > p - 1$, then the existence follows from Theorem 5.5, the moduli space is a complex ball of dimension $\dim S(\xi) - 1$, and these cases are realized by natural automorphisms on the Hilbert scheme of points of a K3 surface (see Section 6.1 for explicit examples). If $\operatorname{rank} S = p-1$, then $\dim \Omega_T^{\rho,\xi} = 0$ and the existence follows again from the existence of a K3 surface with a nonsymplectic automorphism of order p with lattices $\overline{T}$ and S. The remaining cases $(3,11,1)$ and $(3,8,6)$ are realized by automorphisms on Fano varieties of lines on cubic four-folds in Examples 6.4 and 6.6.

Fixed locus. We prove that the fixed locus is smooth and that the maximal dimension of a fixed component is 2. By the local inversion theorem, since σ has finite order its action at the neighborhood of a fixed point x can be linearized as an action of a 4×4-matrix M. The dimension of the fixed locus at x is the multiplicity of 1 as an eigenvalue of M. It follows that the fixed locus is the disjoint union of smooth connected subvarieties. Since σ is nonsymplectic, one has $M^T J M = \xi_p J$ where $J = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$ is the standard symplectic form and ξ_p is some primitive pth root of unity. Since $M^{-1} = \xi_p^{-1} J^{-1} M^T J$ one observes that if λ is an eigenvalue of M, then ξ_p/λ is also an eigenvalue of M with the same multiplicity. It follows that there are two possible sequences of eigenvalues for M: $(1, \xi_p, \xi_p^{(p+1)/2})$ with multiplicities (a,a,b) or $(1, \xi_p, \xi_p^i, \xi_p^{-i+1})$ with $2i \not\equiv 1 \mod (p)$ with multiplicities (a,a,b,b). Since the sum of the multiplicities equals 4, we conclude that $a \le 2$ so the maximal dimension of a fixed component is 2. $\square$

REMARK 7.2

One can get some extra information on the local action of an automorphism σ in the neighborhood of a fixed point by using the Pfaffian. With the same notation as in the proof above, one has

$$\xi_p^2 = \operatorname{Pf}(\xi_p J) = \operatorname{Pf}(M^T J M) = \operatorname{Pf}(J)\det(M) = \det(M).$$

In the first case mentioned in the proof, the eigenvalues are $(1, \xi_p, \xi_p^{(p+1)/2})$ with multiplicities (a,a,b) such that $2a+b=4$ and the equation above gives $a + \frac{p+1}{2} b \equiv 2 \mod (p)$. In the second case, the eigenvalues are $(1, \xi_p, \xi_p^i, \xi_p^{-i+1})$ with $2i \not\equiv 1 \mod (p)$ with multiplicities (a,a,b,b) such that $a+b=2$.

REMARK 7.3

The case $(13,1,0)$ cannot be realized by a natural automorphism for the following reason. Suppose that there exists an irreducible holomorphic symplectic manifold X with a natural automorphism σ with invariants $(p,m,a) = (13,1,0)$, $S = U \oplus U \oplus E_8$, and invariant lattice $T = U \oplus E_8 \oplus \langle -2 \rangle$. If this automorphism is natural, then there exists a K3 surface with a nonsymplectic automorphism of order 13, $\overline{T} = U \oplus E_8$, and $\overline{S} = U \oplus U \oplus E_8$, but by [25, Theorem 4.3] such a K3 surface does not exist. This situation is similar to the case $p = 23$ (the maximum prime order for a nonsymplectic automorphism on such varieties; see [10, Section 5.4]) that cannot be realized by a natural automorphism. It is an open problem to construct (or exclude) such nonnatural automorphisms of order 13 or 23 on $\mathrm{IHS} - \mathrm{K3}^{[2]}$.

REMARK 7.4

In the case $p = 2$ one can do a similar construction as above, but the situation is more complicated. For example, we can have several nonequivalent embeddings of T with different orthogonal complements S, which then give different moduli spaces (see Proposition 8.2 below and [24] for a description of the moduli spaces).

COROLLARY 7.5

Let X be an $\mathrm{IHS}-\mathrm{K3}^{[2]}$ *with a nonsymplectic automorphism σ of prime order $3 \leq p \leq 19$, $p \neq 5$. If $(p,m,a) \notin \{(3,8,6),(3,11,1),(13,1,0)\}$, then the action is unique in the sense of Corollary 5.7.*

Proof

By assumption, using Theorem 3.8 in each case the lattice T is of the form $\overline{T} \oplus \langle -2 \rangle$ so the assertion follows from Theorem 5.7 and from the fact that the cases where the moduli space is zero-dimensional are realized by natural automorphisms on K3 surfaces. □

COROLLARY 7.6

There exist $\mathrm{IHS}-\mathrm{K3}^{[2]}$ *with nonnatural nonsymplectic automorphisms of prime order $p = 3$.*

Proof

We provide two examples of $\mathrm{IHS}-\mathrm{K3}^{[2]}$ admitting a nonsymplectic automorphism that cannot be the deformation of an automorphism on a Hilbert scheme of points induced by an automorphism of the underlying K3 surface.

$(p,m,a) = (3,11,1)$. This is Example 6.4, where $\operatorname{rank} T = 2$. For every automorphism on some $\Sigma^{[2]}$ induced by an automorphism of the K3 surface Σ, the rank of the invariant lattice is at least 2, since it contains the class of an ample divisor on the K3 surface and the class of the exceptional divisor is also invariant. Since T is invariant under equivariant deformation, this automorphism is nonnatural.

$(p,m,a) = (3,8,6)$. This is Example 6.6, where $S = U^{\oplus 2} \oplus A_2^{\oplus 6}$. This lattice is invariant under equivariant deformation. Checking all automorphisms obtained by using K3 surfaces one sees that this lattice cannot be obtained in this way (see [1] or [38]), so this automorphism is nonnatural. □

REMARK 7.7 (DIFFERENT FIXED LOCI)

We have shown that in many cases the action of the automorphism on the lattice L is uniquely determined (see Corollary 5.7), but in general the fixed locus is not uniquely determined, for instance, in the case $(p,m,a) = (3,5,5)$. As explained in Section 6.1, starting with a K3 surface Σ with a nonsymplectic automorphism of order 3 with fixed locus consisting of five isolated fixed points and two rational curves (see [2, Table 2]), we obtain a natural automorphism on the Hilbert scheme $\Sigma^{[2]}$ with fixed locus consisting in 10 isolated fixed points, 17 rational curves, one surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and two surfaces isomorphic to $\mathbb{P}^2$. As explained in Example 6.5, using the Fano variety of lines on a cubic four-fold we construct a nonsymplectic automorphism with a different fixed locus consisting this time of 27 isolated fixed points and three rational cubic surfaces. In contrast, Example 6.7 shows a similar situation where the fixed loci are identical.

REMARK 7.8 (OPEN QUESTIONS)

The study of Examples 6.7 and 6.5 leads us to the following questions, which for the moment are open.

- *Are (some of) the Fano varieties with automorphism (X, σ_4) as in Example 6.7 natural in the "old" sense?*
- *Are (some of) the Fano varieties X in Example 6.5 isomorphic to $\Sigma^{[2]}$ for some K3 surface Σ? If the second question is answered in the affirmative, then the example shows the existence of a nonnatural nonsymplectic automorphism of order 3 on a Hilbert scheme $\Sigma^{[2]}$.*

8. Nonsymplectic involutions

Beauville [5, Proposition 2.2] shows that the fixed locus of a nonsymplectic involution σ on an $\mathrm{IHS}-\mathrm{K3}^{[2]}$ is a smooth Lagrangian surface F (possibly not connected) such that $\chi(\mathcal{O}_F) = \frac{1}{8}(t^2+7)$ and $e(F) = \frac{1}{2}(t^2+23)$, where t is the trace of σ^* on $H^{1,1}(X)$, and he proves that t can take all odd integer values $-19 \le t \le 21$. Moreover, he provides examples for all such cases. These are all natural examples except one: Beauville's nonnatural example from [3] for $t=-19$. Nevertheless, the article contains no information about the invariant lattice and its orthogonal.

Ohashi and Wandel [36] study the case $t=-17$ in detail by classifying all possible conjugacy classes of nonsymplectic involutions. In fact, such conjugacy classes are in bijection with the orbits of primitive embeddings of the invariant sublattice T in L. Moreover, in their paper they show that all conjugacy classes are indeed realized, at least abstractly, and give a new explicit example for one of the families using moduli spaces of sheaves on K3 surfaces.

The proof of [10, Lemma 5.5], with little modification, gives the following result for the case $p=2$.

LEMMA 8.1

Let X be an $\mathrm{IHS}-\mathrm{K3}^{[2]}$, and let G be a finite group of order 2 acting nonsymplectically on X. Then:

- $\frac{H^2(X,\mathbb{Z})}{S\oplus T} \cong (\frac{\mathbb{Z}}{2\mathbb{Z}})^{\oplus a_G(X)}$ *for some integer* $0 \le a_G(X) =: a$;
- *S has signature $(2, r-2)$ and T has signature $(1, 22-r)$ where $r = \operatorname{rank} S$; and*
- *either $A_T \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus a+1}$ and $A_S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$ or vice versa.*

Hence, both T and S are indefinite 2-elementary lattices and T is hyperbolic. Recall that the isomorphism class of an indefinite 2-elementary lattice is classified by the triple (r, a, δ), where r is its rank, a is the length of the discriminant group, and $\delta = 0$ if the discriminant form takes values in $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z}$ and $\delta = 1$ otherwise. We are interested in counting how many nonisomorphic primitive embeddings of T in L there are. Proposition 8.2 below is the analogue of Proposition 2.7 in the case of involutions. We formulate it this time for the lattice T instead of S for compatibility with Nikulin's classification [33] of nonsymplectic involutions on K3 surfaces in terms of hyperbolic 2-elementary lattices.

PROPOSITION 8.2

Let T be an even hyperbolic 2-elementary lattice of signature $(1, t)$ and length $\ell(A_T) = a \geq 0$, and let $L = U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$. Assume that T admits a primitive embedding in L.

(i) *If there is no $x \in A_T$ such that $q_T(x) = 3/2 \mod 2\mathbb{Z}$, then T admits a unique primitive embedding into L whose orthogonal complement is a 2-elementary lattice S of signature $(2, 20 - t)$, length $\ell(A_S) = a + 1$, and $\delta_S = 1$.*

(ii) *Otherwise, nonisomorphic primitive embeddings of T into L are in one-to-one correspondence with nonisometric choices of a 2-elementary lattice S of signature $(2, 20 - t)$ with either $l(A_S) = l(A_T) - 1$, or $l(A_S) = l(A_T) + 1$ and $\delta_S = 1$.*

Proof

We proceed as in the proof of Proposition 2.7. By [32, Proposition 1.15.1] a primitive embedding of T into L is equivalent to the data of a quintuple $(H_T, H_L, \gamma, S, \gamma_S)$ satisfying the following conditions.

- H_T is a subgroup of $A_T = (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$, H_L is a subgroup of $A_L = \mathbb{Z}/2\mathbb{Z}$, and $\gamma\colon H_T \to H_L$ is an isomorphism of groups such that, for any $x \in H_T$, $q_L(\gamma(x)) = q_T(x)$.
- S is a lattice of invariants $(2, 20 - t, q_S)$ with $q_S = ((-q_T) \oplus q_L)|_{\Gamma^\perp/\Gamma}$, where Γ is the graph of γ in $A_T \oplus A_L$, $\Gamma^\perp$ is the orthogonal complement of Γ in $A_T \oplus A_L$ with respect to the bilinear form induced on $A_T \oplus A_L$ and with values in $\mathbb{Q}/\mathbb{Z}$, and γ_S is an automorphism of A_S that preserves q_S. Moreover, S is the orthogonal complement of T in L.

In our case there are only two possibilities.

(1) $H_T = H_L = \{0\}$ and $\gamma = \mathrm{id}$. In this case, $\Gamma = \{(0, 0)\}$, so the discriminant group of the orthogonal complement is $A_T \oplus A_L$ and $q_S = -q_T \oplus q_L$. Recall that $A_L = \frac{\mathbb{Z}}{2\mathbb{Z}}(\frac{3}{2})$, so let $y \in A_L$ be such that $q_L(y) = 3/2$. Then $q_S((0, y)) = q_L(y) \notin \mathbb{Z}/2\mathbb{Z}$ and hence $\delta_S = 1$.

(2) $H_T = H_L = \mathbb{Z}/2\mathbb{Z}$ and $\gamma = \mathrm{id}$. This case can happen only if there is $x \in A_T$ such that $q_T(x) = 3/2$. In this case, $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$, so the discriminant group of S is $A_S = \Gamma^\perp/\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus a-1}$ and $q_S = (-q_T \oplus q_L)_{|A_S}$.

In each case the lattice S is 2-elementary so the natural map $O(S) \to O(q_S)$ is surjective by [32, Theorem 3.6.3]. This implies that different choices of the isometry γ_S produce isomorphic embeddings of T in L. As a consequence, if there is no $x \in A_T$ such that $q_T(x) = 3/2 \mod 2\mathbb{Z}$, then only the first case occurs and the lattice S has signature $(2, 20-t)$, $\ell(A_S) = a+1$, and $\delta_S = 1$. As noted in Remark 2.3, the lattice S is uniquely determined by these invariants so T admits a unique embedding in L. Otherwise, both cases can occur and the primitive embeddings of T in L are classified by S: the signature of S is $(2, 20-t)$, in the first case S has length $a+1$ and $\delta_S = 1$, and in the second case its length is $a-1$. □

REMARK 8.3

As a consequence, the lattice T admits at most three nonisometric embeddings in L.

(1) If $\delta_T = 0$, then case (i) occurs and T admits a unique primitive embedding in L. By [33, Theorem 4.3.2] this implies that $1-t \equiv 0 \mod (4)$ so $\operatorname{rank} T \equiv 2 \mod (4)$.

(2) By [16, Theorem 1.5.2] if $\delta_S = 0$, then $t - 18 \equiv 0 \mod (4)$ so $\operatorname{rank} S \equiv 0 \mod (4)$ and $\operatorname{rank} T \equiv 3 \mod (4)$. Hence, if $\operatorname{rank} T \not\equiv 3 \mod (4)$, then one has necessarily $\delta_S = 1$; hence, T admits at most two embeddings in L.

In Figures 1 and 2 we give all possible values of (r, a, δ) such that a 2-elementary lattice T with these invariants admits a primitive embedding in L. To show all possible embeddings, in Figure 1 we give the cases where the orthogonal S has the property $\ell(A_S) = \ell(A_T) + 1$ and in Figure 2 we give the cases where $\ell(A_S) = \ell(A_T) - 1$. These figures are obtained by using the results of Nikulin on primitive embeddings (see Remark 2.3).

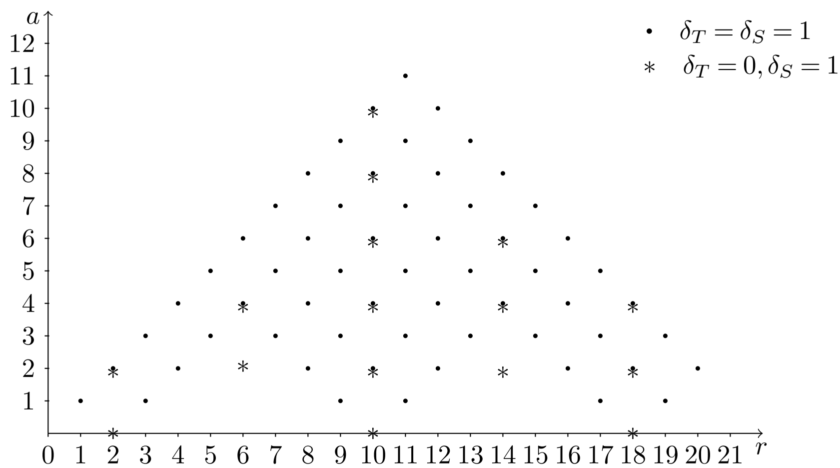

Figure 1. Order 2: The lattice T admits an embedding in L with orthogonal S such that $\ell(A_S) = \ell(A_T) + 1$.

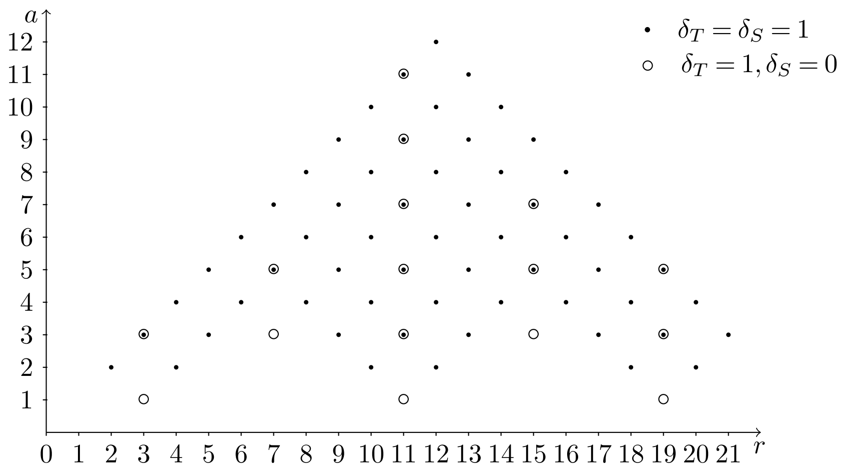

Figure 2. Order 2: The lattice T admits an embedding in L with orthogonal S such that $l(A_S) = l(A_T) - 1$ (all realized by natural automorphisms).

REMARK 8.4

Let Σ be a K3 surface with a nonsymplectic involution ι such that the invariant lattice $T_\iota(\Sigma)$ has invariants (r, a, δ). Then the natural involution $\iota^{[2]}$ induced by ι on $\Sigma^{[2]}$ gives an example of the case where the invariant sublattice T has invariants $(r+1, a+1, 1)$, its orthogonal complement S satisfies $\ell(A_S) = a = \ell(A_T) - 1$, and $\delta_S = \delta$. This gives a realization of all the cases illustrated in Figure 2.

PROPOSITION 8.5

Under the same assumptions as in Proposition 8.2, for each embedding $j : T \hookrightarrow L$ there exists an $\mathrm{IHS} - \mathrm{K3}^{[2]}$ *with a nonsymplectic involution $\sigma \colon X \to X$ such that the invariant lattice $T_\sigma(X) \subset H^2(X, \mathbb{Z})$ is isomorphic to the embedding $j(T) \subset L$.*

Proof

The proof follows the same lines as those of [36, Lemma 2.6] with one exception. The isometry $i = \mathrm{id}_T \oplus (-\mathrm{id}_S)$ of $T \oplus S$ induces the identity on $A_{S\oplus T}$, so it leaves stable the subgroup $\frac{L}{T\oplus S} \subset A_{T\oplus S}$. This implies that i extends to an isometry of L such that $L^i = T$ (see [32, Corollary 1.5.2]). Assume first that $\operatorname{rank} T \leq 20$. By the surjectivity of the period map P_0 and Proposition 5.3, for any $\omega \in \Omega_T^\circ$ there exists an $\mathrm{IHS} - \mathrm{K3}^{[2]}$ with a marking $\eta \colon L \to H^2(X, \mathbb{Z})$ such that $\eta(T) = \mathrm{NS}(X)$ and $\eta(\omega) = H^{2,0}(X)$. Then $\mathrm{NS}(X)$ is hyperbolic, so X is projective by [22, Theorem 3.11]. The action of i on $H^2(X, \mathbb{Z})$ induced by η, which we still denote by i, is a Hodge isometry since $i(\omega) = -\omega$ implies $i(H^{2,0}(X)) = H^{2,0}(X)$. We apply the strong Torelli theorem as stated in [26, Theorem 1.3]. We verify the conditions of [26, Theorem 9.8] to prove that the involution i is a parallel transport operator. The positive vectors of T and S generate a positive 3-dimensional subspace of $L_\mathbb{R}$ whose orientation is preserved by i. Moreover, since the lattice L admits a unique primitive embedding in the Mukai lattice $U^{\oplus 4} \oplus E_8^{\oplus 2}$, we conclude that the involution i is a parallel transport operator. Since X is projective and i acts trivially on $\mathrm{NS}(X)$, it leaves invariant an ample class, so i maps the Kähler cone of X to itself. By the strong Torelli theorem it follows that there exists an automorphism ι of X such that $\iota^* = i$. Since the natural map $\mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z}))$ is injective, ι is an involution.

Assume now that $\operatorname{rank} T = 21$: the previous argument does not work since the period domain is zero-dimensional but we observe that by [33] there exists a K3 surface Σ with an involution ι such that the invariant lattice $T_\iota(\Sigma)$ has invariants $(20,2,1)$. Then by Remark 8.4 the involution $\iota^{[2]}$ on $\Sigma^{[2]}$ is such that $(\iota^{[2]})^* = i$ and realizes the case where the invariant sublattice T has invariants $(21,3,1)$. □

EXAMPLE 8.6

(1) Take $T = \langle 2 \rangle$ of invariants $(1,1,1)$. The unique embedding in L (see Figure 1) corresponds to Beauville's nonnatural involution [3] on the Hilbert scheme of two points of a quartic in $\mathbb{P}^3$ containing no line; here $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2}$.

(2) Take $T = \langle 2 \rangle \oplus \langle -2 \rangle$ of invariants $(2,2,1)$. The embedding in Figure 1 has orthogonal complement $S = U^{\oplus 2} \oplus E_8 \oplus E_7 \oplus \langle -2 \rangle^{\oplus 2}$; it corresponds to Ohashi–Wandel's involution [36]. The embedding in Figure 2 has orthogonal complement $S = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2 \rangle$ and is realized by a natural involution on the Hilbert scheme of two points on a K3 surface (see Remark 8.4).

Appendix: Tables for the invariant lattice and its orthogonal

The isometry classes of the lattices S and T for all admissible values of (p,m,a) are summarized in Tables 1, 3–7 corresponding to $p = 3,7,11,13,17,19$. The excluded values of (p,m,a) are not written in the tables. Theorem 3.3 for the order 5 automorphisms holds only for natural automorphisms, so in Table 2 the list of admissible triples $(5,m,a)$ is given only in this special case. Recall that in Tables 1, 3–7 we mark with a ♣ the cases that are realized with natural automorphisms and with a ♢ the cases that are realized with automorphisms on the Fano variety of lines of a cubic four-fold.

Table 1. Order 3.

p	m	a	χ	h^*	S	T
$\diamondsuit$ 3	11	1	27	67	$U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$	$\langle 6\rangle$
$\clubsuit$ 3	10	0	9	109	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	$U \oplus \langle -2\rangle$
$\clubsuit$ 3	10	2	9	57	$U \oplus U(3) \oplus E_8^{\oplus 2}$	$U(3) \oplus \langle -2\rangle$
$\clubsuit$ 3	9	1	0	96	$U^{\oplus 2} \oplus E_6 \oplus E_8$	$U \oplus A_2 \oplus \langle -2\rangle$
$\clubsuit$ 3	9	3	0	48	$U \oplus U(3) \oplus E_6 \oplus E_8$	$U(3) \oplus A_2 \oplus \langle -2\rangle$
$\clubsuit$ 3	8	2	0	84	$U^{\oplus 2} \oplus E_6 \oplus E_6$	$U \oplus A_2^{\oplus 2} \oplus \langle -2\rangle$
$\clubsuit$ 3	8	4	0	40	$U \oplus U(3) \oplus E_6^{\oplus 2}$	$U(3) \oplus A_2^{\oplus 2} \oplus \langle -2\rangle$
$\diamondsuit$ 3	8	6	0	12	$U^{\oplus 2} \oplus A_2^6$	$\langle 6\rangle \oplus E_6^\vee(3)$
$\clubsuit$ 3	7	1	9	129	$U^{\oplus 2} \oplus A_2 \oplus E_8$	$U \oplus E_6 \oplus \langle -2\rangle$
$\clubsuit$ 3	7	3	9	73	$U \oplus U(3) \oplus A_2 \oplus E_8$	$U \oplus A_2^{\oplus 3} \oplus \langle -2\rangle$
$\clubsuit$ 3	7	5	9	33	$U^{\oplus 2} \oplus A_2^5$	$U(3) \oplus A_2^3 \oplus \langle -2\rangle$
$\diamondsuit$, $\clubsuit$ 3	7	7	9	9	$U \oplus U(3) \oplus A_2^5$	$U(3) \oplus E_6^\vee(3) \oplus \langle -2\rangle$
$\clubsuit$ 3	6	0	27	183	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -2\rangle$
$\clubsuit$ 3	6	2	27	115	$U \oplus U(3) \oplus E_8$	$U \oplus E_6 \oplus A_2 \oplus \langle -2\rangle$
$\clubsuit$ 3	6	4	27	63	$U^{\oplus 2} \oplus A_2^4$	$U \oplus A_2^4 \oplus \langle -2\rangle$
$\clubsuit$ 3	6	6	27	27	$U \oplus U(3) \oplus A_2^4$	$U(3) \oplus A_2^4 \oplus \langle -2\rangle$
$\clubsuit$ 3	5	1	54	166	$U^{\oplus 2} \oplus E_6$	$U \oplus E_8 \oplus A_2 \oplus \langle -2\rangle$
$\clubsuit$ 3	5	3	54	102	$U \oplus U(3) \oplus E_6$	$U \oplus A_2^2 \oplus E_6 \oplus \langle -2\rangle$
$\diamondsuit$, $\clubsuit$ 3	5	5	54	54	$U \oplus U(3) \oplus A_2^3$	$U \oplus A_2^5 \oplus \langle -2\rangle$
$\clubsuit$ 3	4	2	90	150	$U^{\oplus 2} \oplus A_2^2$	$U \oplus E_6^2 \oplus \langle -2\rangle$
$\clubsuit$ 3	4	4	90	90	$U \oplus U(3) \oplus A_2^2$	$U \oplus E_6 \oplus A_2^3 \oplus \langle -2\rangle$
$\clubsuit$ 3	3	1	135	207	$U^{\oplus 2} \oplus A_2$	$U \oplus E_6 \oplus E_8 \oplus \langle -2\rangle$
$\clubsuit$ 3	3	3	135	135	$U \oplus U(3) \oplus A_2$	$U \oplus E_6^{\oplus 2} \oplus A_2 \oplus \langle -2\rangle$
$\clubsuit$ 3	2	0	189	273	$U^{\oplus 2}$	$U \oplus E_8^{\oplus 2} \oplus \langle -2\rangle$
$\clubsuit$ 3	2	2	189	189	$U \oplus U(3)$	$U \oplus E_6 \oplus E_8 \oplus A_2 \oplus \langle -2\rangle$
$\clubsuit$ 3	1	1	252	252	$A_2(-1)$	$U \oplus E_8^{\oplus 2} \oplus A_2 \oplus \langle -2\rangle$

Table 2. Order 5.

m	a	χ	h^*	S	T
5	1	-1	31	$U \oplus E_8^{\oplus 2} \oplus H_5$	$H_5 \oplus \langle -2\rangle$
4	2	14	42	$U \oplus H_5 \oplus E_8 \oplus A_4$	$H_5 \oplus A_4 \oplus \langle -2\rangle$
4	4	14	14	$U(5) \oplus H_5 \oplus E_8 \oplus A_4$	$H_5 \oplus A_4^*(5) \oplus \langle -2\rangle$
3	1	54	102	$U \oplus H_5 \oplus E_8$	$H_5 \oplus E_8 \oplus \langle -2\rangle$
3	3	54	54	$U \oplus H_5 \oplus A_4^2$	$H_5 \oplus A_4^2 \oplus \langle -2\rangle$
2	2	119	119	$U \oplus H_5 \oplus A_4$	$H_5 \oplus A_4 \oplus E_8 \oplus \langle -2\rangle$
1	1	202	202	$U \oplus H_5$	$H_5 \oplus E_8^2 \oplus \langle -2\rangle$

Table 3. Order 7.

p	m	a	χ	h^*	S	T
♣ 7	3	1	9	33	$U^{\oplus 2} \oplus E_8 \oplus A_6$	$U \oplus K_7 \oplus \langle -2\rangle$
♣ 7	3	3	9	9	$U \oplus U(7) \oplus E_8 \oplus A_6$	$U(7) \oplus K_7 \oplus \langle -2\rangle$
♣ 7	2	0	65	117	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -2\rangle$
♣ 7	2	2	65	65	$U \oplus U(7) \oplus E_8$	$U(7) \oplus E_8 \oplus \langle -2\rangle$
♣ 7	1	1	170	170	$U^{\oplus 2} \oplus K_7$	$U \oplus E_8 \oplus A_6 \oplus \langle -2\rangle$

Table 4. Order 11.

p	m	a	χ	h^*	S	T
♣ 11	2	0	5	25	$U^{\oplus 2} \oplus E_8^{\oplus 2}$	$U \oplus \langle -2\rangle$
♣ 11	2	2	5	5	$U \oplus U(11) \oplus E_8^{\oplus 2}$	$U(11) \oplus \langle -2\rangle$
♣ 11	1	1	104	104	$K_{11}(-1) \oplus E_8$	$U \oplus A_{10} \oplus \langle -2\rangle$

Table 5. Order 13.

p	m	a	χ	h^*	S	T
13	1	0	77	103	$U^{\oplus 2} \oplus E_8$	$U \oplus E_8 \oplus \langle -2\rangle$
♣ 13	1	1	77	77	$U \oplus E_8 \oplus H_{13}$	$E_8 \oplus H_{13} \oplus \langle -2\rangle$

Table 6. Order 17.

p	m	a	χ	h^*	S	T
♣ 17	1	1	35	35	$U^{\oplus 2} \oplus E_8 \oplus L_{17}$	$U \oplus L_{17} \oplus \langle -2\rangle$

Table 7. Order 19.

p	m	a	χ	h^*	S	T
♣ 19	1	1	20	20	$K_{19}(-1) \oplus E_8^{\oplus 2}$	$U \oplus K_{19} \oplus \langle -2\rangle$

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References

[1] M. Artebani and A. Sarti, *Non-symplectic automorphisms of order* 3 *on K3 surfaces*, Math. Ann. **342** (2008), 903–921. MR 2443767. DOI 10.1007/s00208-008-0260-1.

[2] M. Artebani, A. Sarti, and S. Taki, *K3 surfaces with non-symplectic automorphisms of prime order*, with an appendix by S. Kondō, Math. Z. **268** (2011), 507–533. MR 2805445. DOI 10.1007/s00209-010-0681-x.

[3] A. Beauville, "Some remarks on Kähler manifolds with $c_1 = 0$" in *Classification of Algebraic and Analytic Manifolds (Katata, 1982)*, Progr. Math. **39**, Birkhäuser, Boston, 1983, 1–26. MR 0728605. DOI 10.1007/BF02592068.

[4] ———, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), 755–782. MR 0730926.

[5] ———, *Antisymplectic involutions of holomorphic symplectic manifolds*, J. Topol. **4** (2011), 300–304. MR 2805992. DOI 10.1112/jtopol/jtr002.

[6] A. Beauville and R. Donagi, *La variété des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 703–706. MR 0818549.

[7] S. Boissière, *Automorphismes naturels de l'espace de Douady de points sur une surface*, Canad. J. Math. **64** (2012), 3–23. MR 2932167. DOI 10.4153/CJM-2011-041-5.

[8] S. Boissière, C. Camere, G. Mongardi, and A. Sarti, *Isometries of ideal lattices and hyperkähler manifolds*, Int. Math. Res. Notes (IMRN) **(2016)**, 963–977. DOI 10.1093/imrn/rnv137.

[9] S. Boissière, M. Nieper-Wisskirchen, and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, J. Math. Pures Appl. (9) **95** (2011), 553–563. MR 2786223. DOI 10.1016/j.matpur.2010.12.003.

[10] ———, *Smith theory and irreducible holomorphic symplectic manifolds*, J. Topol. **6** (2013), 361–390. MR 3065180. DOI 10.1112/jtopol/jtt002.

[11] S. Boissière and A. Sarti, *A note on automorphisms and birational transformations of holomorphic symplectic manifolds*, Proc. Amer. Math. Soc. **140** (2012), 4053–4062. MR 2957195. DOI 10.1090/S0002-9939-2012-11277-8.

[12] C. Camere, *Symplectic involutions of holomorphic symplectic four-folds*, Bull. Lond. Math. Soc. **44** (2012), 687–702. MR 2967237. DOI 10.1112/blms/bdr133.

[13] ———, *Lattice polarized irreducible holomorphic symplectic manifolds*, Ann. Inst. Fourier **66** (2016), 687–709.

[14] C. H. Clemens and P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356. MR 0302652.

[15] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, third ed., Grundlehren Math. Wiss. **290**, Springer, New York, 1999. MR 1662447. DOI 10.1007/978-1-4757-6568-7.

[16] I. V. Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, Astérisque **105**, Soc. Math. France, Paris, 1983, 251–278, Séminaire Bourbaki 1982/1983, no. 611. MR 0728992.

[17] ———, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. **81** (1996), 2599–2630. MR 1420220. DOI 10.1007/BF02362332.

[18] I. V. Dolgachev and S. Kondō, "Moduli of K3 surfaces and complex ball quotients" in *Arithmetic and Geometry around Hypergeometric Functions*, Progr. Math. **260**, Birkhäuser, Basel, 2007, 43–100. MR 2306149. DOI 10.1007/978-3-7643-8284-1_3.

[19] I. V. Dolgachev, B. van Geemen, and S. Kondō, *A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces*, J. Reine Angew. Math. **588** (2005), 99–148. MR 2196731. DOI 10.1515/crll.2005.2005.588.99.

[20] V. González-Aguilera and A. Liendo, *Automorphisms of prime order of smooth cubic n-folds*, Arch. Math. (Basel) **97** (2011), 25–37. MR 2820585. DOI 10.1007/s00013-011-0247-0.

[21] E. Horikawa, *On deformations of holomorphic maps, III*, Math. Ann. **222** (1976), 275–282. MR 0417458.

[22] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), 63–113. MR 1664696. DOI 10.1007/s002220050280.

[23] ———, *A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]*, Astérisque **348** (2012), 375–403, Séminaire Bourbaki 2010/2011, no. 1040. MR 3051203.

[24] M. Joumaah, *Non-symplectic involutions of irreducible symplectic manifolds of $K3^{[n]}$-type*, Math. Z. online 20 January 2016. DOI 10.1007/s00209-16-1620-2.

[25] S. Kondō, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), 75–98. MR 1139659. DOI 10.2969/jmsj/04410075.

[26] E. Markman, "A survey of Torelli and monodromy results for holomorphic-symplectic varieties" in *Complex and Differential Geometry*, Springer Proc. Math. **8**, Springer, Heidelberg, 2011, 257–322. MR 2964480. DOI 10.1007/978-3-642-20300-8_15.

[27] H. Matsumura and P. Monsky, *On the automorphisms of hypersurfaces*, J. Math. Kyoto Univ. **3** (1963/1964), 347–361. MR 0168559.

[28] G. Mongardi, *Symplectic involutions on deformations of $K3^{[2]}$*, Cent. Eur. J. Math. **10** (2012), 1472–1485. MR 2925616. DOI 10.2478/s11533-012-0073-z.

[29] ———, *Automorphisms of hyperkähler manifolds*, Ph.D. dissertation, Universitá degli Studi di Roma 3, Rome, preprint, arXiv:1303.4670v1 [math.AG].

[30] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984), 105–121. MR 0728142. DOI 10.1007/BF01403093.

[31] Y. Namikawa, *Periods of Enriques surfaces*, Math. Ann. **270** (1985), 201–222. MR 0771979. DOI 10.1007/BF01456182.

[32] V. V. Nikulin, *Integer symmetric bilinear forms and some of their applications* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238; English translation in Math. USSR Izv. **14** (1980), 103–167. MR 0525944.

[33] ———, *Factor groups of groups of the automorphisms of hyperbolic forms with respect to subgroups generated by* 2*-reflections* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 1, 109–188; English translation in J. Soviet Math. **22** (1983), 1401–1475. MR 0688920.

[34] K. G. O'Grady, *Irreducible symplectic* 4*-folds and Eisenbud–Popescu–Walter sextics*, Duke Math. J. **134** (2006), 99–137. MR 2239344. DOI 10.1215/S0012-7094-06-13413-0.

[35] K. Oguiso and S. Schröer, *Enriques manifolds*, J. Reine Angew. Math. **661** (2011), 215–235. MR 2863907. DOI 10.1515/CRELLE.2011.077.

[36] H. Ohashi and M. Wandel, *Non-natural non-symplectic involutions on symplectic manifolds of* $K3^{[2]}$*-type*, preprint, arXiv:1305.6353v2 [math.AG].

[37] A. N. Rudakov and I. R. Shafarevich, "Surfaces of type K3 over fields of finite characteristic" in *Current Problems in Mathematics, Vol. 18*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981, 115–207. MR 0633161.

[38] S. Taki, *Non-symplectic automorphisms of* 3*-power order on K3 surfaces*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), 125–130. MR 2721856.

[39] M. Verbitsky, *Cohomology of compact hyper-Kähler manifolds and its applications*, Geom. Funct. Anal. **6** (1996), 601–611. MR 1406664. DOI 10.1007/BF02247112.

[40] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spéc. **10**, Soc. Math. France, Paris, 2002. MR 1988456. DOI 10.1017/CBO9780511615344.

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