

# Saito–Kurokawa lifts of square-free level

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**Abstract** Let  $f \in S_{2\kappa-2}(\Gamma_0(M))$  be a Hecke eigenform with  $\kappa \geq 2$  even and  $M \geq 1$  and odd and square-free. In this paper we survey the construction of the Saito–Kurokawa lifting from the classical point of view. We also provide some arithmetic results on the Fourier coefficients of Saito–Kurokawa liftings. We then calculate the norm of the Saito–Kurokawa lift.

## 1. Introduction

It is well established that one can prove deep theorems in arithmetic by studying liftings of automorphic forms from a reductive algebraic group to a larger reductive algebraic group. For instance, one can see Ribet’s [21] proof of the converse of Herbrand’s theorem, Wiles’s [29] proof of the main conjecture of Iwasawa theory for totally real fields, or Skinner–Urban’s [27] proof of the main conjecture of Iwasawa theory for  $GL(2)$  for three prominent examples of this philosophy. One such lifting that has figured prominently in several such results is the Saito–Kurokawa lifting that lifts a form from  $GL(2)$  to  $GSp(4)$ . One can see [1], [4], [16], and [26] for examples of arithmetic applications of Saito–Kurokawa liftings. It is these liftings that this paper focuses on.

The Saito–Kurokawa lifting in the full level case was established via a series of papers culminating in the work of Zagier [30]. The lifting was established from an automorphic point of view via the work of Piatetski and Shapiro [20] and Schmidt [23], [24]. For the arithmetic applications referenced above, one needs a classical construction of the Saito–Kurokawa lifting of square-free level. This lifting was claimed in a series of papers (see [19], [17], [18]). Unfortunately, there are many omitted proofs in these papers, and the generalized Maass lifting used in these papers is known to be given incorrectly. It was not until recently that a correct treatment of the generalized Maass lifting was given by Ibukiyama [11]. This allows one to give a correct classical construction of the Saito–Kurokawa lifting of square-free level.

In this paper, after introducing some notation in Section 2, in Section 3 we survey the classical construction of a Saito–Kurokawa lift  $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$  from  $f \in S_{2\kappa-2}(\Gamma_0(M))$  for  $M$  square-free and  $\kappa \geq 2$  even. In the same section we also show that with a suitable choice of scalar one can fix a lifting  $F_f$  that has

Fourier coefficients in the same ring as  $f$ 's Fourier coefficients. This is essential for arithmetic applications. In Section 4 we compute the norm of  $F_f$ . Such a calculation originally appeared in [3], but this was based on the incorrect Maass lifting mentioned above, so it is not correct. This norm is needed for the main result of [1].

**2. Definitions and notation**

In this section we fix basic definitions and notations we will use throughout the rest of the paper. Throughout this paper we let  $M \geq 1$  denote a square-free integer, and let  $\kappa \geq 2$  be an even integer.

Given a ring  $R$ , we let  $\text{Mat}_n(R)$  denote the set of  $n$  by  $n$  matrices with entries in  $R$ . As usual we let  $\text{GL}(n, R) \subset \text{Mat}_n(R)$  denote the group of invertible matrices, and let  $\text{SL}(n, R) \subset \text{GL}(n, R)$  denote the matrices with determinant 1. We write  $1_n$  for the identity matrix in  $\text{GL}_n(R)$  and  $0_n$  for the zero matrix in  $\text{Mat}_n(R)$ . Given  $A \in \text{Mat}_n(R)$ , we denote the transpose of  $A$  by  ${}^tA$ . Let  $J = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$ . The symplectic group  $\text{GSp}(4, R)$  is defined by

$$\text{GSp}(4, R) = \{g \in \text{GL}(4, R) : {}^t g J g = \mu(g) J, \mu(g) \in \text{GL}(1, R)\}.$$

We set  $\text{Sp}(4, R) = \ker(\mu)$ . We let  $\text{PGSp}(4, R)$  denote the projective symplectic group. Let  $\Gamma_0(M) \subset \text{SL}(2, \mathbb{Z})$  have its usual meaning, and set

$$\Gamma_0^{(2)}(M) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}) : C \equiv 0 \pmod{M} \right\}.$$

We write  $e(z) = e^{2\pi iz}$ . We let  $\mathfrak{h}^n = \{Z = X + iY \in \text{Mat}_n(\mathbb{C}) : X, Y \in \text{Mat}_n(\mathbb{R}), Y > 0\}$ . We let  $S_\kappa(\Gamma_0(M))$  denote the cusp forms of weight  $\kappa$  and level  $\Gamma_0(M)$ . Let  $f \in S_\kappa(\Gamma_0(M))$  be a normalized eigenform with Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz).$$

Given a ring  $R$ , we write  $S_\kappa(\Gamma_0(M); R)$  to denote the space of cusp forms that have Fourier coefficients in  $R$ . We define the Peterson product on  $S_\kappa(\Gamma_0(M))$  by setting

$$\langle f_1, f_2 \rangle = \frac{1}{[\text{SL}(2, \mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M) \backslash \mathfrak{h}} f_1(z) \overline{f_2(z)} y^{\kappa-2} dx dy$$

for  $f_1, f_2 \in S_\kappa(\Gamma_0(M))$ .

We denote the cuspidal automorphic representation associated to an eigenform  $f$  by  $\pi_f = \otimes' \pi_{f,p}$ . Recall that  $\pi_{f,\infty}$  is the discrete series representation with lowest weight vector of weight  $\kappa$  and for  $p \nmid M$  the representation  $\pi_{f,p}$  is an unramified principal series representation. The local representations for  $p \mid M$  are determined by the Atkin–Lehner eigenvalues of  $f$ . For  $p \mid M$ , recall that the Atkin–Lehner operator at  $p$  is the matrix  $W_p = \begin{pmatrix} pa & b \\ Mc & pd \end{pmatrix}$  with  $p^2 ad - Mbc = p$ . If  $f \in S_\kappa^{\text{new}}(\Gamma_0(M))$ , then we let  $\epsilon_p \in \{\pm 1\}$  denote the Atkin–Lehner eigenvalue of  $f$  at  $p$ , that is,  $W_p f = \epsilon_p f$ . If  $\epsilon_p = -1$ , then  $\pi_{f,p} = \text{St}_{\text{GL}(2)}$ , and if  $\epsilon_p = 1$ , then

$\pi_{f,p} = \xi \text{St}_{\text{GL}(2)}$  where  $\text{St}_{\text{GL}(2)}$  is the Steinberg representation and  $\xi$  is the unique nontrivial unramified quadratic character of  $\mathbb{Q}_p^\times$ .

We will also need  $L$ -functions attached to  $f$  and  $\pi_f$ . For each prime  $p \nmid M$  there exists a character  $\sigma_p$  so that  $\pi_{f,p} = \pi(\sigma_p, \sigma_p^{-1})$  (see [5, Section 4.5]). The  $p$ -Satake parameter of  $f$  is given by  $\alpha_0(p; f) = \sigma_p(p)$ . The  $L$ -function associated to  $\pi_{f,p}$  is

$$L(s, \pi_{f,p}) = (1 - \alpha_0(p; f)p^{-s})^{-1} (1 - \alpha_0(p; f)^{-1}p^{-s})^{-1}.$$

For  $p \mid M$ ,

$$L(s, \pi_{f,p}) = (1 + \epsilon_p p^{-s-\frac{1}{2}})^{-1}.$$

We set

$$L_\infty(s, \pi_{f,\infty}) = (2\pi)^{-(s+(\kappa-1)/2)} \Gamma(s + (\kappa - 1)/2).$$

The  $L$ -function associated to  $\pi_f$  is

$$L(s, \pi_f) = \prod_p L(s, \pi_{f,p}).$$

The functional equation for  $L(s, \pi_f)$  is given by

$$L(s, \pi_f) = \varepsilon(s, \pi_f) L(1 - s, \pi_f),$$

where  $\varepsilon(s, \pi_f) = \prod_p \varepsilon_p(s, \pi_{f,p})$  and

$$\varepsilon_p(s, \pi_{f,p}) = \begin{cases} (-1)^{\kappa/2} & \text{if } p = \infty, \\ -p^{\frac{1}{2}-s} & \text{if } \epsilon_p = -1, p < \infty, \\ p^{\frac{1}{2}-s} & \text{if } \epsilon_p = 1, p < \infty. \end{cases}$$

In particular, the sign of the functional equation is given by  $\varepsilon(\frac{1}{2}, \pi_f) \in \{\pm 1\}$ . The classical  $L$ -function of  $f$  is given by

$$L(s, f) = \prod_{p < \infty} L\left(s + \frac{1}{2} - \kappa/2, \pi_{f,p}\right).$$

Let  $S_\kappa(\Gamma_0^{(2)}(M))$  denote the space of Siegel modular forms of weight  $\kappa$  and level  $\Gamma_0^{(2)}(M)$ . A form  $F \in S_\kappa(\Gamma_0^{(2)}(M))$  has a Fourier expansion

$$F(z) = \sum_{T \in \Lambda_2} a_F(T) e(\text{Tr}(Tz)),$$

where  $\Lambda_2$  is the set of 2 by 2 half-integral positive definite symmetric matrices. As above, for a ring  $R$  we write  $S_\kappa(\Gamma_0^{(2)}(N); R)$  to denote the forms with Fourier coefficients in  $R$ .

Given  $g \in \text{Sp}_4^+(\mathbb{Q})$ , we write  $T(g)$  to denote

$$\Gamma_0^{(2)}(M)g\Gamma_0^{(2)}(M).$$

We define the usual action of  $T(g)$  on Siegel modular forms by setting

$$T(g)F = \sum_i F|_{\kappa} g_i,$$

where  $\Gamma_0^{(2)}(M)g\Gamma_0^{(2)}(M) = \coprod_i \Gamma_0^{(2)}(M)g_i$  and  $F \in M_\kappa(\Gamma_0^{(2)}(M))$ . Let  $p$  be prime, and define

$$T_S(p) = T(\text{diag}(1_2, p1_2)).$$

In the case where  $p \mid M$  we write  $U_S(p)$  for  $T_S(p)$ . Set

$$T'_S(p) = pT(\text{diag}(1, p, p^2, p)) + p(1 + p + p^2)T(\text{diag}(p, p, p, p)).$$

Let  $F \in S_\kappa(\Gamma_0^{(2)}(M))$  be an eigenform. Then  $F$  generates a space of cuspidal automorphic forms on  $\text{GSp}_4(\mathbb{A})$  invariant under right translation. In general this space may not be irreducible but does decompose into a finite number of irreducible, cuspidal, automorphic representations. Let  $\pi_F$  be one of these irreducible pieces. The representation  $\pi_F$  can be decomposed into local components  $\Pi_F = \otimes \Pi_{F,p}$  with  $\Pi_{F,p}$  a representation of  $\text{PGSp}(4, \mathbb{Q}_p)$ . We refer the reader to [2, Section 3] for the details concerning the construction of cuspidal automorphic representations associated to Siegel cusp forms. For all but finitely many places  $p$  the representation  $\Pi_{F,p}$  will be an Iwahori spherical representation  $\Pi(\sigma, \chi_1, \chi_2)$ , which is isomorphic to the Langlands quotient of an induced representation of the form  $\chi_1 \times \chi_2 \rtimes \sigma$  with  $\chi_i$  and  $\sigma$  unramified characters of  $\mathbb{Q}_p^\times$ . One can see [2] and [22] for the definitions and details. For such  $p$  the  $p$ -Satake parameters are defined by  $b_0 = \sigma(p)$  and  $b_i = \chi_i(p)$  for  $i = 1, 2$ . We define

$$L(s, \Pi_{F,p}, \text{spin}) = ((1 - b_0p^{-s})(1 - b_0b_1p^{-s})(1 - b_0b_2p^{-s})(1 - b_0b_1b_2p^{-s}))^{-1}$$

for  $\Pi_{F,p} = \Pi(\sigma, \chi_1, \chi_2)$ . We leave the local  $L$ -functions for  $p = \infty$  and  $p \mid M$  undefined, as defining these would take us too far afield. One can see [24] for the definitions. Set

$$L(s, \Pi_F, \text{spin}) = \prod_p L(s, \Pi_{F,p}, \text{spin}).$$

The classical spinor  $L$ -function is given by

$$L(s, F, \text{spin}) = \prod_{p < \infty} L(s - \kappa + 3/2, \Pi_{F,p}, \text{spin}).$$

Given an Euler product  $L(s) = \prod_p L_p(s)$ , we write  $L^M(s) = \prod_{p \nmid M} L_p(s)$ .

### 3. Classical construction

In this section we gather known results and give a classical construction of the Saito–Kurokawa lifting from  $S_{2\kappa-2}(\Gamma_0(M))$  to  $S_\kappa(\Gamma_0^{(2)}(M))$  for  $\kappa \geq 2$  an even integer and  $M \geq 1$  an odd square-free integer. The existence of a Saito–Kurokawa lifting is known from a representation theory point of view via [24]. In particular, [24, Theorem 5.2] gives that this lift is unique up to a constant multiple. Using the classical construction given in this section we fix the scalar so that the resulting Saito–Kurokawa lift is more useful for arithmetic applications (see, e.g., [1], [16]). In particular, given a newform  $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ , if we let  $\mathcal{O}$  be a ring containing the Hecke eigenvalues of  $f$ , then we can show that the Saito–Kurokawa lift of  $f$  can be normalized so that it has Fourier coefficients lying in  $\mathcal{O}$  as well.

The classical lifting is constructed via a composition of liftings, the first from integral to half-integral weight, then from half-integral weight to Jacobi forms, and finally from Jacobi forms to Siegel modular forms. We begin by recalling results on the lifting from integral to half-integral weight forms.

Let  $S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  denote Kohnen’s  $+$ -space, that is, the cusp forms of weight  $\kappa - \frac{1}{2}$  and level  $\Gamma_0(4M)$  whose  $n$ th Fourier coefficients at infinity vanish for  $(-1)^{\kappa-1}n \equiv 2, 3 \pmod{4}$ . For a prime  $p \nmid M$  we define the  $p^2$ th Hecke operator acting on  $S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  by

$$\begin{aligned} T_H(p) &= \sum_{\substack{n \geq 1 \\ (-1)^{\kappa-1}n \equiv 0,1(4)}} c(n)q^n \\ &= \sum_{\substack{n \geq 1 \\ (-1)^{\kappa-1}n \equiv 0,1(4)}} \left( c(p^2n) + \left( \frac{(-1)^{\kappa-1}n}{p} \right) p^{\kappa-2}c(n) + p^{2\kappa-3}c(n/p^2) \right) q^n. \end{aligned}$$

We follow the literature and denote this by  $T_H(p)$  instead of  $T_H(p^2)$  since the  $p$ th Hecke operator on half-integral weight modular forms vanishes. For  $p \mid M$  we set

$$U_H(p) = \sum_{\substack{n \geq 1 \\ (-1)^{\kappa-1}n \equiv 0,1(4)}} c(n)q^n = \sum_{\substack{n \geq 1 \\ (-1)^{\kappa-1}n \equiv 0,1(4)}} c(p^2n)q^n.$$

The inner product on  $S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  is given by

$$\langle g_1, g_2 \rangle = \frac{1}{[\Gamma_0(4) : \Gamma_0(4M)]} \int_{\Gamma_0(4M) \backslash \mathfrak{h}^1} g_1(z) \overline{g_2(z)} y^{\kappa-5/2} dx dy$$

for  $z = x + iy$ .

One has a Hecke-equivariant isomorphism between  $S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$  and  $S_{\kappa-\frac{1}{2}}^{+, \text{new}}(\Gamma_0(4M))$  (see [12, Section 5, Theorem 2]) where  $S_{\kappa-\frac{1}{2}}^{+, \text{new}}(\Gamma_0(4M))$  denotes the subspace of newforms as defined in [12, Section 5].

For arithmetic applications such as those given in [1] it is important to keep track of the Fourier coefficients of the Saito–Kurokawa lift of  $f$ . Let  $D < 0$  be a fundamental discriminant, and let  $\theta_{\kappa, D} : S_{2\kappa-2}(\Gamma_0(M)) \rightarrow S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  be Shintani’s lifting (see [25]). Let  $\mathcal{O}$  be a ring so that an embedding of  $\mathcal{O}$  into  $\mathbb{C}$  exists. We choose such an embedding and identify  $\mathcal{O}$  with its image in  $\mathbb{C}$ . Let  $f \in S_{2\kappa-2}(\Gamma_0(M))$  be a normalized Hecke eigenform, and assume that  $\mathcal{O}$  contains the Fourier coefficients of  $f$ . The Shintani lifting of  $f$  is determined up to a scalar multiple. Stevens [28] constructs a cohomological version of the Shintani lifting as a step in producing a  $\Lambda$ -adic Shintani lifting. This cohomological Shintani lifting allows one to deduce the following result.

**THEOREM 3.1** ([28, PROPOSITION 2.3.1])

Let  $f \in S_{2\kappa-2}(\Gamma_0(M))$  be a Hecke eigenform. Let  $D < 0$  be a fundamental discriminant. If the Fourier coefficients of  $f$  are in  $\mathcal{O}$ , then there exists a Shintani lifting  $\theta_{\kappa,D}^{\text{alg}}(f)$  with Fourier coefficients in  $\mathcal{O}$ .

This gives what we need for the first part of the construction. We now consider the lifting from half-integral weight forms to Jacobi forms. Let  $\Gamma_0(M)^J = \Gamma_0(M) \times \mathbb{Z}^2$ , that is, the set of pairs  $(\gamma, X)$  with  $\gamma \in \Gamma_0(M)$  and  $X \in \mathbb{Z}^2$  with the group law given by  $(\gamma_1, X_1)(\gamma_2, X_2) = (\gamma_1\gamma_2, X_1\gamma_2 + X_2)$ . We denote the space of Jacobi cusp forms of weight  $\kappa$ , index 1, and level  $\Gamma_0(M)^J$  by  $J_{\kappa,1}^c(\Gamma_0(M)^J)$ . Given a positive integer  $m$ , define

$$\Delta_{M,0}(m) = \left\{ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bcM = m, \gcd(a, M) = 1 \right\}.$$

We make use of the Hecke operators on  $J_{\kappa,1}^c(\Gamma_0(M)^J)$  defined by

$$T_J(p)\phi = p^{\kappa-4} \sum_{\gamma} \sum_{(\lambda, \mu) \in \mathbb{F}_p^2} \phi|_{\kappa,1}(\gamma/p)|_1(\lambda, \mu),$$

where  $\gamma$  runs over

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} : b = 0, \dots, p^2 - 1 \right\} \cup \left\{ \begin{pmatrix} p & b \\ 0 & p \end{pmatrix} : b = 0, \dots, p - 1 \right\} \cup \left\{ \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In the case where  $p \mid M$ , we define  $U_J(p)$  as we defined  $T_J(p)$  with the only difference being that here we only take  $\gamma$  running over

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} : b = 0, \dots, p^2 - 1 \right\}.$$

The inner product on  $J_{\kappa,1}^c(\Gamma_0(M)^J)$  is defined by

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &= \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \\ &\times \int_{\Gamma_0(M)^J \backslash \mathfrak{h}^1 \times \mathbb{C}} \phi_1(\tau, z) \overline{\phi_2(\tau, z)} v^{\kappa-3} e^{-4\pi y^2/v} dx dy du dv \end{aligned}$$

for  $\tau = u + iv$  and  $z = x + iy$ .

We recall the isomorphism between  $J_{\kappa,1}^c(\Gamma_0(M)^J)$  and  $S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  for  $M$  an odd integer.

Let  $g \in S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$ . Define  $g_j(\tau)$  by

$$g_j(\tau) = \sum_{\substack{n > 0 \\ n \equiv j \pmod{4}}} a_g(n) e(n\tau/4)$$

for  $j = 0, 1$  where the  $a_g(n)$ 's are the Fourier coefficients of  $g$ . It is clear that  $g(\tau) = g_0(4\tau) + g_1(4\tau)$ . Define

$$\vartheta_j(\tau, z) = \sum_{n \in \mathbb{Z}} e\left(\frac{2n-j^2}{4}\tau + (2n-j)z\right).$$

Define a map  $\mathcal{J}$  by

$$\mathcal{J}(g)(\tau, z) = g_0(\tau)\vartheta_0(\tau, z) + g_1(\tau)\vartheta_1(\tau, z)$$

for  $g \in S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$ . The following theorem provides the necessary connection between half-integral weight forms and Jacobi forms. Note that the compatibility with Hecke operators follows immediately from the definitions and the map given.

**THEOREM 3.2** ([15, COROLLARY 3])

*The map  $\mathcal{J}$  gives a Hecke-equivariant isomorphism between  $S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  and  $J_{\kappa,1}^c(\Gamma_0(M)^J)$ . Moreover, if  $g \in S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$  has Fourier coefficients in  $\mathcal{O}$ , then so does  $\mathcal{J}$ .*

Finally, we recall the Maass lifting from the Jacobi forms to Siegel forms. Let

$$V_m : J_{\kappa,t}^c(\Gamma_0(M)^J) \rightarrow J_{\kappa,mt}^c(\Gamma_0(M)^J)$$

be the index-shifting operator defined by

$$(V_m\phi)(\tau, z) = m^{\kappa-1} \sum_{g \in \Gamma_0(M) \backslash \Delta_{M,0}(m)} (\phi|_{\kappa,t}g)(\tau, z).$$

If the Fourier expansion of  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$  is given by

$$\phi(\tau, z) = \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4}}} c(D, r) e\left(\frac{r^2 - D}{4}\tau + rz\right),$$

then

$$(V_m\phi)(\tau, z) = \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4m}}} \left( \sum_{\substack{d | \gcd(r, m) \\ \gcd(d, M) = 1 \\ D \equiv r^2 \pmod{4md}}} d^{\kappa-1} c\left(\frac{D}{d^2}, \frac{r}{d}\right) \right) e\left(\frac{r^2 - D}{4m}\tau + rz\right).$$

Define a function on  $\mathfrak{h}^2$  by

$$(\mathcal{V}_M\phi)(Z) = \sum_{m=1}^{\infty} (V_m\phi)(\tau, z) e(m\tau')$$

where  $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ . One has the following result of Ibukiyama.

**THEOREM 3.3** ([11, THEOREMS 3.2, 4.1])

*The map  $\mathcal{V}_M$  is an injective linear map from  $J_{\kappa,1}^c(\Gamma_0(M)^J)$  to  $S_{\kappa}(\Gamma_0^{(2)}(M))$ . Moreover, one has for  $p \nmid M$ ,*

$$\begin{aligned} T_S(p)(\mathcal{V}_M\phi) &= \mathcal{V}_M(T_J(p)\phi + (p^{\kappa-1} + p^{\kappa-2})\phi), \\ T'_S(p)(\mathcal{V}_M\phi) &= \mathcal{V}_M((p^{\kappa-2} + p^{\kappa-1})T_J(p)\phi + (2p^{2\kappa-3} + p^{2\kappa-4})\phi). \end{aligned}$$

If  $p \nmid M$ , then

$$U_S(p)(\mathcal{V}_M\phi) = \mathcal{V}_M(U_J(p)\phi),$$

where  $T_J(p)$  and  $U_J(p)$  are as defined above and the Siegel Hecke operators  $T_S(p)$  and  $U_S(p)$  were defined in the previous section.

An immediate consequence of the calculation of the Fourier coefficients of  $\mathcal{V}_M$  carried out in [11] is the following corollary.

**COROLLARY 3.4**

Let  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J, \mathcal{O})$  for some ring  $\mathcal{O}$ . Then  $\mathcal{V}_M\phi \in S_\kappa(\Gamma_0^{(2)}(M), \mathcal{O})$ .

The following theorem gives the existence of a Saito–Kurokawa lifting by combining the result on the Shintani lifting with Theorems 3.2 and 3.3.

**THEOREM 3.5**

Let  $\kappa \geq 2$  be an even integer, and let  $M \geq 1$  be an odd square-free integer. Let  $f \in S_{2\kappa-2}(\Gamma_0(M))$  be a normalized Hecke eigenform. Then there exists a nonzero cuspidal Siegel eigenform  $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$  satisfying

$$L^M(s, F_f, \text{spin}) = \zeta^M(s - \kappa + 1)\zeta^M(s - \kappa + 2)L^M(s, f).$$

Moreover, if  $\mathcal{O}$  is a ring that can be embedded into  $\mathbb{C}$  and  $f$  has Fourier coefficients in  $\mathcal{O}$ , then the lift  $F_f$  can be normalized to have Fourier coefficients in  $\mathcal{O}$ . If  $\mathcal{O}$  is a discrete valuation ring (DVR), then  $F_f$  can be normalized to have Fourier coefficients in  $\mathcal{O}$  with at least one Fourier coefficient in  $\mathcal{O}^\times$ .

*Proof*

The only thing that remains to prove is that if  $\mathcal{O}$  is a DVR, then we can normalize  $F_f$  so that it has Fourier coefficients in  $\mathcal{O}$  with at least one in  $\mathcal{O}^\times$ . The point is that the Shintani lifting is determined only up to a scalar multiple. By choosing the Shintani lifting  $\theta_{\kappa,D}^{\text{alg}}(f)$  to have Fourier coefficients in  $\mathcal{O}$ , it certainly follows that  $F_f$  has Fourier coefficients in  $\mathcal{O}$  from the above construction. Now suppose that  $m = \min_{T \in \Lambda_2} \text{val}_\varpi(a_F(T))$  where  $\varpi$  is a uniformizer of  $\mathcal{O}$ . Thus, if we normalize the Shintani lifting by  $\varpi^{-m}$ , the resulting  $F_f$  will still have Fourier coefficients in  $\mathcal{O}$  but will have at least one coefficient in  $\mathcal{O}^\times$ . □

**4. Norm of  $F_f$**

We now calculate the norm of  $F_f$  in terms of the norm of  $f$ . This forms a key step in the main result of [1], but is also of independent interest. We do this by relating the norm of the image of each lift to the norm of the form being lifted for each of the three lifts composed to give the Saito–Kurokawa lift. Again, we fix  $\kappa \geq 2$  to be an even integer and  $M \geq 1$  to be odd and square-free.

Let  $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$  be a newform. For each prime  $\ell \mid M$ , recall that  $W_\ell$  is the Atkin–Lehner involution on  $S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$ . Define  $\epsilon_\ell \in \{\pm 1\}$  by

$$f \mid W_\ell = \epsilon_\ell f.$$

The following theorem relates the norm of  $f$  to that of  $\theta_{\kappa,D}^{\text{alg}}(f)$ .



THEOREM 4.1 ([13, COROLLARY 1])

Let  $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$  be a newform, and let  $D < 0$  be a fundamental discriminant. Suppose that  $(\frac{D}{\ell}) = \epsilon_\ell$  for all primes  $\ell \mid M$ . Then

$$(1) \quad \frac{|a_{\theta_{\kappa,D}^{\text{alg}}}(f)(|D|)|^2}{\langle \theta_{\kappa,D}^{\text{alg}}(f), \theta_{\kappa,D}^{\text{alg}}(f) \rangle} = 2^{\nu(M)} \frac{(\kappa-2)!}{\pi^{\kappa-1}} |D|^{\kappa-3/2} \frac{L(\kappa-1, f, \chi_D)}{\langle f, f \rangle},$$

where  $\chi_D = (\frac{D}{\cdot})$  and  $\nu(M)$  is the number of prime divisors of  $M$ .

One should note that if there is a prime  $\ell \mid M$  so that  $(\frac{D}{\ell}) \neq \epsilon_\ell$ , then one has  $a_{\theta_{\kappa,D}^{\text{alg}}}(f)(|D|) = 0$ .

Now let  $g \in S_{\kappa-\frac{1}{2}}^+(\Gamma_0(4M))$ , and let  $\mathcal{J}(g)$  be the associated form in  $J_{\kappa,1}^c(\Gamma_0(M)^J)$ . Let  $g(z) = \sum_{n=1}^\infty a_g(n)e(nz)$  be the Fourier expansion of  $g$ . Consider the summation  $\sum_{n=1}^\infty a_g(n)^2/(n^{s+\kappa-3/2})$ . Applying the Rankin–Selberg method to this summation we have for sufficiently large  $s$ ,

$$\begin{aligned} \frac{\Gamma(s+\kappa-3/2)}{(4\pi)^{s+\kappa-\frac{1}{2}}} \sum_{n=1}^\infty \frac{a_g(n)^2}{n^{s+\kappa-3/2}} &= \int_{\mathfrak{h}^1/\Gamma_\infty} |g(z)|^2 y^{s+\kappa-5/2} dx dy \\ &= \int_{\mathfrak{h}^1/\Gamma_0(4M)} y^{\kappa-\frac{1}{2}} |g(z)|^2 E_s^{4M}(z) \frac{dx dy}{y^2}, \end{aligned}$$

where  $E_s^{4M}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4M)} (\text{Im}(\gamma z))^s$  and  $\Gamma_\infty$  is the stabilizer of  $\infty$ . In other words,

$$(2) \quad \sum_{n=1}^\infty \frac{a_g(n)^2}{n^{s+\kappa-3/2}} = \frac{(4\pi)^{s+\kappa-\frac{1}{2}}}{\Gamma(s+\kappa-3/2)} \int_{\Gamma_0(4M) \backslash \mathfrak{h}^1} E_s^{4M}(z) g(z) \overline{g(z)} y^{\kappa-\frac{1}{2}} \frac{dx dy}{y^2}.$$

Taking residues at  $s = 1$  we obtain that

$$\begin{aligned} \text{res}_{s=1} \left( \sum_{n=1}^\infty \frac{a_g(n)^2}{n^{s+\kappa-3/2}} \right) &= \frac{(4\pi)^{1+\kappa-\frac{1}{2}} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]}{\Gamma(\kappa-\frac{1}{2})} \langle g, g \rangle \text{res}_{s=1} E_s^{4M}(z) \\ &= \frac{3 \cdot 2^{\kappa-1} (4\pi)^{\kappa-\frac{1}{2}}}{\pi^{3/2} (2\kappa-3)!!} \langle g, g \rangle, \end{aligned}$$

where

$$n!! = \begin{cases} n(n-2) \cdots 5 \cdot 3 \cdot 1 & \text{for } n > 0 \text{ and odd,} \\ n(n-2) \cdots 6 \cdot 4 \cdot 2 & \text{for } n > 0 \text{ and even,} \end{cases}$$

and we have used that

$$\begin{aligned} \text{res}_{s=1} E_s^{4M}(z) &= \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]} \text{res}_{s=1} E_s(z) \\ &= \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(4M)]} \left( \frac{3}{\pi} \right), \end{aligned}$$

where  $E_s(z) = E_s^1(z)$  and one can see [9] for the calculation of the residue of the Eisenstein series.

Solving the above residue calculation for  $\langle g, g \rangle$  we obtain that

$$(3) \quad \langle g, g \rangle = \frac{(2\kappa - 3)!!}{3 \cdot 2^{3\kappa-2}\pi^{\kappa-2}} \operatorname{res}_{s=1} \left( \sum_{n=1}^{\infty} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} \right).$$

Recall the two half-integral weight modular forms  $g_0$  and  $g_1$  defined in Section 3. Applying the same process to  $g_0$  and  $g_1$  we obtain that

$$\langle g_j, g_j \rangle = \frac{(2\kappa - 3)!!}{3 \cdot 2^{3\kappa-2}\pi^{\kappa-2}} \cdot 2^{2\kappa-1} \operatorname{res}_{s=1} \left( \sum_{n \equiv j} \frac{a_g(n)^2}{n^{s+\kappa-3/2}} \right).$$

Thus

$$(4) \quad \langle g_0, g_0 \rangle + \langle g_1, g_1 \rangle = 2^{2\kappa-1} \langle g, g \rangle.$$

We need a slight generalization of [8, Theorem 5.3]. In [8], the formula given only deals with the case  $M = 1$ . However, the proof carries through verbatim to the general case.

**THEOREM 4.2 ([8, THEOREM 5.3])**

For  $\mathcal{J}(g)$  and  $g_j$  as defined above, one has that

$$(5) \quad \langle \mathcal{J}(g), \mathcal{J}(g) \rangle = \frac{1}{2[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)]} \int_{\Gamma_0(M) \backslash \mathfrak{h}^1} \sum_{j=0}^1 g_j(z) \overline{g_j(z)} v^{\kappa-3/2} \frac{du dv}{v^2}.$$

Combining (4) and (5) we obtain the following result.

**LEMMA 4.3**

For  $\mathcal{J}(g)$  and  $g$  defined as above,

$$(6) \quad \langle \mathcal{J}(g), \mathcal{J}(g) \rangle = \frac{2^{2\kappa-2}}{[\Gamma_0(M) : \Gamma_0(4M)]} \langle g, g \rangle.$$

In light of Theorem 4.1 and Proposition 4.3, it only remains to calculate the ratio of  $\langle \phi, \phi \rangle$  and  $\langle \mathcal{V}_M \phi, \mathcal{V}_M \phi \rangle$  for  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ . We follow the arguments used in [14] where this ratio is computed when  $M = 1$ . One should note that this inner product was originally given in [3]. However, that result cited a theorem in [6, Theorem 4.2] which in turn was based on the incorrect definition of the  $\mathcal{V}_M$  map used in [19]. Thus, we give the computation here using the corrected definition given in [11]. The argument given in [3, Section 4] is correct up until the point the result of [6] is invoked; however, we include the complete argument here to have it given in one place.

Let  $F, G \in S_{\kappa}(\Gamma_0^{(2)}(M))$  be eigenforms with Fourier–Jacobi expansions given by

$$F(Z) = \sum_{N \geq 1} \phi_N(\tau, z) e(N\tau')$$

and

$$G(Z) = \sum_{N \geq 1} \psi_N(\tau, z)e(N\tau').$$

Define a Dirichlet series attached to  $F$  and  $G$  by

$$D_{F,G}(s) = \zeta^M(2s - 2\kappa + 4) \sum_{N \geq 1} \langle \phi_N, \psi_N \rangle N^{-s},$$

and set

$$(7) \quad D_{F,G}^*(s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - \kappa + 2) D_{F,G}(s).$$

It is shown in [10] that  $D_{F,G}^*(s)$  has meromorphic continuation to  $\mathbb{C}$ , is entire if  $\langle F, G \rangle = 0$ , and otherwise has a simple pole at  $s = \kappa$ . The first step in calculating the ratio of inner products we desire is calculating the residue of  $D_{F,G}$  at  $s = \kappa$ . We do this by writing  $D_{F,G}$  as the Petersson product of  $F(Z)G(\bar{Z})|Y|^\kappa$  against a certain nonholomorphic Klingen Eisenstein series  $E_{s,M}(Z)$ . Define a Klingen Eisenstein series

$$E_{s,M}(Z) = \sum_{\gamma \in C_{2,1}(M) \backslash \Gamma_0^{(2)}(M)} \left( \frac{\det(\operatorname{Im} \gamma Z)}{\operatorname{Im}(\gamma Z)_1} \right)^s,$$

where  $(\gamma Z)_1$  denotes the upper left entry of  $\gamma Z$  and

$$C_{2,1}(M) = \left\{ \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^{(2)}(M) \right\}, \quad (\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Set

$$(8) \quad E_{s,M}^*(Z) = \pi^{-s} \Gamma(s) \zeta(2s) \prod_{p|M} (1 - p^{-2s}) E_{s,M}(Z).$$

One has that  $E_{s,M}^*(Z)$  has meromorphic continuation to  $\mathbb{C}$  with possible simple poles at  $s = 0, 2$  (see [10]). It is known that  $\operatorname{res}_{s=2} E_{s,1}^*(Z) = 1$  (see [14]). Note that this is independent of  $Z$ , so  $\operatorname{res}_{s=2} E_{s,1}^*(NZ) = 1$  for all positive integers  $N$ . Then (8) gives that  $\operatorname{res}_{s=2} E_{s,1}(Z) = 90/\pi^2$ . As above, this residue is independent of  $Z$ , so  $\operatorname{res}_{s=2} E_{s,1}(NZ) = 90/\pi^2$  for all positive integers  $N$ . The following formula is given in [10]:

$$E_{s,1}(MZ) = \frac{1}{M^s} \sum_{d|M} d^{2s} \prod_{p|d} (1 - p^{-2s}) E_{s,d}(Z).$$

This formula allows one to calculate the residue of  $E_{s,M}(Z)$  inductively in terms of  $E_{s,d}(Z)$  for  $d | M$ . In fact, for  $M = p_1^{m_1} \cdots p_n^{m_n}$ ,

$$(9) \quad \operatorname{res}_{s=2} E_{s,M}(Z) = \left( \frac{90}{\pi^2} \right) h(p_1, \dots, p_n) \prod_{i=1}^n \left( \frac{1}{p_i^{2m_i-2} (p_i^4 - 1)} \right),$$

where  $h$  is a polynomial with coefficients in  $\mathbb{Z}$  uniquely determined by  $M$ . For example, if  $M = p^n$  for a prime  $p$ , then

$$h(p) = p^2 - 1,$$

and if  $M = p_1 \cdots p_n$  is a product of distinct primes, then

$$h(p_1, \dots, p_n) = \prod_{i=1}^n (p_i^2 - 1).$$

We will be mainly interested in the case in which  $M = p_1 \cdots p_n$  is odd and square-free. Appealing to (8) we obtain that

$$(10) \quad \operatorname{res}_{s=2} E_{s,M}^*(Z) = \prod_{i=1}^n \left( \frac{1 - p_i^{-4}}{p_i^2 + 1} \right).$$

We now turn our attention back to calculating the residue of  $D_{F,G}(s)$  at  $s = \kappa$ . We make use of the following equation (see [10]):

$$\pi^{-\kappa+2} [\operatorname{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)] \langle FE_{s-\kappa+2,M}^*, G \rangle = M^s D_{F,G}^*(s).$$

Taking the residue of this equation at  $s = \kappa$  and solving for  $\operatorname{res}_{s=\kappa} D_{F,G}^*(s)$  we obtain that

$$\begin{aligned} \operatorname{res}_{s=\kappa} D_{F,G}^*(s) &= \frac{\pi^{2-\kappa} [\operatorname{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)]}{M^\kappa} \operatorname{res}_{s=2} E_{s,M}^*(Z) \langle F, G \rangle \\ &= \frac{\pi^{2-\kappa} [\operatorname{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)]}{M^\kappa} \prod_{p|M} \left( \frac{1 - p_i^{-4}}{p_i^2 + 1} \right) \langle F, G \rangle \\ &= \frac{\pi^{2-\kappa} [\operatorname{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)]}{M^\kappa \zeta_M(4)} \prod_{p|M} \left( \frac{1}{p_i^2 + 1} \right) \langle F, G \rangle. \end{aligned}$$

On the other hand, taking the residue at  $s = \kappa$  of (7) gives

$$\operatorname{res}_{s=\kappa} D_{F,G}^*(s) = (2\pi)^{-2\kappa} (\kappa - 1)! \operatorname{res}_{s=\kappa} D_{F,G}(s).$$

Combining these two results and solving for  $\operatorname{res}_{s=\kappa} D_{F,G}(s)$  we obtain

$$(11) \quad \operatorname{res}_{s=\kappa} D_{F,G}(s) = \frac{2^{2\kappa} \pi^{\kappa+2} [\operatorname{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)]}{M^\kappa \zeta_M(4) (\kappa - 1)! \prod_{p|M} (p_i^2 + 1)} \langle F, G \rangle.$$

Following [14], our next step is to calculate the adjoint of the operator  $V_m$ . We will need the following lemma.

LEMMA 4.4

Let  $\Delta_{M,0}^*(m) \subset \Delta_{M,0}(m)$  be the matrices  $\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}$  with  $\gcd(a, b, c, d) = 1$ . The map

$$\begin{aligned} \varphi_M : \Gamma_0(M, m) \backslash \Gamma_0(M) &\rightarrow \Gamma_0(M) \backslash \Delta_{M,0}^*(m), \\ \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \end{aligned}$$

gives a bijection where

$$\Gamma_0(M, m) = \left\{ \begin{pmatrix} a & mb \\ Mc & d \end{pmatrix} \in \Gamma_0(M) \right\}.$$

*Proof*

The proof amounts to showing that the cardinality of  $\Gamma_0(M, m) \backslash \Gamma_0(M)$  is the same as the cardinality of  $\Gamma_0(M) \backslash \Delta_{M,0}^*(m)$  and then showing that the map  $\varphi_M$  is injective by using the explicit coset representatives given for  $\Gamma_0(M, m) \backslash \Gamma_0(M)$ . Write  $M = \prod_{i=1}^r p^{e_i}$  and  $m = \prod_{i=1}^r p^{f_i}$  where the  $e_i$ 's and  $f_i$ 's are nonnegative integers. One has that

$$\#(\Gamma_0(M) \backslash \Delta_{M,0}(m)) = \prod_{i=1}^r \#(\Gamma_0(M) \backslash \Delta_{M,0}(p_i^{f_i}))$$

and so

$$\#(\Gamma_0(M) \backslash \Delta_{M,0}^*(m)) = \prod_{i=1}^r \#(\Gamma_0(M) \backslash \Delta_{M,0}^*(p_i^{f_i})).$$

Thus, we only need to calculate

$$\#(\Gamma_0(M) \backslash \Delta_{M,0}^*(p^f))$$

for a prime  $p$ . This breaks into two cases depending upon whether  $p$  divides  $M$  or not. The main input is the fact that

$$\Gamma_0(M) \backslash \Delta_{M,0}(m) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : ad = m, \gcd(a, M) = 1, 0 \leq b \leq d - 1 \right\}.$$

Using this, one sees that if  $p \mid M$ , then

$$\Gamma_0(M) \backslash \Delta_{M,0}^*(p^f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & p^f \end{pmatrix} : 0 \leq b < p^f \right\},$$

and so there are  $p^f$  elements. If  $p \nmid M$ , then one shows by induction on  $f$  and counting as above that

$$\#(\Gamma_0(M) \backslash \Delta_{M,0}^*(p^f)) = p^f + p^{f-1}.$$

Recall that the map

$$\lambda_T : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/T\mathbb{Z})$$

is surjective with kernel  $\Gamma(T)$ . From this, one sees that there is a bijection between  $\Gamma_0(M, m) \backslash \Gamma_0(M)$  and  $\lambda_T(\Gamma_0(M, m)) \backslash \lambda_T(\Gamma_0(M))$  where  $T = \mathrm{lcm}(M, m)$ . Moreover, one has that

$$\mathrm{SL}_2(\mathbb{Z}/T\mathbb{Z}) \cong \prod_{p^f \parallel T} \mathrm{SL}_2(\mathbb{Z}/p^f\mathbb{Z}).$$

Thus, one only needs to work with prime powers to compute the cosets. Write  $M = p^e$  and  $m = p^f$  for nonnegative integers  $e, f$ . We break into cases for this:

(1) Suppose that  $e = 0$ . In this case  $\lambda_T(\Gamma_0(M, m)) \setminus \lambda_T(\Gamma_0(M)) = \lambda_T(\Gamma^0(m)) \setminus \lambda_T(\text{SL}_2(\mathbb{Z}))$ . In this case the coset representatives are given by  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  for  $0 \leq b < p^f$  and  $\begin{pmatrix} 1 & b \\ (1+b)p^{-1} & (1+b)p \end{pmatrix}$  for  $0 \leq b < p^{f-1}$  (see [7, Section 3.7]).

(2) Suppose that  $e$  is positive. In this case the representatives are given by  $\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : 0 \leq b < p^f\}$ .

Thus, the cardinalities match up and the map  $\varphi_M$  is clearly injective. □

**PROPOSITION 4.5**

Let  $V_m^* : J_{\kappa, m}^c(\Gamma_0(M)^J) \rightarrow J_{\kappa, 1}^c(\Gamma_0(M)^J)$  be the adjoint of  $V_m$  with respect to the Petersson inner product. Let  $\psi \in J_{\kappa, m}^c(\Gamma_0(M)^J)$  with

$$\psi(\tau, z) = \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2(4m)}} c(D, r) e\left(\frac{r^2 - D}{4m} \tau + rz\right).$$

The action of  $V_m^*$  on Fourier coefficients is given by

$$\begin{aligned} V_m^* \psi(\tau, z) &= \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2(4)}} \left( \sum_{\substack{d|m \\ \gcd(d, M)=1}} d^{\kappa-2} \sum_{s \in S(r, d, D)} c\left(\frac{m^2}{d^2} D, \frac{m}{d} s\right) \right) \\ &\quad \cdot e\left(\frac{r^2 - D}{4} \tau + rz\right), \end{aligned}$$

where  $S(r, d, D) = \{s \pmod{2d} : s \equiv r \pmod{2}, s^2 \equiv D \pmod{4d}\}$ .

*Proof*

The proof here is analogous to the one given in [14] for the level 1 case. We include a proof for  $M \geq 1$  and square-free here with more details for the reader's convenience. Let  $\phi \in J_{\kappa, 1}^c(\Gamma_0(M)^J)$ . Given  $a \in \mathbb{C}$ , write  $\phi_a(\tau, z)$  for the function  $\phi(\tau, az)$ . For  $m' \mid m$ , we write  $m/m' = \square$  to denote that  $m/m'$  is a perfect square. One has immediately from the definition and the lemma above that

$$\begin{aligned} V_m \phi &= m^{\kappa/2-1} \sum_{g \in \Gamma_0(M) \setminus \Delta_{M, 0}(m)} \phi_{\sqrt{m} | \kappa, m} \left( \frac{g}{\sqrt{m}} \right) \\ &= m^{\kappa/2-1} \sum_{\substack{m' | m \\ m/m' = \square}} \sum_{g \in \Gamma_0(M) \setminus \Delta_{M, 0}^*(m)} \phi_{\sqrt{m} | \kappa, m} \left( \frac{g}{\sqrt{m'}} \right) \\ &= m^{\kappa/2-1} \sum_{\substack{m' | m \\ m/m' = \square}} \sum_{g \in \Gamma_0(M, m) \setminus \Gamma_0(M)} \phi_{\sqrt{m} | \kappa, m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} g. \end{aligned}$$

Note that  $\phi_{\sqrt{m} | \kappa, m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} \in J_{\kappa, m}^c(\Gamma_0(M, m)^J)$ .

Given  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$  and  $\psi \in J_{\kappa,m}^c(\Gamma_0(M)^J)$ ,

$$\begin{aligned} \langle V_m \phi, \psi \rangle &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} \sum_{g \in \Gamma_0(M,m) \backslash \Gamma_0(M)} \left\langle \phi_{\sqrt{m}|\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix} g, \psi \right\rangle \\ &= m^{\kappa/2-1} \sum_{\substack{m'|m \\ m/m'=\square}} [\Gamma_0(M) : \Gamma_0(M,m)] \left\langle \phi_{\sqrt{m}|\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix}, \psi \right\rangle, \end{aligned}$$

where we have used  $\langle \phi | \gamma, \psi \rangle = \langle \phi, \psi | \gamma^{-1} \rangle$  and  $\psi|_{\kappa,m} g = \psi$  as  $\psi$  has level  $\Gamma_0(M)^J$ . Note that  $\psi_{\frac{1}{\sqrt{m}}|\kappa,1} \begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} \in J_{\kappa,1}^c(\Gamma_0(M,m)^J)$ . Moreover,

$$\begin{aligned} &\left\langle \phi_{\sqrt{m}|\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix}, \psi \right\rangle \\ &= \left\langle \phi, \psi_{\frac{1}{\sqrt{m}}|\kappa,1} \begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} \right\rangle. \end{aligned}$$

Observe that we can write

$$\begin{aligned} &\left\langle \phi_{\sqrt{m}|\kappa,m} \begin{pmatrix} \frac{1}{\sqrt{m'}} & 0 \\ 0 & \sqrt{m'} \end{pmatrix}, \psi \right\rangle \\ &= \left\langle \phi, \psi_{\frac{1}{\sqrt{m}}|\kappa,1} \begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} \right\rangle \\ &= \frac{1}{m'^2 [\Gamma_0(M) : \Gamma_0(M,m')]} \\ &\quad \times \sum_{X \in (\mathbb{Z}/m'\mathbb{Z})^2} \sum_{g \in \Gamma_0(M,m') \backslash \Gamma_0(M)} \left\langle \phi, \psi_{\frac{1}{\sqrt{m}}|\kappa,1} \begin{pmatrix} \sqrt{m'} & 0 \\ 0 & \frac{1}{\sqrt{m'}} \end{pmatrix} gX \right\rangle. \end{aligned}$$

Thus, by essentially reversing the above argument we obtain that

$$\langle V_m \phi, \psi \rangle = \left\langle \phi, m^{\kappa/2-3} \sum_{X \in (\mathbb{Z}/m\mathbb{Z})^2} \sum_{g \in \Gamma_0(M,m) \backslash \Gamma_0(M)} \psi_{\frac{1}{\sqrt{m}}|\kappa,1} \left( \frac{gX}{\sqrt{m}} \right) \right\rangle.$$

One then checks that in fact

$$m^{\kappa/2-3} \sum_{X \in (\mathbb{Z}/m\mathbb{Z})^2} \sum_{g \in \Gamma_0(M) \backslash \Delta_{M,0}^*(m)} \psi_{\frac{1}{\sqrt{m}}|\kappa,1} \left( \frac{gX}{\sqrt{m}} \right) \in J_{\kappa,1}^c(\Gamma_0(M)^J),$$

and so we obtain the formula for  $V_m^* : J_{\kappa,m}^c(\Gamma_0(M)^J) \rightarrow J_{\kappa,1}^c(\Gamma_0(M)^J)$ .

Thus, it just remains to compute the Fourier expansion of  $V_m^* \psi$ . Let  $\psi \in J_{\kappa,m}^c(\Gamma_0(M)^J)$  with Fourier expansion given by

$$\psi(\tau, z) = \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2(4)}} c(D, r) e\left(\frac{r^2 - D}{4m} \tau + rz\right).$$

We use the same representatives for  $\Gamma_0(M)\backslash\Delta_{M,0}(m)$  as above. Thus,

$$\begin{aligned} V_m^*\psi(\tau, z) &= m^{\kappa/2-3} \sum_{\lambda, \mu(m)} \sum_{\substack{ad=m \\ \gcd(a, M)=1 \\ b(d)}} \left(\frac{\sqrt{m}}{d}\right)^\kappa \\ &\quad \cdot e(\lambda^2\tau + 2\lambda z) \psi\left(\frac{a\tau + b}{d}, \frac{z + \lambda\tau + \mu}{d}\right) \\ &= m^{\kappa-3} \sum_{\lambda, \mu(m)} \sum_{\substack{ad=m \\ \gcd(a, M)=1 \\ b(d)}} d^{-\kappa} \\ &\quad \times \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2(4m)}} c(D, r) e\left(\left(\frac{r^2 - D}{4m} \frac{a}{d} + \frac{\lambda r}{d} + \lambda^2\right)\tau\right) \\ &\quad \cdot e\left(\left(\frac{r}{d} + 2\lambda\right)z + \frac{r^2 - D}{4m} \frac{b}{d} + \frac{r\mu}{d}\right). \end{aligned}$$

Note that

$$\sum_{\substack{\mu(m) \\ b(d)}} e\left(\frac{r^2 - D}{4m} \frac{b}{d} + \frac{r\mu}{d}\right) = \begin{cases} md & \text{if } d \mid r \text{ and } d \mid (r^2 - D)/(4m), \\ 0 & \text{otherwise.} \end{cases}$$

Setting  $r_1 = dr$  and  $D_1 = d^2D$ , we obtain that

$$\begin{aligned} V_m^*\psi(\tau, z) &= m^{\kappa-2} \sum_{\lambda(m)} \sum_{\substack{d \mid m \\ \gcd(\frac{m}{d}, M)=1}} d^{1-\kappa} \sum_{\substack{D_1 < 0, r_1 \in \mathbb{Z} \\ D_1 \equiv r_1^2(4m/d)}} c(d^2D_1, dr_1) \\ &\quad \cdot e\left(\frac{(r_1 + 2\lambda)^2 - D_1}{4}\tau + (r_1 + 2\lambda)z\right). \end{aligned}$$

Letting  $r_2 = r_1 + 2\lambda$ , we have that

$$\begin{aligned} V_m^*\psi(\tau, z) &= m^{\kappa-2} \sum_{\lambda(m)} \sum_{\substack{d \mid m \\ \gcd(\frac{m}{d}, M)=1}} d^{1-\kappa} \sum_{\substack{D_1 < 0, r_2 \in \mathbb{Z} \\ D_1 \equiv (r_2 - 2\lambda)^2(4m/d)}} c(d^2D_1, d(r_2 - 2\lambda)) \\ &\quad \cdot e\left(\frac{r_2^2 - D_1}{4}\tau + r_2z\right). \end{aligned}$$

We can write

$$\lambda \equiv s + \frac{m}{d}s' \pmod{m},$$

where  $s$  runs over  $\mathbb{Z}/(m/d)\mathbb{Z}$  and  $s'$  runs over  $\mathbb{Z}/d\mathbb{Z}$ . We immediately obtain

$$d(r_2 - 2\lambda) \equiv d(r_2 - 2s) \pmod{2m}, \quad D_1 \equiv (r_2 - 2s)^2 \pmod{4m/d}.$$

We now use the fact that the coefficients  $c(D, r)$  depend only on the pair  $(D, r)$  with  $r \pmod{2m}$  and  $D \equiv r^2 \pmod{4m}$  to write



$$V_m^* \psi(\tau, z) = m^{\kappa-2} \sum_{\substack{d|m \\ \gcd(\frac{m}{d}, M)=1}} d^{2-\kappa} \sum_{s(m/d)} \sum_{\substack{D_1 < 0, r_1 \in \mathbb{Z} \\ D_1 \equiv (r_2 - 2s)^2 \pmod{4m/d}}} c(d^2 D_1, d(r_2 - 2s)) \\ \cdot e\left(\frac{r_2^2 - D_1}{4} \tau + r_2 z\right).$$

Finally, we change variables and replace  $D_1$  by  $D$ ,  $d$  by  $m/d$ , and  $r_2 - 2s$  by  $s$  to obtain that

$$V_m^* \psi(\tau, z) = \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4}}} \left( \sum_{\substack{d|m \\ \gcd(d, M)=1}} d^{\kappa-2} \sum_{s \in S(r, d, D)} c\left(\frac{m^2}{d^2} D, \frac{m}{d} s\right) \right) \\ \cdot e\left(\frac{r^2 - D}{4} \tau + rz\right),$$

where  $S(r, d, D) = \{s \pmod{2d} : s \equiv r \pmod{2}, s^2 \equiv D \pmod{4d}\}$ . □

**PROPOSITION 4.6**

The map  $V_m^* V_m : J_{\kappa,1}^c(\Gamma_0(M)^J) \rightarrow J_{\kappa,1}^c(\Gamma_0(M)^J)$  is given by

$$V_m^* V_m = \sum_{\substack{d|m \\ \gcd(d, M)=1}} \varsigma(d) d^{\kappa-2} T_J\left(\frac{m}{d}\right),$$

where  $T_J(n)$  is the  $n$ th Hecke operator on  $J_{\kappa,1}^c(\Gamma_0(M)^J)$  (we write  $U_J(p)$  for  $T_J(p)$  if  $p \mid M$ ) and

$$\varsigma(d) = d \prod_{p|d} \left(1 + \frac{1}{p}\right).$$

*Proof*

The proof of this proposition follows along the same lines as the proof of the analogous result in the  $M = 1$  case given in [14]. Note that it is enough to check this fact on Fourier coefficients indexed by fundamental discriminants, as is pointed out in [14]. We show this for a representative case that is not too computationally cumbersome, but leave the proof of the general case to the reader.

Let  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ , and put  $\psi = V_m \phi$ ,  $\varphi = V_m^* \psi$ . Write  $c_\phi$  for the Fourier coefficients of  $\phi$ , write  $c_\psi$  for the Fourier coefficients of  $\psi$ , and write  $c_\varphi$  for the Fourier coefficients of  $\varphi$ .

We begin by recalling the definition of the action of  $T_J(p)$  and  $U_J(p)$  on the Fourier coefficients indexed by fundamental discriminants. Let  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$  with

$$\phi(\tau, z) = \sum_{\substack{D < 0, r \in \mathbb{Z} \\ D \equiv r^2 \pmod{4}}} c_\phi(D, r) e\left(\frac{r^2 - D}{4} \tau + rz\right).$$

Then

$$c_{U_J(p)\phi}(D, r) = c_\phi(p^2 D, pr)$$

and

$$c_{T_J(p)\phi}(D, r) = c_\phi(p^2D, pr) + \chi_D(p)p^{\kappa-2}c_\phi(D, r),$$

where  $\chi_D$  is the quadratic character associated to the fundamental discriminant  $D$ .

Consider the case where  $m = pq$  with  $p \mid M, q \nmid M$ . In this case

$$\begin{aligned} \sum_{\substack{d \mid m \\ \gcd(d, M) = 1}} \zeta(d)d^{\kappa-2}T_J\left(\frac{m}{d}\right) &= T_J(m) + q^{\kappa-2}(q+1)U_J(p) \\ &= T_J(q)U_J(p) + q^{\kappa-2}(q+1)U_J(p). \end{aligned}$$

We have that the  $(D, r)$ th Fourier coefficient of  $(T_J(m) + q^{\kappa-2}(q+1)U_J(p))\phi$  is given by

$$c_\phi(m^2D, mr) + q^{\kappa-2}(1+q+\chi_D(q))c_\phi(p^2D, pr).$$

We now calculate the  $(D, r)$ th Fourier coefficient of  $\varphi$ . Observe that

$$c_\psi(D, r) = c_\phi(D, r) + q^{\kappa-1}c_\phi\left(\frac{D}{q^2}, \frac{r}{q}\right).$$

Using the above formula for  $V_m^*$  on the Fourier coefficients we obtain that

$$\begin{aligned} c_\varphi(D, r) &= \sum_{\substack{d \mid m \\ \gcd(d, M) = 1}} d^{\kappa-2} \sum_{s \in S(r, d, D)} c_\psi\left(\frac{m^2}{d^2}D, \frac{m}{d}s\right) \\ &= \sum_{s \in S(r, 1, D)} c_\psi(m^2D, ms) + q^{\kappa-2} \sum_{s \in S(r, q, D)} c_\psi(p^2D, ps). \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{s \in S(r, 1, D)} c_\psi(m^2D, ms) &= c_\psi(m^2D, mr) \\ &= c_\phi(m^2D, mr) + q^{\kappa-1}c_\phi(p^2D, pr). \end{aligned}$$

For the second summation we have that

$$\begin{aligned} \sum_{s \in S(r, q, D)} c_\psi(p^2D, ps) &= \sum_{s \in S(r, q, D)} \left( c_\phi(p^2D, ps) + q^{\kappa-1}c_\phi\left(\frac{p^2D}{q^2}, \frac{ps}{q}\right) \right) \\ &= \sum_{s \in S(r, q, D)} c_\phi(p^2D, ps), \end{aligned}$$

where we used that  $c_\phi(x, y) = 0$  unless  $x$  and  $y$  are both integers, and for  $p^2D/q^2$  to be an integer we must have  $q^2 \mid D$ , which cannot happen because  $D$  is assumed to be a fundamental discriminant so it cannot be divisible by the square of a prime. Hence

$$\begin{aligned} \sum_{s \in S(r, q, D)} c_\phi(p^2D, ps) &= c_\phi(p^2D, pr) \sum_{s \in S(r, q, D)} 1 \\ &= (1 + \chi_D(q))c_\phi(p^2D, pr) \end{aligned}$$

as the sum is counting whether  $D$  is a square modulo  $q$  or not. Thus, combining these, we obtain that

$$c_\varphi(D, r) = c_\phi(m^2D, mr) + q^{\kappa-2}(1 + q + \chi_D(q))c_\phi(p^2D, pr),$$

which is exactly what we were trying to prove. □

Let  $F = \mathcal{V}_M\phi$  for  $\phi \in J_{\kappa,1}^c(\Gamma_0(M)^J)$ . Then

$$D_{F,F}(s) = \zeta^M(2s - 2\kappa + 4) \sum_{m \geq 1} \langle V_m\phi, V_m\phi \rangle m^{-s}.$$

Using the previous proposition we calculate

$$\begin{aligned} \langle V_m\phi, V_m\phi \rangle &= \langle V_m^*V_m\phi, \phi \rangle \\ &= \left\langle \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}T_J\left(\frac{m}{d}\right)\phi, \phi \right\rangle \\ &= \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}\lambda_f\left(\frac{m}{d}\right)\langle \phi, \phi \rangle, \end{aligned}$$

where we recall that  $T_J(n)\phi = \lambda_f(n)\phi$ . Thus,

$$D_{F,F}(s) = \zeta^M(2s - 2\kappa + 4)\langle \phi, \phi \rangle \sum_{m \geq 1} \left( \sum_{\substack{d|m \\ \gcd(d,M)=1}} \varsigma(d)d^{\kappa-2}\lambda_f\left(\frac{m}{d}\right) \right) m^{-s}.$$

If we set

$$\begin{aligned} A(s) &= \sum_{\substack{d \geq 1 \\ \gcd(d,M)=1}} a(d)d^{-s}, \\ B(s) &= \sum_{t \geq 1} b(t)t^{-s}, \end{aligned}$$

and

$$C(s) = \sum_{m \geq 1} \left( \sum_{\substack{dt=m \\ \gcd(d,M)=1}} a(d)b(t) \right) m^{-s},$$

then

$$C(s) = A(s)B(s).$$

We can apply this with  $a(d) = \varsigma(d)d^{\kappa-2}$  and  $b(t) = \lambda_f(t)$  to obtain that

$$\begin{aligned} D_{F,F}(s) &= \zeta^M(2s - 2\kappa + 4)\langle \phi, \phi \rangle \left( \sum_{\substack{d \geq 1 \\ \gcd(d,M)=1}} \varsigma(d)d^{-s+\kappa-2} \right) \left( \sum_{t \geq 1} \lambda_f(t)t^{-s} \right) \\ &= \zeta^M(2s - 2\kappa + 4)\langle \phi, \phi \rangle L(s, f) \left( \sum_{\substack{d \geq 1 \\ \gcd(d,M)=1}} \varsigma(d)d^{-s+\kappa-2} \right). \end{aligned}$$

One can check immediately by expanding the right-hand side that

$$\sum_{\substack{d \geq 1 \\ \gcd(d, M) = 1}} \varsigma(d) d^{-s} = \frac{\zeta^M(s-1) \zeta^M(s)}{\zeta^M(2s)}.$$

Thus,

$$D_{F, F}(s) = \zeta^M(s - \kappa + 1) \zeta^M(s - \kappa + 2) \langle \phi, \phi \rangle L(s, f).$$

In particular, taking the residue of each side at  $s = \kappa$  we obtain that

$$\begin{aligned} \operatorname{res}_{s=\kappa} D_{F, F}(s) &= \operatorname{res}_{s=1} \zeta^M(s) \zeta^M(2) \langle \phi, \phi \rangle L(\kappa, f) \\ &= \frac{\pi^2}{6} \left( \prod_{p|M} (1 - p^{-1})^2 (1 + p^{-1}) \right) \langle \phi, \phi \rangle L(\kappa, f). \end{aligned}$$

We now combine this with (1), (6), and (11) to obtain the following corollary.

#### COROLLARY 4.7

Let  $\kappa \geq 2$  be an even integer, let  $M$  be an odd square-free integer, and let  $f \in S_{2\kappa-2}^{\text{new}}(\Gamma_0(M))$  be a newform. Let  $F_f \in S_\kappa(\Gamma_0^{(2)}(M))$  be the Saito–Kurokawa lift of  $f$ . Then we have that

$$\langle F_f, F_f \rangle = \mathcal{A}_{\kappa, M} \frac{|a_{\theta_{\kappa, D}^{\text{alg}}(f)}(|D|)|^2}{|D|^{\kappa-3/2}} \frac{L(\kappa, f)}{\pi L(\kappa-1, f, \chi_D)} \langle f, f \rangle,$$

where

$$\mathcal{A}_{\kappa, M} = \frac{M^\kappa \zeta_M(4) \zeta_M(1)^2 (\kappa-1) (\prod_{p|M} (1+p^2)(1+p^{-1}))}{2^{\nu(M)+3} [\Gamma_0(M) : \Gamma_0(4M)] [\text{Sp}_4(\mathbb{Z}) : \Gamma_0^{(2)}(M)]}.$$

One should note that one can give a similar expression for general odd  $M$  that depends upon the function  $h(p_1, \dots, p_n)$  which shows up in the calculation of the residue of the Eisenstein series  $E_{s, M}(Z)$  above. As we will only be interested in the case of  $M$  odd and square-free, we restrict our attention to this case.

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