Linear flags and Koszul filtrations

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Abstract We show that the graded maximal ideal of a graded K-algebra R has linear quotients for a suitable choice and order of its generators if the defining ideal of R has a quadratic Gröbner basis with respect to the reverse lexicographic order, and we show that this linear quotient property for algebras defined by binomial edge ideals characterizes closed graphs. Furthermore, for algebras defined by binomial edge ideals attached to a closed graph and for join-meet rings attached to a finite distributive lattice we present explicit Koszul filtrations.

Introduction

Let K be a field. In this paper we consider standard graded K-algebras. Any such algebra is isomorphic to a K-algebra of the form S/I where $S = K[x_1, \dots, x_n]$ is the polynomial ring in the indeterminates $x_1, \dots, x_n$ and $I \subset S$ is a graded ideal with $I \subset (x_1, \dots, x_n)^2$. Let $\mathfrak{m}$ be the graded maximal ideal of S/I. The K-algebra $R = S/I$ is called Koszul if $K = R/\mathfrak{m}$ has a linear resolution. In other words, R is Koszul if the chain maps in the minimal graded free R-resolution of the residue class field of R are given by matrices whose entries are linear forms.

Obviously, the polynomial ring S itself is a Koszul algebra since the Koszul complex attached to the sequence $x_1, \dots, x_n$ provides a linear (and finite) resolution of K. For $R = S/I$ with $I \neq 0$ the graded minimal free R-resolution F of the residue class field $R/\mathfrak{m}$ is infinite, and there are examples by Roos [13] which show that F may be linear up to any given homological degree and then becomes nonlinear. Thus, it is not surprising that no algorithm for testing Koszulness is known, and in fact, there may not exist any such algorithm. It is of more interest to have some necessary conditions and also some sufficient conditions for Koszulness at hand. It is well known that I must be quadratically generated if S/I is Koszul, and that S/I is indeed Koszul if I has a quadratic Gröbner basis.

More recently, filtrations have been considered to check whether a standard graded K-algebra is Koszul. This strategy has first been applied in the paper [9] in which the authors introduced strongly Koszul algebras which are defined via sequential conditions. Inspired by this concept, Conca, Trung, and Valla [3] intro-

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duced the more flexible notion of Koszul filtration, which is defined as follows. Let R be a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. A *Koszul filtration* of R is a family $\mathcal{F}$ of ideals generated by linear forms with the property that $\mathfrak{m} \in \mathcal{F}$ and that for each $I \in \mathcal{F}$ with $I \neq 0$ there exists $J \in \mathcal{F}$ with $J \subset I$ such that I/J is a cyclic module whose annihilator belongs to $\mathcal{F}$. It is shown in [3, Proposition 1.2] that all the ideals I belonging to a Koszul filtration have a linear resolution. In particular, any standard graded K-algebra admitting a Koszul filtration is Koszul. It has been shown by an example on [2, p. 101] that not each Koszul algebra has a Koszul filtration.

The question is how the property of a standard graded K-algebra to admit a Koszul filtration is related to the property that its defining ideal admits a quadratic Gröbner basis. In this paper we will be mainly concerned with this question. At present it seems to us that none of these properties implies the other one. Indeed, in Section 2 we give an example (Example 2.4) of a binomial edge ideal whose residue class ring has a Koszul filtration, while in the given coordinates the ideal has no quadratic Gröbner basis for any monomial order. Other examples arise from the work of Ohsugi and Hibi [11]. On the other hand, if the Koszul filtration $\mathcal{F}$ is of a very special nature, namely, if $\mathcal{F}$ is a flag, then the defining ideal of the algebra has a quadratic Gröbner basis (see [2, Theorem 2.4]).

For the moment we do not know any example of a standard graded K-algebra which does not admit a Koszul filtration, even though its defining ideal has a quadratic Gröbner basis. As a generalization of [14, Lemma 12.1] of Sturmfels ([14, Chapter 12]) we prove in Section 1 the following result (Theorem 1.1). Let $I \subset S$ be a graded ideal which has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n$. Then, for all i, the colon ideals $(I, x_{i+1}, \ldots, x_n) : x_i$ are generated, modulo I, by linear forms. Thus, the flag of ideals $0 \subset (\bar{x}_n) \subset (\bar{x}_n, \bar{x}_{n-1}) \subset \cdots \subset (\bar{x}_n, \bar{x}_{n-1}, \ldots, \bar{x}_1)$ has the potential to belong to a Koszul filtration of S/I. Here $\bar{f}$ denotes the residue class of a polynomial $f \in S$ modulo I. We call any chain of ideals $(0) = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = (\bar{x}_1, \ldots, \bar{x}_n)$ generated by linear forms a *linear flag* if, for all j, I_{j+1}/I_j is cyclic and the annihilator of I_{j+1}/I_j is generated by linear forms. Thus, Theorem 1.1 says that if I has a quadratic Gröbner basis with respect to the reverse lexicographic order, then S/I admits a linear flag.

In general, even if I is a binomial ideal with quadratic Gröbner basis with respect to the reverse lexicographic order, the colon ideals $(\bar{x}_{i+1}, \ldots, \bar{x}_n) : \bar{x}_i$ are not generated by subsets of $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$. However, this is the case in various combinatorial contexts and, in particular, for toric ideals. This is the content of Theorem 1.3 in which we give an algebraic condition for binomial ideals which ensures that all the colon ideals under consideration are generated by variables. There we also show that, under the given conditions, the colon ideals modulo I and modulo $\mathrm{in}_<(I)$ are generated by the residue classes of the same sets of variables. This observation makes it much easier to compute these colon ideals and, in some cases, allows a combinatorial interpretation.

Suppose now that $0 \subset (\bar{x}_n) \subset (\bar{x}_n, \bar{x}_{n-1}) \subset \cdots \subset (\bar{x}_n, \bar{x}_{n-1}, \ldots, \bar{x}_1)$ is a linear flag. One may ask whether under this assumption I has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n$. In general, this is not the case. Indeed, let $R_{5,2} = K[x_1, \ldots, x_{10}]/I$ be the K-algebra generated by all squarefree monomials $t_i t_j \subset K[t_1, \ldots, t_5]$ in 5 variables with $\bar{x}_k = t_{i_k} t_{j_k}$ and such that $k < \ell$ if $t_{i_k} t_{j_k} > t_{i_\ell} t_{j_\ell}$ in the lexicographic order. Then I does not have a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $x_1 > x_2 > \cdots > x_n$. Nevertheless, the sequence $x_n, x_{n-1}, \ldots, x_1$ has linear quotients modulo I and hence defines a linear flag. Actually, it is shown in [10] that $R_{m,2}$, the algebra generated by all squarefree monomials of degree 2 in m variables, even has a Koszul filtration.

On the other hand, in Theorem 1.6 we show that if G is a finite simple graph on the vertex set $[n]$ and $J_G \subset K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is its binomial edge ideal, then G is closed, that is, J_G has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $y_1 > y_2 > \cdots > y_n > x_1 > x_2 > \cdots > x_n$ if and only if all the colon ideals $(\bar{x}_{i+1}, \ldots, \bar{x}_n) : \bar{x}_i$ have linear quotients. In Section 2 we then show in Theorem 2.1 that, for a closed graph, the linear flag $0 \subset (\bar{x}_n) \subset \cdots \subset (\bar{x}_n, \bar{x}_{n-1}, \ldots, \bar{x}_1)$ can be extended to a Koszul filtration of S/J_G. We close this section by proving in Corollary 2.6 that the family of poset ideals of a finite distributive lattice defines a Koszul filtration of the join-meet ring attached to the lattice. As a consequence one obtains that all the poset ideals generate ideals with linear resolution in the join-meet ring.

1. Gröbner bases and linear quotients

Let K be a field, and let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$. Any standard graded K-algebra R of embedding dimension n is isomorphic to S/I where I is a graded ideal with $I \subset (x_1, \ldots, x_n)^2$. Let $\mathfrak{m}$ be the graded maximal ideal of R. As explained in the introduction, a *Koszul filtration* of R is a finite set $\mathcal{F}$ of ideals generated by linear forms such that

(1) $\mathfrak{m} \in \mathcal{F}$;
(2) for any $I \in \mathcal{F}$ with $I \neq 0$, there exists $J \in \mathcal{F}$ with $J \subset I$ such that I/J is cyclic and $J : I \in \mathcal{F}$.

As shown in [3, Proposition 1.2], any $I \in \mathcal{F}$ has a linear resolution and, in particular, R is Koszul if it admits a Koszul filtration. Obviously, if $\mathcal{F}$ is a Koszul filtration, then $\mathcal{F}$ contains a flag of ideals

$$0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n = \mathfrak{m},$$

where $I_j \in \mathcal{F}$ for all j (and I_j/I_{j+1} is cyclic for all j). If it happens that for all j there exists k such that $I_{j+1} : I_j = I_k$, then $\{I_0, I_1, \ldots, I_n\}$ is a Koszul filtration. Such Koszul filtrations are called *Koszul flags*. Conca, Rossi, and Valla [2, Theorem 2.4] showed that if S/I has a Koszul flag, then I has a quadratic Gröbner basis. The following theorem is a partial converse of this result.

THEOREM 1.1

Let $I \subset S$ be a graded ideal which has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n$. Then, for all i, the colon ideals

$$(I, x_{i+1}, \ldots, x_n) : x_i$$

are generated, modulo I, by linear forms.

For the proof of the theorem we need to recall the following result from [14, Chapter 12].

LEMMA 1.2

Let $\mathcal{G}$ be the reduced Gröbner basis of the graded ideal $I \subset S$ with respect to the reverse lexicographic order induced by $x_1 > \cdots > x_n$. Then

$$\mathcal{G}' = \{f \in \mathcal{G} : x_n \nmid f\} \cup \{f/x_n : f \in \mathcal{G} \text{ and } x_n \mid f\}$$

is a Gröbner basis of $I : x_n$.

Proof of Theorem 1.1

Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be the reduced Gröbner basis of I with respect to the reverse lexicographic order, and fix $i \leq n$. Let $f_j = g_j \operatorname{mod}(x_{i+1}, \ldots, x_n)$, $f_j \in K[x_1, \ldots, x_i]$ for all j. We may assume that $\mathrm{in}_<(g_1) > \cdots > \mathrm{in}_<(g_m)$ and, therefore, that there exists an $s \leq m$ such that $f_s \neq 0$ and $f_{s+1} = \cdots = f_m = 0$. In addition, we have $\mathrm{in}_<(f_j) = \mathrm{in}_<(g_j)$ for $1 \leq j \leq s$. It then follows that $(I, x_{i+1}, \ldots, x_n) = (f_1, \ldots, f_s, x_{i+1}, \ldots, x_n)$ and the set $\mathcal{F} = \{f_1, \ldots, f_s, x_{i+1}, \ldots, x_n\}$ is a Gröbner basis since by [7, Lemma 4.3.7]

$$\mathrm{in}_<(I, x_{i+1}, \ldots, x_n) = \big(\mathrm{in}_<(I), x_{i+1}, \ldots, x_n\big).$$

Moreover, $\mathcal{F}$ is reduced since $\mathcal{G}$ is reduced. Let $J = (f_1, \ldots, f_s)$. Then

$$(I, x_{i+1}, \ldots, x_n) : x_i = (J, x_{i+1}, \ldots, x_n) : x_i = (J : x_i) + (x_{i+1}, \ldots, x_n).$$

By applying Lemma 1.2 for $J \cap K[x_1, \ldots, x_i]$, it follows that, modulo J, $(J : x_i)$ is generated by linear forms in $K[x_1, \ldots, x_i]$, which implies that $(I, x_{i+1}, \ldots, x_n)$ is also generated by linear forms modulo I. $\square$

Consider the ideal I which is generated by the binomial $x_1x_3 - x_2x_3$. Then $I : x_3 = (I, x_1 - x_2) = (x_1 - x_2)$. Thus, in general, one cannot expect that under the assumptions of Theorem 1.1 the ideals $(I, x_{i+1}, \ldots, x_n) : x_i$ are generated by a subset of the variables modulo I, even when I is a binomial ideal. Therefore some additional assumptions on the Gröbner basis of I are required to have monomial colon ideals.

For a graded ideal $J \subset S = K[x_1, \ldots, x_n]$ we denote by J_j the jth graded component of J.

THEOREM 1.3

Let $I \subset S = K[x_1, \ldots, x_n]$ be an ideal generated by quadratic binomials, and let $<$ be the reverse lexicographic order induced by $x_1 > x_2 > \cdots > x_n$. Let $f_1, \ldots, f_m$ be the degree 2 binomials of the reduced Gröbner basis of I with respect to $<$. Let $f_i = u_i - v_i$ for $i = 1, \ldots, m$, and assume that $\gcd(u_i, v_i) = 1$ for all i. Then, for all i, we have the following.

(a) $[(I, x_{i+1}, \ldots, x_n) : x_i]_1 = [(\mathrm{in}_<(I), x_{i+1}, \ldots, x_n) : x_i]_1$.

(b) *Suppose that I has a quadratic Gröbner basis with respect to $<$. Then*

$$(I, x_{i+1}, \ldots, x_n) : x_i = \big(I, x_{i+1}, \ldots, x_n, \{x_j : j \le i, x_j x_i \in \mathrm{in}_<(I)\}\big),$$

$$\big(\mathrm{in}_<(I), x_{i+1}, \ldots, x_n\big) : x_i = \big(\mathrm{in}_<(I), x_{i+1}, \ldots, x_n, \{x_j : j \le i, x_j x_i \in \mathrm{in}_<(I)\}\big).$$

Proof

(a) Let $\ell = \sum_{k=1}^n a_k x_k$ be a linear form with $\ell x_i \in (I, x_{i+1}, \ldots, x_n)$. We may assume that $a_k = 0$ for $k > i$. Let $x_j = \mathrm{in}_<(\ell)$. Then $j \le i$ and $x_j x_i \in \mathrm{in}_<(I, x_{i+1}, \ldots, x_n) = (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n)$. Therefore, there exists f_k with $\mathrm{in}_<(f_k) = x_j x_i$. Thus, if $f_k = x_j x_i - x_r x_s$, then $s \ge i$. However, since $\gcd(u_k, v_k) = 1$, we see that $s > i$. This implies that $x_j x_i \in (I, x_{i+1}, \ldots, x_n)$ and, consequently, $(\ell - a_j x_j) x_i \in (I, x_{i+1}, \ldots, x_n)$. Since $x_j \in (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n) : x_i$, induction on $\mathrm{in}_<(\ell)$ shows that $\ell \in (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n) : x_i$.

Conversely, suppose that $\ell \in (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n) : x_i$. Since $(\mathrm{in}_<(I), x_{i+1}, \ldots, x_n)$ is a monomial ideal, we may assume that ℓ is a monomial and $\ell \notin (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n)$, say, $\ell = x_j$. Then $j \le i$ and $x_j x_i \in (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n)$. Then, as before, there exists $f_k = x_j x_i - x_r x_s$ with $r \le s$ and $s > i$. It follows that $x_j x_i \in (I, x_{i+1}, \ldots, x_n)$, and hence $x_j \in (I, x_{i+1}, \ldots, x_n) : x_i$.

(b) Suppose that $\mathcal{G} = \{f_1, \ldots, f_m\}$ is the reduced Gröbner basis of I with respect to $<$. Let $J_i = (I, x_{i+1}, \ldots, x_n) : x_i$, and let

$$J_i' = \big(I, x_{i+1}, \ldots, x_n, \{x_j : j \le i, x_j x_i \in \mathrm{in}_<(I)\}\big).$$

One has that $J_i' \subset J_i$. To see why this is true, suppose that $x_j x_i \in \mathrm{in}_<(I)$ with $j \le i$. Then there is $f_k = x_j x_i - x_p x_q \in \mathcal{G}$ with $\mathrm{in}_<(f) = x_j x_i$. Since $j \le i$, it follows that either $p > i$ or $q > i$. Hence $x_p x_q \in (I, x_{i+1}, \ldots, x_n)$. Thus $x_j x_i \in (I, x_{i+1}, \ldots, x_n)$ and $x_j \in J_i$, as required.

Now, let $\mathcal{A}$ denote the set of homogeneous polynomials $f \in S$ of degree at least 1 which belong to J_i with the property that none of the monomials appearing in f belongs to J_i'. Suppose that $\mathcal{A} \ne \emptyset$. Among the polynomials belonging to $\mathcal{A}$, we choose $f \in \mathcal{A}$ such that $\mathrm{in}_<(f) \le \mathrm{in}_<(g)$ for all $g \in \mathcal{A}$. Let $u = \mathrm{in}_<(f)$. Since $x_i f \in (I, x_{i+1}, \ldots, x_n)$, one has that $x_i u \in (\mathrm{in}_<(I), x_{i+1}, \ldots, x_n)$. Since $u \notin J_i'$, it follows that $x_i u \in \mathrm{in}_<(I)$. Thus there is $f_\ell = x_p x_q - x_r x_s$ with $\mathrm{in}_<(f) = x_p x_q$ such that $x_p x_q$ divides $x_i u$. If, say, $p = i$, then x_q divides u. Thus $q \le i$. Since $x_i x_q \in \mathrm{in}_<(I)$, one has $x_q \in J_i'$. This contradicts that $u \notin J_i'$. Thus $p \ne i$, $q \ne i$, and $x_p x_q$ divides u. Let $w = (u/x_p x_q) x_r x_s$, and let $f' = f - a(u - w)$, where $a \ne 0$ is the coefficient of u in f. Since $u - w \in I$, one has $f' \in J_i$. Since $u \notin J_i'$, one has

$w \notin J'_i$. Thus $f' \in \mathcal{A}$ and $\mathrm{in}_<(f') < \mathrm{in}_<(f)$. This contradicts the choice of $f \in \mathcal{A}$. Hence $\mathcal{A} = \emptyset$ and $J_i = J'_i$, as desired.

The proof of the corresponding statement for $\mathrm{in}_<(I)$ is obvious. □

In [1] a standard graded K-algebra R is called *universally Koszul* if the set consisting of all ideals generated by linear forms is a Koszul filtration of R. In combinatorial contexts it is natural to consider standard graded K-algebras R whose set of ideals consists of all ideals which are generated by subsets of the variables as a Koszul filtration of R. We call such algebras *c-universally Koszul*. It is clear that any universally Koszul algebra or any strongly Koszul algebra is c-universally Koszul.

COROLLARY 1.4

Let $I \subset S = K[x_1, \ldots, x_n]$ be a toric ideal with the property that I has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by any given order of the variables. Then S/I is c-universally Koszul.

Proof

The binomials in a minimal set of binomial generators of a toric ideal are all irreducible, since I is a prime ideal. Hence, the conclusion follows immediately from Theorem 1.3. □

REMARK 1.5

In view of Theorems 1.1 and 1.3 one may expect that the following more general statement may be true. Let $I \subset S = K[x_1, \ldots, x_n]$ be a graded ideal. Then the following are equivalent: (a) I has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $x_1 > x_2 > \cdots > x_n$; (b) the sequence $x_n, x_{n-1}, \ldots, x_1$ has linear quotients modulo I. In general, however, (b) does not imply (a). Indeed, let $R_{5,2} = K[x_1, \ldots, x_{10}]/I$ be the K-algebra generated by all squarefree monomials $t_i t_j \subset K[t_1, \ldots, t_5]$ in 5 variables with $\bar{x}_k = t_{i_k} t_{j_k}$ and such that $k < \ell$ if $t_{i_k} t_{j_k} > t_{i_\ell} t_{j_\ell}$ in the lexicographic order. Then I does not have a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $x_1 > x_2 > \cdots > x_n$. Nevertheless, the sequence $x_n, x_{n-1}, \ldots, x_1$ has linear quotients modulo I.

Surprisingly, for any binomial edge ideal, Remarks 1.5(a) and 1.5(b) turn out to be equivalent, as will be shown in the next theorem.

Let G be a finite simple graph on the vertex set $[n]$. The *binomial edge ideal* J_G associated with G is the ideal generated by the quadrics $f_{ij} = x_i y_j - x_j y_i$ in $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\{i, j\}$ an edge of G. This class of ideals was introduced in [8] and [12].

The graph G is called *closed* with respect to the given labeling if G satisfies the following condition: whenever $\{i, j\}$ and $\{i, k\}$ are edges of G and either $i < j$, $i < k$ or $i > j$, $i > k$, then $\{j, k\}$ is also an edge of G. It is shown in [8, Theorem 1.1] that G is closed with respect to the given labeling if and only if J_G has a quadratic

Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. It is easily seen that any binomial ideal J_G has the same reduced Gröbner basis with respect to the lexicographic order induced by the natural order of the variables and with respect to the reverse lexicographic order induced by $y_1 > \cdots > y_n > x_1 > \cdots > x_n$. Therefore, G is closed with respect to the given labeling if and only if J_G has a quadratic Gröbner basis with respect to the reverse lexicographic order induced by $y_1 > \cdots > y_n > x_1 > \cdots > x_n$.

One calls a graph G closed if it is closed with respect to some labeling of its vertices. D. A. Cox and A. Erskine [4, Theorem 1.4] showed that a connected graph G is closed if and only if G is chordal, claw-free, and narrow.

We will often use the following notation. For $k \in [n]$, we let

$$N^<(k) = \{j : j < k, \{j,k\} \in E(G)\} \qquad \text{and}$$
$$N^>(k) = \{j : j > k, \{k,j\} \in E(G)\}.$$

For some of the following proofs it will be useful to note that, provided that they are nonempty, each of these sets is an interval if the graph G is closed with respect to its labeling. Indeed, let us take $i \in N^<(k)$. In particular, we have $\{i,k\} \in E(G)$. Then, as all the maximal cliques of G are intervals (see [5, Theorem 2.2]), it follows that, for any $i \le j < k$, $\{j,k\} \in E(G)$; thus, $j \in N^<(k)$. A similar argument works for $N^>(k)$.

THEOREM 1.6

Let G be a connected finite simple graph on the vertex set $[n]$. The following conditions are equivalent:

(a) *G is closed with respect to the given labeling;*
(b) *the sequence $x_n, x_{n-1}, \dots, x_1$ has linear quotients modulo J_G.*

Proof

(a) $\Rightarrow$ (b). Let G be closed with respect to the given labeling. It follows that the generators of J_G form the reduced Gröbner basis of J_G with respect to the reverse lexicographic order induced by $y_1 > \cdots > y_n > x_1 > \cdots > x_n$. Let $i \le n$. The generators of $\mathrm{in}_<(J_G)$ which are divisible by x_i are exactly $x_i y_j$ where $i < j$ and $\{i,j\} \in E(G)$. Hence, by using Theorem 1.3(b), we get that

$$(\bar{x}_n, \bar{x}_{n-1}, \dots, \bar{x}_{i+1}) : \bar{x}_i = (\bar{x}_n, \bar{x}_{n-1}, \dots, \bar{x}_{i+1}, \{\bar{y}_j : j \in N^>(i)\}). \tag{1}$$

Here $\bar{f}$ denotes the residue class for a polynomial $f \in S$ modulo J_G.

(b) $\Rightarrow$ (a). We may suppose that $\bar{x}_n, \bar{x}_{n-1}, \dots, \bar{x}_1$ has linear quotients and show that G is closed with respect to the given labeling. In fact, assume that G is not closed. Then there exist $\{i,j\}, \{i,k\} \in E(G)$ with $i < j < k$ or $i > j > k$ and such that $\{j,k\} \notin E(G)$.

Let us first consider the case in which $i < j < k$. Since

$$\bar{x}_j \bar{y}_i \bar{y}_k = \bar{x}_i \bar{y}_j \bar{y}_k = \bar{x}_k \bar{y}_i \bar{y}_j,$$

we see that $\bar{y}_i \bar{y}_k \in (\bar{x}_n, \dots, \bar{x}_{j+1}) : \bar{x}_j$.

We claim that $\bar{y}_i\bar{y}_k$ is a minimal generator of $(\bar{x}_n,\ldots,\bar{x}_{j+1}):\bar{x}_j$, contradicting the assumption that $\bar{x}_n,\bar{x}_{n-1},\ldots,\bar{x}_1$ has linear quotients. Indeed, suppose that $\bar{y}_i\bar{y}_k$ is not a minimal generator of $(\bar{x}_n,\ldots,\bar{x}_{j+1}):\bar{x}_j$; then there exist linear forms ℓ_1 and ℓ_2 in S such that $\bar{\ell}_1\bar{\ell}_2=\bar{y}_i\bar{y}_k$ and at least one of the forms $\bar{\ell}_1,\bar{\ell}_2$ belongs to $(\bar{x}_n,\ldots,\bar{x}_{j+1}):\bar{x}_j$.

Now we observe that J_G is $\mathbb{Z}^n$-graded with $\deg x_i=\deg y_i=\varepsilon_i$ for all i, where ε_i is the ith canonical unit vector of $\mathbb{Z}^n$. It follows that the $\bar{\ell}_i$'s are multihomogeneous as well with $\deg\bar{\ell}_1\bar{\ell}_2=\varepsilon_i+\varepsilon_k$, say, $\deg\bar{\ell}_1=\varepsilon_i$ and $\deg\bar{\ell}_2=\varepsilon_k$. Thus $\ell_1=ax_i+by_i$ and $\ell_2=cx_k+dy_k$ with $a,b,c,d\in K$. Let us first assume that $\bar{\ell}_1\in(\bar{x}_n,\ldots,\bar{x}_{j+1}):\bar{x}_j$. We get that

$$ax_ix_j+bx_jy_i\in(J_G,x_n,\ldots,x_{j+1}),$$

which implies that

$$\mathrm{in}_<(ax_ix_j+bx_jy_i)\in\mathrm{in}_<(J_G,x_n,\ldots,x_{j+1})=\big((\mathrm{in}_<J_G),x_n,\ldots,x_{j+1}\big).$$

Here $<$ denotes the reverse lexicographic order induced by $y_1>\cdots>y_n>x_1>\cdots>x_n$. It follows that $x_ix_j\in\mathrm{in}_<(J_G)$ or $x_jy_i\in\mathrm{in}_<(J_G)$, which is impossible since the generators of degree 2 of $\mathrm{in}_<(J_G)$ are of the form x_ky_ℓ with $\{k,\ell\}\in E(G)$ and $k<\ell$.

Let us consider now that $\bar{\ell}_2\in(\bar{x}_n,\ldots,\bar{x}_{j+1}):\bar{x}_j$. We get $cx_kx_j+dx_jy_k\in(J_G,x_n,\ldots,x_{j+1})$. If $d\neq 0$, then we obtain $x_jy_k\in(J_G,x_n,\ldots,x_{j+1})$ and, therefore, $x_jy_k\in(\mathrm{in}_<(J_G),x_n,\ldots,x_{j+1})$, which implies that $x_jy_k\in\mathrm{in}_<(J_G)$, a contradiction since $\{j,k\}\notin E(G)$ by assumption. Therefore, we must have $\ell_2=cx_k$ for some $c\in K\setminus\{0\}$. The equation $\bar{\ell}_1\bar{\ell}_2=\bar{y}_i\bar{y}_k$ implies that $cx_k(ax_i+by_i)-y_iy_k\in J_G$. It follows that one of the monomials x_ix_k,x_ky_i,y_iy_k belongs to $\mathrm{in}_<(J_G)$, a contradiction.

Finally, we consider the case in which $i>j>k$. Then $x_if_{jk}\in J_G$, and so $\bar{f}_{jk}\in(\bar{x}_n,\ldots,\bar{x}_{i+1}):\bar{x}_i$. By similar arguments as above, we show that $\bar{f}_{jk}$ is a minimal generator of $(\bar{x}_n,\ldots,\bar{x}_{i+1}):\bar{x}_i$. Suppose that there exist linear forms $\ell_1=ax_j+by_j$ and $\ell_2=cx_k+dy_k$ such that $g=f_{jk}-\ell_1\ell_2\in J_G$. Since no monomial in the support of g belongs to $\mathrm{in}_<(G)$ (with the monomial order as in the previous paragraph), it follows that $g\notin J_G$, a contradiction. Hence, we see that $(\bar{x}_n,\ldots,\bar{x}_{i+1}):\bar{x}_i$ is not generated by linear forms. $\square$

2. Classes of ideals with Koszul filtration

In this section we present two large classes of K-algebras which admit Koszul filtrations. In both cases their defining ideal also admits a quadratic Gröbner basis.

THEOREM 2.1
Let G be a closed graph. Then $R=S/J_G$ has a Koszul filtration.

For the proof of this theorem we need a preparatory result.

LEMMA 2.2

Let $0 \le k \le n-1$, let $N^>(k) = \{k+1, \dots, \ell\}$ for some $\ell \ge k+1$, and let $N^<(k+1) = \{i, i+1, \dots, k\}$ for some $i \le k$. Then:

(a) $(J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell) : y_{k+1} = (J_G, x_n, \dots, x_{k+1}, x_k, \dots, x_i, y_{k+2}, \dots, y_\ell)$,

(b) *for* $k+2 \le s \le \ell$, y_s *is regular on* $(J_G, x_n, \dots, x_i, y_{s+1}, \dots, y_\ell)$.

Proof

(a) Let $r \in N^<(k+1)$. Then $x_r y_{k+1} = (x_r y_{k+1} - x_{k+1} y_r) + x_{k+1} y_r \in (J_G, x_n, \dots, x_{k+1})$. This shows the inclusion $\supseteq$.

For the other inclusion, let $f \in S$ be such that $f y_{k+1} \in (J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell)$. If H is the restriction of G to the set $[k]$, then $(J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell) = (J_H, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell, \{x_r y_j : r \le k < j, \{r, j\} \in E(G)\})$. Let us observe that if $\{r, j\} \in E(G)$ with $r \le k < j$, then, as G is closed, we have $\{k, j\} \in E(G)$; thus, $j \in \{k+1, \dots, \ell\}$. Therefore, we get that

$$\begin{aligned}&(J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell)\\ &\quad = (J_H, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell, x_i y_{k+1}, \dots, x_k y_{k+1}).\end{aligned}$$

By inspecting the S-polynomials of the generators on the right-hand side of the above equality of ideals, it follows that

$$\begin{aligned}&\mathrm{in}_<(J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell)\\ &\quad = \big(\mathrm{in}_<(J_H), x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell, x_i y_{k+1}, \dots, x_k y_{k+1}\big).\end{aligned}$$

Here $<$ denotes the lexicographic order on $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ induced by the natural order of the variables.

It follows that $\mathrm{in}_<(f) y_{k+1} \in (\mathrm{in}_<(J_H), x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell, x_i y_{k+1}, \dots, x_k y_{k+1})$, which implies that $\mathrm{in}_<(f) \in (\mathrm{in}_<(J_H), x_n, \dots, x_{k+1}, x_k, \dots, x_i, y_{k+2}, \dots, y_\ell)$. Hence, either $\mathrm{in}_<(f) \in (x_n, \dots, x_{k+1}, x_k, \dots, x_i, y_{k+2}, \dots, y_\ell)$ or $\mathrm{in}_<(f) \in \mathrm{in}_<(J_H)$. In both cases we may proceed by induction on $\mathrm{in}_<(f)$. In the first case, let a be the coefficient of $\mathrm{in}_<(f)$ in f. Then $g = f - a\,\mathrm{in}_<(f)$ has $\mathrm{in}_<(g) < \mathrm{in}_<(f)$ and $g y_{k+1} \in (J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell)$. In the second case, let $h \in J_H$ and $c \in K \setminus \{0\}$ be such that $\mathrm{in}_<(h - cf) < \mathrm{in}_<(f)$. Thus, if $g = h - cf$, then it follows that $g y_{k+1} \in (J_G, x_n, \dots, x_{k+1}, y_{k+2}, \dots, y_\ell)$ as well.

(b) Let $k+2 \le s \le \ell$. It is enough to show that y_s is regular on the initial ideal of $(J_G, x_n, \dots, x_i, y_{s+1}, \dots, y_\ell)$. If H is the restriction of G to the set $[i]$, then we get that

$$\begin{aligned}&\mathrm{in}_<(J_G, x_n, \dots, x_i, y_{s+1}, \dots, y_\ell)\\ &\quad = \mathrm{in}_<\big(J_H, x_n, \dots, x_i, y_{s+1}, \dots, y_\ell, \{x_r y_j : r < i < j, \{r, j\} \in E(G)\}\big)\\ &\quad = \big(\mathrm{in}_<(J_H), x_n, \dots, x_i, y_{s+1}, \dots, y_\ell, \{x_r y_j : r < i < j, \{r, j\} \in E(G)\}\big).\end{aligned}$$

The last equality from above may be easily checked by observing that the S-polynomials $S(f_{r\ell}, x_r y_j)$ reduce to 0 for any $r < \ell \le i < j$ with $\{r, \ell\} \in E(H)$. We claim that y_s does not divide any of the generators of

$$\big(\mathrm{in}_<(J_H), x_n, \ldots, x_i, y_{s+1}, \ldots, y_\ell, \{x_r y_j : r < i < j, \{r,j\} \in E(G)\}\big).$$

Obviously, y_s does not divide any of the generators of $\mathrm{in}_<(J_H)$. Next, if $\{r,s\} \in E(G)$ for some $r < i < k+1 < s$, then, as G is closed, we get $\{r, k+1\} \in E(G)$, contradicting the fact that $i = \min N^<(k+1)$. This shows that none of the generators $x_r y_j$ is divisible by y_s. $\square$

Proof of Theorem 2.1

Let G be closed with respect to its labeling. We set $\bar{f}$ for $f \bmod (J_G) \in R = S/J_G$. For $k \in [n-1]$, let $N^>(k) = \{k+1, \ldots, \ell_k\}$ and $N^<(k+1) = \{i_k, i_k+1, \ldots, k\}$.

Let us consider the following families of ideals:

$$\mathcal{F}_1 = \{(\bar{x}_n, \ldots, \bar{x}_1, \bar{y}_n, \ldots, \bar{y}_k) : 1 \le k \le n\} \cup \{(\bar{x}_n, \ldots, \bar{x}_k) : 1 \le k \le n\},$$

$$\mathcal{F}_2 = \bigcup_{k=1}^{n-1} \{(\bar{x}_n, \ldots, \bar{x}_{k+1}, \bar{y}_{k+1}, \ldots, \bar{y}_{\ell_k}), (\bar{x}_n, \ldots, \bar{x}_{k+1}, \bar{y}_{k+2}, \ldots, \bar{y}_{\ell_k})\},$$

and

$$\mathcal{F}_3 = \bigcup_{k=1}^{n-1} \{(\bar{x}_n, \ldots, \bar{x}_{i_k}, \bar{y}_s, \ldots, \bar{y}_{\ell_k}) : k+2 \le s \le \ell_k\}.$$

We claim that the family $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \{(0)\}$ is a Koszul filtration of R. We have to check that, for every $I \in \mathcal{F}$, there exists $J \in \mathcal{F}$ such that I/J is cyclic and $J : I \in \mathcal{F}$.

Let us consider $I = (\bar{x}_n, \ldots, \bar{x}_1, \bar{y}_n, \ldots, \bar{y}_k) \in \mathcal{F}_1$. Then, for $J = (\bar{x}_n, \ldots, \bar{x}_1, \bar{y}_n, \ldots, \bar{y}_{k+1}) \in \mathcal{F}_1$, we have $J : I = J$ since $\bar{y}_k$ is obviously regular on R/J.

For $I = (\bar{x}_n, \ldots, \bar{x}_k) \in \mathcal{F}_1$ with $1 \le k \le n-1$, we take $J = (\bar{x}_n, \ldots, \bar{x}_{k+1}) \in \mathcal{F}_1$. Then, by (1), we get $J : I = (\bar{x}_n, \ldots, \bar{x}_{k+1}, \bar{y}_{k+1}, \ldots, \bar{y}_{\ell_k}) \in \mathcal{F}_2$. In addition, for $I = (\bar{x}_n)$, we have $(0) : I = (0)$ since $\bar{x}_n$ is regular on R.

Let us now choose $I \in \mathcal{F}_2$, $I = (\bar{x}_n, \ldots, \bar{x}_{k+1}, \bar{y}_{k+1}, \ldots, \bar{y}_{\ell_k})$ for some $1 \le k \le n-1$. Then, $J = (\bar{x}_n, \ldots, \bar{x}_{k+1}, \bar{y}_{k+2}, \ldots, \bar{y}_{\ell_k}) \in \mathcal{F}_2$ and, by Lemma 2.2(a), we have $J : I = (\bar{x}_n, \ldots, \bar{x}_{i_k}, \bar{y}_{k+2}, \ldots, \bar{y}_{\ell_k}) \in \mathcal{F}_3$.

Finally, if $I \in \mathcal{F}_3$, $I = (\bar{x}_n, \ldots, \bar{x}_{i_k}, \bar{y}_s, \ldots, \bar{y}_{\ell_k})$ for some $k+2 \le s \le \ell_k$, then we take $J = (\bar{x}_n, \ldots, \bar{x}_{i_k}, \bar{y}_{s+1}, \ldots, \bar{y}_{\ell_k}) \in \mathcal{F}_3$. By Lemma 2.2(b), we get that $J : I = J$ since $\bar{y}_s$ is regular on R/J. $\square$

One may ask for which binomial edge ideals J_G is the K-algebra S/J_G c-universally Koszul? The following result answers this question.

PROPOSITION 2.3

Let G be a finite simple graph. Then S/J_G is c-universally Koszul if and only if G is a complete graph.

Proof

In [9, Example 1.6] it has been shown that S/J_G is strongly Koszul if G is a complete graph. In particular, G is c-universally Koszul. For the converse implication,

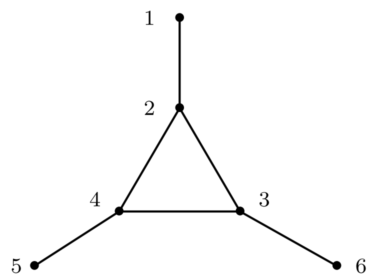

Figure 1.

assume that G is not complete. Then there exist edges $\{i,j\}$ and $\{i,k\}$ of G such that $\{j,k\} \notin E(G)$. This implies that $x_j y_k - x_k y_j \notin J_G$. On the other hand, $x_j y_k - x_k y_j \in J_G : x_i$, because $x_i(x_j y_k - x_k y_j) = x_j(x_i y_k - x_k y_i) - x_k(x_i y_j - x_j y_i)$. $\square$

The following example shows that the converse of Theorem 2.1 is not true. In other words, there exist Koszul nonclosed graphs G such that $R = S/J_G$ has a Koszul filtration.

EXAMPLE 2.4

Let G be the graph given in Figure 1. Then G is not closed, but it is Koszul (see [6]).

The ring $R = K[x_1,\ldots,x_6,y_1,\ldots,y_6]/J_G$ possesses the following Koszul filtration:

(0), (y_6), (y_6,x_6),
(y_6,y_3), (y_6,x_6,x_5), (y_6,x_6,y_5,x_5),
(y_6,x_6,x_5,x_4), (y_6,y_4,x_6,x_5,x_4), (y_6,x_6,x_5,x_4,x_3),
$(y_6,x_6,x_5,x_4,x_3,x_2)$, $(y_6,y_4,x_6,x_5,x_4,x_3)$, $(y_6,y_4,y_3,x_6,x_5,x_4,x_3)$,
$(y_6,x_6,x_5,\ldots,x_1)$, $(y_6,y_2,x_6,x_5,\ldots,x_2)$, $(y_6,y_4,x_6,x_5,\ldots,x_2)$,
$(y_6,y_5,x_6,x_5,\ldots,x_1)$, $(y_6,y_5,y_4,x_6,x_5,\ldots,x_1)$, $(y_6,y_5,y_4,y_3,x_6,x_5,\ldots,x_1)$,
$(y_6,y_5,\ldots,y_2,x_6,x_5,\ldots,x_1)$, $(y_6,y_5,\ldots,y_1,x_6,x_5,\ldots,x_1)$.

In view of this example it would be of interest to classify all finite simple graphs for which S/J_G has a Koszul filtration.

Let K be field, and let L be a finite distributive lattice. We denote by $K[L]$ the polynomial ring over K whose variables are the elements of L, and we denote by I_L the binomial ideal in $K[L]$ generated by the binomials $ab - (a \wedge b)(a \vee b)$ with $a, b \in L$ incomparable. The ideal I_L is called the *join-meet ideal*, and $A(L) = K[L]/I_L$ is called the *join-meet ring* of L (also known as the Hibi ring). The residue class $f + I_L \in A(L)$ of a polynomial $f \in K[L]$ will be denoted by $\bar{f}$. Let $I \subset L$ be a poset ideal of L. We denote by $\bar{I}$ the ideal in $A(L)$ generated by the elements $\bar{a}$ with $a \in I$.

THEOREM 2.5

Let $I \subset J$ be poset ideals of L with $J \setminus I = \{a\}$. Then $\bar{I} : \bar{J} = \bar{H}$, where H is the poset ideal $\{b \in L : b \not\geq a\}$.

Proof
Let $b \in L$ with $b \not> a$. Then $\bar{a}\bar{b} = (\bar{a} \wedge \bar{b})(\bar{a} \vee \bar{b}) \in \bar{I}$, since $a \wedge b < a$. This shows that $\bar{H} \subset \bar{I} : (\bar{a}) = \bar{I} : \bar{J}$. In order to prove the converse inclusion, we let $\bar{f} \in \bar{I} : (\bar{a})$ and may assume that $\bar{f} \notin \bar{I}$. Thus, $af \in (I_L, I)$ and $f \notin (I_L, I)$. Now we choose a total order $\prec$ of the variables such that $a \prec b$ if $a > b$ in L and such that $a \prec b$ if $a \notin I$ and $b \in I$, and we denote again by $\prec$ the reverse lexicographic order induced by $\prec$. Then $\mathrm{in}_\prec(I_L, I) = (\mathrm{in}_\prec(I_L), I)$, and it follows that $a\,\mathrm{in}_\prec(f) \in (\mathrm{in}_\prec(I_L), I)$. By a classical result of Hibi (see [7, Theorem 10.1.3], $\mathrm{in}_\prec(I_L)$ is generated by all monomials bc with $b, c \in L$ incomparable. Thus,

$$\big(\mathrm{in}_\prec(I_L), I\big) = \big(\{bc : b, c \in L \setminus I \text{ with } b, c \text{ incomparable}\}, I\big).$$

Since $f \notin (I_L, I)$, we may assume that f is in standard form with respect to (I_L, I) and $\prec$. In other words, we may assume that no monomial in the support of f belongs to $(\mathrm{in}_\prec(I_L), I)$. On the other hand, since $a\,\mathrm{in}(f) \in (\mathrm{in}_\prec(I_L), I)$ it follows that one of the generating monomials of $(\mathrm{in}_\prec(I_L), I)$ divides $a\,\mathrm{in}_<(f)$. The only monomials among the monomial generators which can divide $a\,\mathrm{in}_\prec(f)$ must be of the form ab with a and b incomparable. Thus b divides $\mathrm{in}_\prec(f)$ and $b \in H$. Let $g = f - \lambda\,\mathrm{in}_\prec(f)$ where λ is the leading coefficient of f. Since $\bar{g} \in \bar{I} : (\bar{a})$ and $\mathrm{in}_\prec(g) \prec \mathrm{in}_\prec(f)$, induction completes the proof. □

COROLLARY 2.6
Let L be a finite distributive lattice. Then the family

$$\mathcal{F} = \{\bar{I} : I \text{ is a poset ideal of } L\}$$

of ideals is a Koszul filtration of $A(L)$. In particular, for each poset ideal I of L, the ideal $\bar{I} \subset A(L)$ has a linear resolution.

In [9] all finite distributive lattices L for which $A(L)$ is strongly Koszul are classified. Among them are the Boolean lattices. Thus, if B is a Boolean lattice, then B admits the Koszul filtration consisting of all ideals of the form $\bar{U}$ with U a subset of B, and by Corollary 2.6, B also admits the Koszul filtration consisting of all poset ideals of B. These are already two different Koszul filtrations of $A(B)$.

An *upset* in a partially ordered set P is a subset J with the property that if $x \in J$ and $y \geq x$, then $y \in J$. Since reversion of the partial order in a distributive lattice L defines again a distributive lattice, it follows from Corollary 2.6 that the collection of ideals $\bar{J}$ with $J \subset L$ an upset forms a Koszul filtration. So for any Boolean lattice we have now three different Koszul filtrations. One obtains even more Koszul filtrations by observing that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are Koszul filtrations of a standard graded K-algebra R, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a Koszul filtration of R as well. Thus, any standard graded K-algebra R which admits a Koszul filtration also admits a Koszul filtration which among all Koszul filtrations of R is *maximal* with respect to inclusion. The maximal Koszul filtration is of interest because it gives a large family of ideals of linear forms with linear resolution.

We say a Koszul filtration $\mathcal{F}$ of R is *minimal* if no proper subset of $\mathcal{F}$ is a Koszul filtration of R. In general, the Koszul filtration of $A(L)$ given in

Corollary 2.6 is not minimal. To see this, first notice that each of the poset ideals H in Theorem 2.5 is cogenerated by a single element. Thus, the following observation is immediate. Suppose that $\mathcal{I}$ is a set of poset ideals of L satisfying the following conditions:

(1) all poset ideals are cogenerated by an element of L, and L belongs to $\mathcal{I}$;
(2) for all $I \in \mathcal{I}$ there exists $J \subset I$ such that $|I \setminus J| = 1$.

Then $\mathcal{F} = \{\bar{I} : I \in \mathcal{I}\}$ is a Koszul filtration of $A(L)$.

In general a set $\mathcal{I}$ of poset ideals of L satisfying (1) and (2) may be different from the set of all poset ideals of L. For example, let B_3 be the Boolean lattice of rank 3 whose elements we may identify with the subsets of [2]. Then the set $\mathcal{F}$ consisting of all poset ideals of B_3 except the poset ideal $\{\{3\}, \emptyset\}$ satisfies (1) and (2).

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