Virtual Gorensteinness over group algebras

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Abstract Let Γ be a finite group, and let Λ be any Artin algebra. It is shown that the group algebra $\Lambda\Gamma$ is virtually Gorenstein if and only if $\Lambda\Gamma'$ is virtually Gorenstein, for all elementary abelian subgroups Γ' of Γ. We also extend this result to cover the more general context. Precisely, assume that Γ is a group in Kropholler's hierarchy $\mathbf{H}\mathfrak{F}$, Γ' is a subgroup of Γ of finite index, and R is any ring with identity. It is proved that, in certain circumstances, that $R\Gamma$ is virtually Gorenstein if and only if $R\Gamma'$ is so.

1. Introduction

Let Λ be an Artin algebra. The algebra Λ is called left (resp., right) Gorenstein if Λ viewed as a left (resp., right) module has finite injective dimension. Note that it is an open problem whether or not a left Gorenstein algebra is right Gorenstein. Λ is called Gorenstein if it is both left and right Gorenstein. The problem of understanding the Gorenstein left–right symmetry, which is referred as the Gorenstein symmetry conjecture (see [7, Conjecture 13]), provided a motivation for studying the class of virtually Gorenstein algebras which has been introduced in [13]. We recall from [12] that an algebra Λ is said to be *virtually Gorenstein* if for every Λ-module X, the functor $\mathrm{Ext}^i_\Lambda(X,-)$ vanishes for all $i>0$ on all Gorenstein injective Λ-modules if and only if $\mathrm{Ext}^i_\Lambda(-,X)$ vanishes for all $i>0$ on all Gorenstein projective Λ-modules. It is known that if Λ is virtually Gorenstein, then the Gorenstein symmetry conjecture is true for Λ (see [11, Theorem 11.4]). Virtually Gorenstein algebras provide a natural enlargement of the class of Gorenstein algebras giving at the same time a homological generalization of algebras of finite representation type and more generally of algebras of finite Cohen–Macaulay type. We would like to stress that all Artin algebras are "locally," that is, at the finitely generated level, virtually Gorenstein (see [11]). However, in [12] an example of an Artin algebra which is not virtually Gorenstein is presented. The main result of [12] provides a remarkable characterization of virtually Gorenstein algebras in terms of finitely generated modules. Precisely, it is shown in [12, Theorem 1] that Λ being virtually Gorenstein is equivalent to saying that $\mathrm{Thick}(\mathrm{proj}\,\Lambda \cup \mathrm{inj}\,\Lambda)$, the smallest thick subcategory of $\mathrm{mod}\,\Lambda$ containing $\mathrm{proj}\,\Lambda \cup \mathrm{inj}\,\Lambda$, is functorially finite (i.e., both contravariantly and covariantly

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finite). Here, $\operatorname{mod}\Lambda$ denotes the class of all finitely generated (left) Λ-modules, $\operatorname{proj}\Lambda$ (resp., $\operatorname{inj}\Lambda$) denotes the full subcategory of $\operatorname{mod}\Lambda$ which consists of all projective (resp., injective) Λ-modules. As a corollary of this result, one immediately deduces that virtual Gorensteinness is left–right symmetric (see also [11, Theorem 8.7]).

Elementary abelian subgroup induction plays a crucial role in cohomology and representation theory of finite groups (see [2], [14], [16], [17], [26]). Roughly speaking, the results say that important cohomological properties hold for a group ring $R\Gamma$, Γ finite, and R an arbitrary ring with identity, if and only if they hold for $R\Gamma'$ where Γ' runs over all elementary abelian subgroups of Γ. It is shown in [17] that if M is any module over $R\Gamma$, then it is weakly projective (projective) if and only if it is weakly projective (projective) over all subrings $R\Gamma'$ where Γ' is an elementary abelian subgroup of Γ. Moreover, it is known that if M is an arbitrary $R\Gamma$-module, Γ is finite and R is an arbitrary ring, then a given element x of the cohomology ring $\operatorname{Ext}^*_{R\Gamma}(M,M)$ (with Yoneda's product) is nilpotent if and only if its restriction to $\operatorname{Ext}^*_{R\Gamma'}(M,M)$ is nilpotent where Γ' runs over all elementary abelian subgroups of Γ (see [16], [25]). These important results exhibit the role of the elementary abelian subgroups. In this direction, we investigate virtual Gorensteinness over group algebras. Indeed, it is shown that virtual Gorensteinness over Γ can be determined by its elementary abelian subgroups. Precisely, our main result in this context is as follows.

THEOREM 1.1

Let Γ be a finite group, and let Λ be any Artin algebra. Then $\Lambda\Gamma$ is a virtually Gorenstein algebra if and only if $\Lambda\Gamma'$ is virtually Gorenstein for every elementary abelian subgroup Γ' of Γ.

Our second task in this paper is to generalize Theorem 1.1 to infinite groups. Let R be an associative ring with identity. Inspired by the definition of virtually Gorenstein algebra, we say that R is a virtually Gorenstein ring, provided $\mathrm{GP}(R)^{\perp} = {}^{\perp}\mathrm{GI}(R)$, where $\mathrm{GP}(R)$ and $\mathrm{GI}(R)$ denote the subcategories of Gorenstein projective and Gorenstein injective modules, respectively, and the symbol $^{\perp}$ refers to the Ext^1_R-orthogonal classes. We study the descent and ascent of virtual Gorensteinness between Γ and its subgroups of finite index.

Actually, we establish the following result.

THEOREM 1.2

Let Γ be an $\mathbf{H}\mathfrak{F}$-group, let Γ' be its subgroup of finite index, and let R be any ring with identity. Assume that the triple (Γ,Γ',R) satisfies Moore's condition; that is, for all $x\in(\Gamma-\Gamma')$, at least one of the following holds:

(1) *there is an integer n such that $1\neq x^n\in\Gamma'$;*
(2) $\operatorname{ord}(x)$ *is finite and invertible in R.*

Moreover, assume that any Gorenstein projective (resp., Gorenstein injective) $R\Gamma$-module is also Gorenstein projective (resp., Gorenstein injective) over R. Then, $R\Gamma$ is a virtually Gorenstein ring if and only if so is $R\Gamma'$.

Restricting the above theorem to finite groups yields the following.

COROLLARY 1.3
Let Γ be a finite group, let Γ' be its subgroup, and let Λ be any Artin algebra. Assume that the triple $(\Gamma, \Gamma', \Lambda)$ satisfies Moore's condition. Then, $\Lambda\Gamma$ is a virtually Gorenstein algebra if and only if so is $\Lambda\Gamma'$.

Throughout the paper, Γ is a group, R is an associative ring with identity, and $R\Gamma$ is the group algebra (of Γ over R); in fact, $R\Gamma$ is the ring $R \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma$. We also fix that Λ is an Artin algebra. If Γ is assumed to be finite, then $\Lambda\Gamma$ will be an Artin algebra. All modules are supposed to be left modules unless otherwise stated. Also, $\mathrm{pd}_{R\Gamma} M$ stands for the projective dimension of a module M over a group ring $R\Gamma$.

2. Preliminaries

In this section, we recall basic definitions and fundamental facts for later use.

2.1. Orthogonal classes

Let $\mathcal{X}$ be a class of objects in $\mathrm{Mod}\, R\Gamma$, the category of all left $R\Gamma$-modules. The left orthogonal of $\mathcal{X}$ in $\mathrm{Mod}\, R\Gamma$, denoted by ${}^{\perp}\mathcal{X}$, is defined by

$$ {}^{\perp}\mathcal{X} = \left\{ M \in \mathrm{Mod}\, R\Gamma \;\middle|\; \mathrm{Ext}^i_{R\Gamma}(M, X) = 0, \text{ for all } X \in \mathcal{X} \text{ and all } i > 0 \right\}. $$

The right orthogonal of $\mathcal{X}$ in $\mathrm{Mod}\, R\Gamma$ is defined similarly.

2.2.

Let Γ be an arbitrary group, and let Γ' be a subgroup of Γ. Since $R\Gamma$ is a free $R\Gamma'$-module, any projective $R\Gamma$-module is also projective over $R\Gamma'$. Consequently, for any $R\Gamma$-module M, one has the inequality $\mathrm{pd}_{R\Gamma'} M \leq \mathrm{pd}_{R\Gamma} M$. Moreover, analogous to [15, Proposition VIII.2.4(a)], one may show that the equality holds if $\mathrm{pd}_{R\Gamma} M < \infty$ and Γ' is of finite index in Γ.

2.3.

Let Γ be a group, and let Γ' be a subgroup of Γ of finite index. Let M be a left $R\Gamma'$-module. Then a verbatim pursuit of the proof of [15, Proposition III.5.9], gives rise to an isomorphism $R\Gamma \otimes_{R\Gamma'} M \cong \mathrm{Hom}_{R\Gamma'}(R\Gamma, M)$, as left $R\Gamma$-modules. One should note that the left-hand side is a left $R\Gamma$-module, since $R\Gamma$ is an $R\Gamma$-$R\Gamma'$-bimodule. However, the case for the right-hand side follows from the $R\Gamma'$-$R\Gamma$-bimodule structure of $R\Gamma$.

2.4. Gorenstein modules

An $R\Gamma$-module M is said to be Gorenstein projective if it is a syzygy of some exact sequence of projective $R\Gamma$-modules

$$\mathbf{T}_\bullet : \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow T_{-1} \longrightarrow \cdots ,$$

which remains exact after applying the functor $\operatorname{Hom}_{R\Gamma}(-, P)$, for any projective $R\Gamma$-module P. The exact sequence $\mathbf{T}_\bullet$ is called a totally acyclic complex of projectives. Gorenstein injective modules are defined dually. The class of all Gorenstein projective and Gorenstein injective $R\Gamma$-modules will be denoted by $\mathrm{GP}(R\Gamma)$ and $\mathrm{GI}(R\Gamma)$, respectively. The reader is advised to look at [21] for the basic properties of these modules.

REMARK 2.5

Gorenstein projective modules, which are a refinement of projective modules, were defined by Enochs and Jenda [20]. This concept even goes back to Auslander and Bridger [5], who introduced the G-dimension of a finitely generated module M over a two-sided Noetherian ring; then Avramov, Martisinkovsky, and Reiten proved that M is Goreinstein projective if and only if the G-dimension of M is zero (see also the remark following [18, Theorem 4.2.6] for the historical information).

EXAMPLE 2.6

The following are examples of Gorenstein projective and injective modules.

(i) Every projective (resp., injective) $R\Gamma$-module is Gorenstein projective (resp., Gorenstein injective).

(ii) Let Γ' be a subgroup of Γ, and let M be a Gorenstein projective (resp., Gorenstein injective) $R\Gamma'$-module. Then $R\Gamma \otimes_{R\Gamma'} M$ (resp., $\operatorname{Hom}_{R\Gamma'}(R\Gamma, M)$) is a Gorenstein projective (resp., Gorenstein injective) $R\Gamma$-module.

(iii) Let Γ be a finite group, and let M be an $R\Gamma$-module. Then M is Gorenstein projective (resp., Gorenstein injective) if and only if it is Gorenstein projective (resp., Gorenstein injective) as an R-module. In particular, R is a Gorenstein projective $R\Gamma$-module.

Proof

(i) The result follows from the definition.

(ii) To prove the assertion for Gorenstein projective module, one only needs to apply the same arguments which have been used in [8, Example 2.1(c)]. The case for Gorenstein injective modules can be obtained dually.

(iii) This is a direct consequence of [9, Theorem 2.9]. □

REMARK 2.7

Let Γ be a group, and let Γ' be an arbitrary subgroup of Γ. Then by using the adjointness of Hom and $\otimes$ in conjunction with Example 2.6(ii), one may deduce that $\mathrm{GP}(R\Gamma)^{\perp} \subseteq \mathrm{GP}(R\Gamma')^{\perp}$ and ${}^{\perp}\mathrm{GI}(R\Gamma) \subseteq {}^{\perp}\mathrm{GI}(R\Gamma')$.

3. Results and proofs

We begin this section by the following result, which says that virtual Gorensteinness descends from a group Γ to its subgroups of finite index.

PROPOSITION 3.1

Let Γ be a group, and let Γ' be a subgroup of Γ of finite index. If $R\Gamma$ is a virtually Gorenstein ring, then so is $R\Gamma'$.

Proof

We only show the inclusion $\mathrm{GP}(R\Gamma')^{\perp} \subseteq {}^{\perp}\mathrm{GI}(R\Gamma')$, since the reverse inclusion is proved similarly. Assume that M is an $R\Gamma'$-module which belongs to $\mathrm{GP}(R\Gamma')^{\perp}$. It should be noted that, by virtue of [15, Section III 3.6], we infer that the functor $\mathrm{Hom}_{R\Gamma'}(R\Gamma, -)$ is right adjoint to the restriction functor, and according to the isomorphism given in Section 2.3, it takes projective $R\Gamma'$-modules to projective $R\Gamma$-modules. These facts indeed yield that every Gorenstein projective $R\Gamma$-module is also Gorenstein projective over $R\Gamma'$. Now take an arbitrary Gorenstein projective $R\Gamma$-module X. Using the adjoitness of Hom and $\otimes$ together with Section 2.3 gives rise to the following isomorphisms:

$$\mathrm{Ext}^i_{R\Gamma}(X, R\Gamma \otimes_{R\Gamma'} M) \cong \mathrm{Ext}^i_{R\Gamma'}(R\Gamma \otimes_{R\Gamma} X, M) \cong \mathrm{Ext}^i_{R\Gamma'}(X, M),$$

implying that $R\Gamma \otimes_{R\Gamma'} M \in \mathrm{GP}(R\Gamma)^{\perp}$. Hence, another use of the isomorphism presented in Section 2.3, combining with the hypothesis made on $R\Gamma$, induces that $\mathrm{Hom}_{R\Gamma'}(R\Gamma, M) \in {}^{\perp}\mathrm{GI}(R\Gamma)$. Consequently, by virtue of Remark 2.7, we deduce that this module belongs to ${}^{\perp}\mathrm{GI}(R\Gamma')$ as well. This, in turn, implies that $M \in {}^{\perp}\mathrm{GI}(R\Gamma')$, since M is an $R\Gamma'$-direct summand of $\mathrm{Hom}_{R\Gamma'}(R\Gamma, M)$ (see [15, Section III 3.7]). The proof then is complete. □

LEMMA 3.2

Let Γ be a group, and let Γ' be its subgroup of finite index such that the index of Γ' in Γ is invertible in R. Then $R\Gamma$ is virtually Gorenstein if and only if $R\Gamma'$ is so.

Proof

According to Proposition 3.1, we only need to show the "if" part. Assuming $M \in \mathrm{GP}(R\Gamma)^{\perp}$, Remark 2.7 yields that it also belongs to $\mathrm{GP}(R\Gamma')^{\perp}$. Hence, by the assumption, $M \in {}^{\perp}\mathrm{GI}(R\Gamma')$, and so $\mathrm{Hom}_{R\Gamma'}(R\Gamma, M) \in {}^{\perp}\mathrm{GI}(R\Gamma)$. In addition, since the index of Γ' in Γ is invertible in R, we infer that M is an $R\Gamma$-direct summand of $\mathrm{Hom}_{R\Gamma'}(R\Gamma, M)$ ensuring that $M \in {}^{\perp}\mathrm{GI}(R\Gamma)$, as needed. The inclusion ${}^{\perp}\mathrm{GI}(R\Gamma) \subseteq \mathrm{GP}(R\Gamma)^{\perp}$ can be proved similarly. So we are done. □

REMARK 3.3

Let Γ be a finite group, and let Λ be a Gorenstein Artin algebra. Since the functor $\mathrm{Hom}_{\Lambda}(\Lambda\Gamma, -)$ carries left (resp., right) injective Λ-modules to right (resp., left) injective $\Lambda\Gamma$-modules, applying this functor to an injective resolution of Λ

and using $\Lambda\Gamma$-isomorphisms $\mathrm{Hom}_\Lambda(\Lambda\Gamma,\Lambda)\cong\Lambda\Gamma\otimes_\Lambda\Lambda\cong\Lambda\Gamma$ yields that the Artin algebra $\Lambda\Gamma$ is Gorenstein as well. However, we do not know whether the same statement is true if one replaces Gorensteinness with virtual Gorensteinness. It follows from Lemma 3.2 that this is the case whenever the order of Γ is invertible in Λ.

Let Γ be a finite group, and let p be a prime integer dividing the order of Γ. Recall that a *Sylow p-subgroup* of Γ is a maximal p-subgroup of Γ. It is known that such subgroups exist for every prime divisor of the order of Γ, and in general, they are not unique (but any two such are conjugate). However, when Γ is abelian, the Sylow p-subgroup is unique. By a Sylow subgroup we mean a Sylow p-subgroup, for some prime integer p dividing the order of Γ.

THEOREM 3.4

Let Γ be a finite group, and let Λ be any Artin algebra. The following statements are equivalent.

(1) $\Lambda\Gamma$ *is a virtually Gorenstein algebra.*
(2) $\Lambda\Gamma'$ *is virtually Gorenstein for all Sylow subgroups* Γ' *of* Γ.

Proof

$(1\Rightarrow 2)$. This follows from Proposition 3.1.

$(2\Rightarrow 1)$. Suppose that $M\in{}^\perp\mathrm{GI}(\Lambda\Gamma)$ and X is an arbitrary Gorenstein projective $\Lambda\Gamma$-module. We must show that $\mathrm{Ext}^i_{\Lambda\Gamma}(X,M)=0$ for all $i>0$. Our assumption combining with Proposition 3.1, induces that $\mathrm{Ext}^i_\Lambda(X,M)=0$ for all $i>0$ and hence, by [17, Lemma 3.1], $\mathrm{Ext}^i_{\Lambda\Gamma}(X,M)\cong\mathrm{H}^i(\Gamma,\mathrm{Hom}_\Lambda(X,M))$. On the other hand, assuming that Γ' is an arbitrary Sylow subgroup of Γ, the hypothesis enforces $\mathrm{Ext}^i_{\Lambda\Gamma'}(X,M)=0$ for all $i>0$, since we know that X and M also belong to $\mathrm{GP}(\Lambda\Gamma')$ and ${}^\perp\mathrm{GI}(\Lambda\Gamma')$, respectively. So, $\mathrm{H}^i(\Gamma',\mathrm{Hom}_\Lambda(X,M))=0$, for all $i>0$. Hence, we may invoke [28, Corollary 9.90(iii)] and conclude that $\mathrm{H}^i(\Gamma,\mathrm{Hom}_\Lambda(X,M))=0$, implying $\mathrm{Ext}^i_{\Lambda\Gamma}(X,M)=0$ for all $i>0$, as required. Likewise, one can show the inclusion $\mathrm{GP}(\Lambda\Gamma)^\perp\subseteq{}^\perp\mathrm{GI}(\Lambda\Gamma)$. So the proof is complete. □

Let Γ be a finite group. Recall that a $\mathbb{Z}\Gamma$-module M is said to be *cohomologically trivial* provided $\hat{\mathrm{H}}^i(\Gamma',M)=0$, for all $i\in\mathbb{Z}$ and all subgroups Γ' of Γ, where $\hat{\mathrm{H}}^i(\Gamma',-)$ denotes the doubly infinite Tate cohomology of Γ'. We refer the reader to [15] and also [27] for a detailed discussion on cohomologically trivial modules.

We are now in a position to present the proof of Theorem 1.1, which is stated in the introduction.

Proof of Theorem 1.1

We only need to prove the "if" part; the "only if" part follows from Proposition 3.1. According to Theorem 3.4, we may assume that Γ is a p-group, for some prime integer p. Suppose that $M \in {}^{\perp}\mathrm{GI}(\Lambda\Gamma)$ and X is an arbitrary Gorenstein projective $\Lambda\Gamma$-module. We would like to show that $\mathrm{Ext}^i_{\Lambda\Gamma}(X,M)=0$, for all $i>0$. Let Γ' be an elementary abelian subgroup of Γ. As we have seen previously, $M \in {}^{\perp}\mathrm{GI}(\Lambda\Gamma')$ and $X \in \mathrm{GP}(\Lambda\Gamma')$. So, by the hypothesis, $\mathrm{Ext}^i_{\Lambda\Gamma'}(X,M)=0$ for all $i>0$. Consequently, in view of Proposition 3.1, one has $\mathrm{Ext}^i_{\Lambda}(X,M)=0$ for all $i>0$. Now, by invoking [17, Lemma 3.1] one may conclude that $\mathrm{Ext}^i_{\Lambda\Gamma'}(X,M) \cong \mathrm{H}^i(\Gamma',\mathrm{Hom}_{\Lambda}(X,M))=0$, for all $i>0$. In particular, $\hat{\mathrm{H}}^i(\Gamma',\mathrm{Hom}_{\Lambda}(X,M))=0$ for all $i>0$. Hence, by making use of [15, Theorem VI.8.7], we infer that $\mathrm{Hom}_{\Lambda}(X,M)$, with diagonal action, is a cohomologically trivial $\mathbb{Z}\Gamma'$-module. Thus, [15, Theorem VI.8.12] implies that $\mathrm{pd}_{\mathbb{Z}\Gamma'}(\mathrm{Hom}_{\Lambda}(X,M)) \leq 1$, for all elementary abelian subgroups Γ' of Γ. Now apply [17, Corollary 1.1] in order to conclude that $\mathrm{pd}_{\mathbb{Z}\Gamma}(\mathrm{Hom}_{\Lambda}(X,M)) < \infty$. Especially, $\hat{\mathrm{H}}^i(\Gamma,\mathrm{Hom}_{\Lambda}(X,M)) \cong \mathrm{Ext}^i_{\Lambda\Gamma}(X,M)=0$ for all $i>0$, as needed. Similarly, one can verify that $\mathrm{GP}(\Lambda\Gamma)^{\perp} \subseteq {}^{\perp}\mathrm{GI}(\Lambda\Gamma)$. Hence, $\Lambda\Gamma$ is virtually Gorenstein, as required. □

As an immediate consequence of Theorem 1.1 in conjunction with [15, Proposition VI.9.5], we include the following result.

COROLLARY 3.5

Let Γ be a finite group such that every Sylow subgroup of Γ is a cyclic or generalized quaternion group. Then $\Lambda\Gamma$ is virtually Gorenstein if and only if $\Lambda\mathbb{Z}_p$ is virtually Gorenstein for all prime integers p dividing the order of Γ.

The class $\mathbf{H}\mathfrak{F}$ was defined by Kropholler in [24] as the smallest class of groups which contains the class of finite groups, and whenever a group Γ admits a finite-dimensional contractible Γ-CW-complex with stabilizers in $\mathbf{H}\mathfrak{F}$, then Γ is in $\mathbf{H}\mathfrak{F}$. It is worth pointing out that $\mathbf{H}\mathfrak{F}$ is a very large class which is extension closed and contains all countable linear and countable soluble groups.

3.6.

Let Γ be a group, let Γ' be a subgroup of Γ, and let M be a Gorenstein projective $R\Gamma$-module. It is worth noting that, unlike the projectivity, we do not know whether or not M is Gorenstein projective as an $R\Gamma'$-module. However, as we have seen in the proof of Proposition 3.1, M is Gorenstein projective as an $R\Gamma'$-module, provided Γ' is of finite index in Γ. In the next result, we impose mild assumptions on M and Γ' ensuring the Gorenstein projectivity of M over $R\Gamma'$. We also point out that the same statement for Gorenstein injective modules can be obtained dually; hence we skip it.

PROPOSITION 3.7

Let Γ be a group, and let M be a Gorenstein projective $R\Gamma$-module. Additionally, let M be Gorenstein projective as an R-module. Then M is a Gorenstein projective $R\Gamma'$-module, for every $\mathbf{H}\mathfrak{F}$-subgroup Γ' of Γ.

Proof
First one should observe that, according to Example 2.6(iii), M is a Gorenstein projective $R\Gamma'$-module whenever Γ' is a finite group. Now, the proof actually follows from the same lines as the proof of [8, Lemma 4.4]. □

Our next result provides an example of a virtually Gorenstein ring.

PROPOSITION 3.8
Let R be a virtually Gorenstein ring and Γ be an infinite cyclic group. Let every Gorenstein projective (resp., Gorenstein injective) $R\Gamma$-module be also Gorenstein projective (resp., Gorenstein injective) over R. Then $R\Gamma$ is a virtually Gorenstein ring.

Proof
Assume that $M \in {}^{\perp}\mathrm{GI}(R\Gamma)$ and that X is an arbitrary Gorenstein projective $R\Gamma$-module. We must show that $\mathrm{Ext}^i_{R\Gamma}(X,M)=0$, for all $i>0$. For this purpose, one should note that, according to Remark 2.7, $M \in {}^{\perp}\mathrm{GI}(R)$. So, in view of the hypothesis, $\mathrm{Ext}^i_R(X,M)=0$, for all $i>0$. Consequently, [17, Lemma 3.1] gives rise to an isomorphism $\mathrm{Ext}^i_{R\Gamma}(X,M) \cong \mathrm{H}^i(\Gamma, \mathrm{Hom}_R(X,M))$ implying $\mathrm{Ext}^i_{R\Gamma}(X,M)=0$ for all $i \geq 2$, since $\mathrm{pd}_{\mathbb{Z}\Gamma}\,\mathbb{Z}=1$. Hence, it remains to show that $\mathrm{Ext}^1_{R\Gamma}(X,M)=0$. To do this, take a short exact sequence of $R\Gamma$-modules $0 \longrightarrow X \longrightarrow P \longrightarrow X' \longrightarrow 0$, where P is projective and X' is Gorenstein projective, and consequently, $\mathrm{Ext}^i_{R\Gamma}(X',M)=0$ for all $i \geq 2$. Hence, by applying the functor $\mathrm{Hom}_{R\Gamma}(-,M)$ to this sequence, one obtains the isomorphism $\mathrm{Ext}^1_{R\Gamma}(X,M) \cong \mathrm{Ext}^2_{R\Gamma}(X',M)$ implying that $\mathrm{Ext}^1_{R\Gamma}(X,M)=0$. Therefore, ${}^{\perp}\mathrm{GI}(R\Gamma) \subseteq \mathrm{GP}(R\Gamma)^{\perp}$. The inverse inclusion can be obtained similarly. So, the proof is complete. □

REMARK 3.9
Let Γ be any group, and let Γ' be its subgroup of finite index. Following Aljadeff [1], we say that the triple (Γ,Γ',R) satisfies Moore's condition if for all $x \in (\Gamma - \Gamma')$ at least one of the following holds:

(1) there is an integer n, such that $1 \neq x^n \in \Gamma'$;
(2) $\mathrm{ord}(x)$ is finite and invertible in R.

In 1976, J. Moore posed the following conjecture which concerns a criterion for modules over group rings to be projective.

Moore's conjecture (see [17]). *Let Γ be a group, let Γ' be a subgroup of Γ of finite index, and let R be any ring with identity. Assume that Moore's condition holds for the triple (Γ,Γ',R). Then every $R\Gamma$-module M which is projective over $R\Gamma'$ is projective over $R\Gamma$ as well.*

This conjecture then has been the subject of several expositions. Recently, it is shown by Aljadeff and Meir that Moore's conjecture is valid for groups which belong to Kropholler's hierarchy $\mathbf{LH}\mathfrak{F}$ (see [3]).

Proof of Theorem 1.2
In view of Proposition 3.1, we only need to show the *if* part. For this purpose, we proceed by induction on the ordinal number α such that the group Γ belongs to $\mathbf{H}_\alpha\mathfrak{F}$. If $\alpha=0$, then Γ is a finite group. Since $R\Gamma'$ is virtually Gorenstein, R is also virtually Gorenstein. Let p be a prime integer, and let Γ'' be an elementary abelian p-subgroup of Γ. If p is invertible in R, then it follows from Lemma 3.2 that $R\Gamma''$ is virtually Gorenstein. Moreover, if p is not invertible in R, then the hypothesis implies that Γ'' is contained in Γ' and hence $R\Gamma''$ is virtually Gorenstein, thanks to Proposition 3.1. Now Theorem 1.1 implies that $R\Gamma$ is virtually Gorenstein. Next assume that the result is true for $\Gamma\in\mathbf{H}_\beta\mathfrak{F}$, for all $\beta<\alpha$, and let $\Gamma\in\mathbf{H}_\alpha\mathfrak{F}$. Suppose that $M\in{}^\perp\operatorname{GI}(R\Gamma)$ and that X is a Gorenstein projective $R\Gamma$-module. We would like to show that $\operatorname{Ext}^i_{R\Gamma}(X,M)=0$, for all $i>0$. Since $\Gamma\in\mathbf{H}_\alpha\mathfrak{F}$, by the definition, there is a finite-dimensional contractible Γ-CW-complex X such that each cell stabilizer belongs to $\mathbf{H}_\beta\mathfrak{F}$ for some $\beta<\alpha$. Tensoring the cellular chain complex X by R, we obtain an exact sequence of $R\Gamma$-modules

$$0\longrightarrow C_r\longrightarrow\cdots\longrightarrow C_0\longrightarrow R\longrightarrow 0,$$

whereas for any $0\le t\le r$, C_t is a direct sum of modules of the form $R[\Gamma/H]$ for $H\in\mathbf{H}_\beta\mathfrak{F}$ with $\beta<\alpha$ (see the proof of [19, Theorem C]). Applying the functor $\operatorname{Hom}_R(-,M)$ to this sequence gives the following exact sequence of $R\Gamma$-modules;

$$0\longrightarrow M\longrightarrow\operatorname{Hom}_R(C_0,M)\longrightarrow\cdots\longrightarrow\operatorname{Hom}_R(C_r,M)\longrightarrow 0.$$

So, assuming that $C_t=\bigoplus R[\Gamma/H_{j_t}]$ for any t, one has the following $R\Gamma$-isomorphisms;

$$\begin{aligned}\operatorname{Hom}_R(C_t,M)&\cong\operatorname{Hom}_R\big(\oplus R[\Gamma/H_{j_t}],M\big)\cong\prod\operatorname{Hom}_R\big(R[\Gamma/H_{j_t}],M\big)\\&\cong\prod\operatorname{Hom}_R(R\Gamma\otimes_{RH_{j_t}}R,M)\cong\prod\operatorname{Hom}_{RH_{j_t}}\big(R\Gamma,\operatorname{Hom}_R(R,M)\big)\\&\cong\prod\operatorname{Hom}_{RH_{j_t}}(R\Gamma,M).\end{aligned}$$

By invoking induction hypothesis in conjunction with Proposition 3.7 and Remark 2.7, one may obtain that for any $i>0$,

$$\operatorname{Ext}^i_{R\Gamma}\Big(X,\prod\operatorname{Hom}_{RH_{j_t}}(R\Gamma,M)\Big)\cong\prod\operatorname{Ext}^i_{RH_{j_t}}(X,M)=0,$$

implying $\operatorname{Ext}^i_{R\Gamma}(X,M)=0$, for all $i>r$. Now, take the exact sequence of $R\Gamma$-modules,

$$0\longrightarrow X\longrightarrow P_0\longrightarrow\cdots\longrightarrow P_r\longrightarrow X'\longrightarrow 0,$$

where for each j, P_j is projective and X' is Gorenstein projective. Applying the functor $\operatorname{Hom}_{R\Gamma}(-,M)$ to this sequence gives rise to an isomorphism $\operatorname{Ext}^i_{R\Gamma}(X,M)\cong\operatorname{Ext}^{i+r}_{R\Gamma}(X',M)$, for all $i>0$. In addition, as we have seen earlier, $\operatorname{Ext}^i_{R\Gamma}(X',M)=0$, for all $i>r$ implying that $\operatorname{Ext}^i_{R\Gamma}(X,M)=0$, for all $i>0$. The inclusion $\operatorname{GP}(R\Gamma)^\perp\subseteq{}^\perp\operatorname{GI}(R\Gamma)$ holds true in a similar way. Then the proof is completed. □

As a direct consequence of Theorem 1.2, we include the following result, which is stated as Corollary 1.3 in the introduction.

COROLLARY 3.10
Let Γ be a finite group, let Γ' be its subgroup, and let Λ be any Artin algebra. Assume that the triple $(\Gamma, \Gamma', \Lambda)$ satisfies Moore's condition. Then, $\Lambda\Gamma$ is a virtually Gorenstein algebra if and only if so is $\Lambda\Gamma'$.

Recall from [12] that a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda\Gamma$ is contravariantly finite if every finitely generated $\Lambda\Gamma$-module C has a right $\mathcal{X}$-approximation $X \longrightarrow C$; that is, $X \in \mathcal{X}$ and the induced map $\operatorname{Hom}_{\Lambda\Gamma}(X', X) \longrightarrow \operatorname{Hom}_{\Lambda\Gamma}(X', C)$ is surjective for every $X' \in \mathcal{X}$. Covariantly finite subcategories are defined dually.

Combining Corollary 3.10 and [12, Theorem 1] yields the following result.

COROLLARY 3.11
Let Γ be a finite group, and let Γ' be a subgroup of Γ. Assume that $\operatorname{Thick}(\operatorname{proj}\Lambda\Gamma' \cup \operatorname{inj}\Lambda\Gamma')$ is contravariantly finite. Then $\operatorname{Thick}(\operatorname{proj}\Lambda\Gamma \cup \operatorname{inj}\Lambda\Gamma)$ is also contravariantly finite provided the triple $(\Gamma, \Gamma', \Lambda)$ satisfies Moore's condition.

REMARK 3.12
As we have mentioned in the introduction, virtually Gorenstein algebras are common generalizations of algebras of finite representation type and of finite Cohen–Macaulay type (see [11, Example 8.4]). Assuming that Γ is a finite group and Γ' is its subgroup, [7, Lemma VI 3.1] implies that finite representation type and finite Cohen–Macaulay type ascends and descends between $\Lambda\Gamma$ and $\Lambda\Gamma'$, if the index of Γ' in Γ is invertible in Λ. However, Corollary 3.10 provides a weaker criterion that guarantees ascent and descent of virtual Gorensteinness between $\Lambda\Gamma$ and $\Lambda\Gamma'$.

3.7. Finitistic dimension

For a ring R, the little finitistic dimension, $\operatorname{fin\,dim} R$, is defined as the supremum of the projective dimensions attained on the category of all finitely generated left R-modules having finite projective dimension. The big finitistic dimension, $\operatorname{Fin\,dim} R$, is defined correspondingly on the category of arbitrary left R-modules of finite projective dimension. It is well known that these dimensions may be infinite. Moreover, they do not coincide in general (see [4]). It is important to mention that there is a tie connection between the Gorensteiness of Artinian algebra Λ and the finiteness of $\operatorname{fin\,dim}\Lambda$. Precisely, a left Gorenstein algebra Λ is right Gorenstein if and only if $\operatorname{fin\,dim}\Lambda$ is finite (see [6]). We end this paper by exploring whether finiteness of finitistic dimensions carries over from $R\Gamma'$ to $R\Gamma$, and vice versa, whenever Γ' is a subgroup of finite index. We should remark that, although this result seems to be rather isolated from the main theme of the paper, we include it because of the fact that the techniques which have been used in this paper yield interesting results in this context.

PROPOSITION 3.13

Let R be any ring, let Γ be a group, and let Γ' be a subgroup of finite index. Then, one has the equalities $\operatorname{Fin}\dim R\Gamma' = \operatorname{Fin}\dim R\Gamma$ *and* $\operatorname{fin}\dim R\Gamma' = \operatorname{fin}\dim R\Gamma$.

Proof

We only prove the first equality. The proof of the second one follows the same lines. To do this, we first show that $\operatorname{Fin}\dim R\Gamma' \leq \operatorname{Fin}\dim R\Gamma$. If $\operatorname{Fin}\dim R\Gamma = \infty$, there is nothing to prove. So assume that $\operatorname{Fin}\dim R\Gamma$ is finite, say, t. Take an arbitrary $R\Gamma'$-module M with finite projective dimension. According to the $R\Gamma$-isomorphism $R\Gamma \otimes_{R\Gamma'} R\Gamma' \cong R\Gamma$, we conclude that $R\Gamma \otimes_{R\Gamma'} M$ has finite projective dimension as an $R\Gamma$-module. Hence, our assumption yields that $\operatorname{pd}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} M) \leq t$, and so Section 2.2 induces $\operatorname{pd}_{R\Gamma'}(R\Gamma \otimes_{R\Gamma'} M) \leq t$. Now M being an $R\Gamma'$-direct summand of $R\Gamma \otimes_{R\Gamma'} M$ implies that $\operatorname{pd}_{R\Gamma'} M \leq t$ and consequently, $\operatorname{Fin}\dim R\Gamma' \leq t$. Next we would like to show that $\operatorname{Fin}\dim R\Gamma \leq \operatorname{Fin}\dim R\Gamma'$. To that end, we may assume that $\operatorname{Fin}\dim R\Gamma'$ is finite. In view of Section 2.2, we have that for any $R\Gamma$-module M of finite projective dimension, the equality $\operatorname{pd}_{R\Gamma'} M = \operatorname{pd}_{R\Gamma} M$ holds true. This, in turn, deduces the claim. □

REMARK 3.14

Let Λ be an Artin algebra. In [10] Bass posed two dimension conjectures: the first one asserts that the little and big finitistic dimension of Λ are equal, that is, $\operatorname{Fin}\dim \Lambda = \operatorname{fin}\dim \Lambda$, and the second one, which is called *the finitistic dimension conjecture*, says that $\operatorname{fin}\dim \Lambda$ is finite. Because of [29], the first conjecture does not hold in general. However, the finitistic dimension conjecture is still open. Some of the known cases in which the finitistic dimension conjecture holds are the radical cubed zero case (see [22]), algebras of representation dimension at most three (see [23]), and algebras in which the category of modules of finite projective dimension is contravariantly finite in $\operatorname{mod} \Lambda$ (see [6]). It is worth pointing out that, according to Proposition 3.13, the validity of the finitistic dimension conjecture ascends and descends between Λ and $\Lambda\Gamma$, provided Γ is a finite group. More generally, the finiteness of finitistic dimension carries over from a finite group Γ to its subgroups, and vice versa. In conclusion, if $\operatorname{Fin}\dim \Lambda = \operatorname{fin}\dim \Lambda$, then the same is true for $\Lambda\Gamma$. Precisely, the validity of the first conjecture stated above carries over from Λ to $\Lambda\Gamma$.

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