On the cofiniteness of generalized local cohomology modules

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Abstract Let R be a commutative Noetherian ring, let I be an ideal of R, and let M, N be two finitely generated R-modules. The aim of this paper is to investigate the I-cofiniteness of generalized local cohomology modules $H_I^j(M,N) = \varinjlim_n \operatorname{Ext}_R^j(M/I^nM,N)$ of M and N with respect to I. We first prove that if I is a principal ideal, then $H_I^j(M,N)$ is I-cofinite for all M, N and all j. Secondly, let t be a nonnegative integer such that $\dim \operatorname{Supp}(H_I^j(M,N)) \leq 1$ for all $j < t$. Then $H_I^j(M,N)$ is I-cofinite for all $j < t$ and $\operatorname{Hom}(R/I, H_I^t(M,N))$ is finitely generated. Finally, we show that if $\dim(M) \leq 2$ or $\dim(N) \leq 2$, then $H_I^j(M,N)$ is I-cofinite for all j.

1. Introduction

Throughout this note the ring R is commutative Noetherian. Let N be finitely generated R-modules, and let I be an ideal of R. In [15], A. Grothendieck conjectured that if I is an ideal of R and N is a finitely generated R-module, then $\operatorname{Hom}_R(R/I, H_I^j(N))$ is finitely generated for all $j \geq 0$. R. Hartshorne provides a counterexample to this conjecture in [16]. He also defined an R-module K to be I-cofinite if $\operatorname{Supp}_R(K) \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I,K)$ is finitely generated for all $j \geq 0$, and he asked the following question.

QUESTION
For which rings R and ideals I are the modules $H_I^j(N)$ is I-cofinite for all j and all finitely generated modules N?

Hartshorne showed that if N is a finitely generated R-module, where R is a complete regular local ring, then $H_I^j(N)$ is I-cofinite in two cases:

(i) I is a principal ideal (see [16, Corollary 6.3]);
(ii) I is a prime ideal with $\dim(R/I) = 1$ (see [16, Corollary 7.7]).

K. I. Kawasaki has proved that if I is a principal ideal in a commutative Noetherian ring, then $H_I^j(N)$ are I-cofinite for all finitely generated R-modules N and all $j \geq 0$ (see [22, Theorem 1]). D. Delfino and T. Marley [12, Theorem 1] and K. I. Yoshida [33, Theorem 1.1] refined result (ii) to more general situation that if

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N is a finitely generated module over a commutative Noetherian local ring R and I is an ideal of R such that $\dim(R/I) = 1$, then $H_I^j(N)$ are I-cofinite for all $j \geq 0$. Recently, K. Bahmanpour and R. Naghipour have extended this result to the case of nonlocal rings; more precisely, they showed that if t is a nonnegative integer such that $\dim \operatorname{Supp}(H_I^j(N)) \leq 1$ for all $j < t$, then $H_I^0(N), H_I^1(N), \dots, H_I^{t-1}(N)$ are I-cofinite and $\operatorname{Hom}(R/I, H_I^t(N))$ is finitely generated (see [2, Theorem 2.6]).

There are some generalizations of the theory of local cohomology modules. The following generalization of local cohomology theory is given by J. Herzog in [17]. Let j be a nonnegative integer, and let M be a finitely generated R-module. Then the jth generalized local cohomology module of M and N with respect to I is defined by

$$H_I^j(M,N) = \varinjlim_n \operatorname{Ext}_R^j(M/I^nM, N).$$

These modules were studied further in many research papers such as [31], [32], [3], [18], [20], [19], [10], [8], [7], and [4]. It is clear that $H_I^j(R,N)$ is just the ordinary local cohomology module $H_I^j(N)$ (cf. [5], [6]).

The purpose of this paper is to investigate a question similar to the one above for the theory of generalized local cohomology. Our first main result is the following theorem.

THEOREM 1.1
If I is a principal ideal, then $H_I^j(M,N)$ is I-cofinite for all finitely generated R-modules M, N and all j.

As an immediate consequence of this theorem, we obtain again a theorem of K. I. Kawasaki [22, Theorem 1] (see Corollary 3.2). Moreover, Theorem 1.1 is an improvement of [14, Theorem 2.8], since we do not need the hypothesis that M has finite projective dimension as in [14]. It should be noticed that the arguments of local cohomology that are used in the proof of K. I. Kawasaki [22] cannot apply to proving Theorem 1.1 because, for the case of local cohomology, if I is a principal ideal, then $H_I^j(N) = 0$ for all $j > 1$. But this does not happen in the theory of generalized local cohomology; that is, $H_I^j(M,N)$ may not vanish for $j > 1$ even if I is principal ideal. Therefore, we have to use a criterion on the cofiniteness which was invented by L. Melkersson in [30]. Here we also give a more elementary proof for this criterion (see Lemma 3.1). The next theorem is our second main result in this paper.

THEOREM 1.2
Let t be a nonnegative integer such that $\dim \operatorname{Supp}(H_I^j(M,N)) \leq 1$ *for all $j < t$. Then $H_I^j(M,N)$ is I-cofinite for all $j < t$ and* $\operatorname{Hom}(R/I, H_I^t(M,N))$ *is finitely generated.*

This theorem is an extension for generalized local cohomology modules of a result of K. Bahmanpour and R. Naghipour [2, Theorem 2.6]. In [2], they used a basic property of local cohomology that $H_I^j(N) \cong H_I^j(N/\Gamma_I(N))$ for all $j > 0$; then it is easy to reduce to the case of $\Gamma_I(N)=0$. But, it is not true that $H_I^j(M,N)\cong H_I^j(M,N/\Gamma_{I_M}(N))$ for all $j>0$ in general, where $I_M=\operatorname{ann}_R(M/IM)$. Hence, we need to establish Lemma 2.4, which says that if t and k are nonnegative integers such that $\dim\operatorname{Supp}(H_I^j(M,N))\le k$ for all $j<t$, then so is $H_I^j(M,N/\Gamma_{I_M}(N))$. Moreover, in order to prove Theorem 1.2, we also need some more auxiliary lemmas such as Lemmas 2.3 and 2.5 on minimax modules. Especially, by Lemma 4.2, instead of studying the cofiniteness of $H_I^j(M,N)$, we need only to prove the cofiniteness of these modules with respect to I_M. As a consequence of Theorem 1.2, we prove that if $\dim\operatorname{Supp}(H_I^j(M,N))\le 1$ for all j (this is the case, e.g., if $\dim(N/I_MN)\le 1$), then $H_I^j(M,N)$ is I-cofinite for all j (see Corollary 4.3). This is an improvement of [14, Theorem 2.9] and [23, Corollary 3], because our theorem does not need the hypothesis that R is complete local, M is of finite projective dimension, and I is prime ideal with $\dim(R/I)=1$. Another consequence of Theorem 1.2 on the finiteness of Bass numbers is Corollary 4.4, which is a stronger result than the main result of S. Kawakami and K. I. Kawasaki in [20].

On the other hand, in the case of small dimension, the third author in [19, Lemma 3.1] proved that if $\dim(N)\le 2$, then any quotient of $H_I^j(M,N)$ has only finitely many associated prime ideals for all finitely generated R-modules M and all $j\ge 0$. We can now prove a stronger result in the following theorem.

THEOREM 1.3
Assume that $\dim(M)\le 2$ *or* $\dim(N)\le 2$. *Then* $H_I^j(M,N)$ *is I-cofinite for all j.*

As an immediate consequence of Theorem 1.3, we get a result on the cofiniteness of local cohomology modules (see Corollary 5.2). Moreover, by application of Theorems 1.2 and 1.3, we obtain a finiteness result on the set of associated prime ideals of $\operatorname{Ext}_R^i(R/I,H_I^j(M,N))$ for all $i,j\ge 0$ when $(R,\mathfrak{m})$ is a Noetherian local ring and $\dim(M)\le 3$ or $\dim(N)\le 3$ (Corollary 5.3).

The paper is divided into five sections. In Section 2, we prove some auxiliary lemmas which will be used in the sequel. Sections 3, 4, and 5 are devoted to proving three main results and their consequences.

2. Auxiliary lemmas

Let R be a commutative Noetherian ring, let I be an ideal of R, and let M, N be finitely generated R-modules. We always denote by I_M the annihilator of R-module M/IM, that is, $I_M=\operatorname{ann}_R(M/IM)$. We first recall the following lemmas.

LEMMA 2.1 (CF. [9, LEMMA 2.3], [10, LEMMA 2.1])

(i) *If* $I\subseteq\operatorname{ann}(M)$ *or* $\Gamma_I(N)=N$, *then* $H_I^j(M,N)\cong\operatorname{Ext}_R^j(M,N)$ *for all* $j\ge 0$.

(ii) $H_I^j(M,N)$ *is I_M-torsion.*

In [34], H. Zöschinger introduced the class of minimax modules. An R-module K is said to be a *minimax module*, if there is a finitely generated submodule T of K, such that K/T is Artinian. Thus the class of minimax modules includes all finitely generated and all Artinian modules.

LEMMA 2.2 (CF. [4, THEOREM 3.6], [1, THEOREM 2.3])
Let t be a nonnegative integer such that $H_I^j(M,N)$ is minimax for all $j<t$. Then $\operatorname{Hom}_R(R/I, H_I^t(M,N))$ is finitely generated.

We next prove some auxiliary lemmas which will be used in the sequel.

LEMMA 2.3
Let t be a nonnegative integer such that $H_I^j(M,N)$ is minimax for all $j<t$. Then $H_I^j(M,N)$ are I-cofinite for all $j<t$.

Proof
We proceed by induction on j. It is clear that $H_I^0(M,N)$ is I-cofinite. Assume that $j>0$ and that the result holds true for smaller values than j. Thus we obtain that $H_I^0(M,N),\dots,H_I^{j-1}(M,N)$ are I-cofinite minimax by the inductive hypothesis and by the hypothesis. It follows by Lemma 2.2 that $\operatorname{Hom}(R/I, H_I^j(M,N))$ is finitely generated, so that $H_I^j(M,N)$ is I-cofinite by [30, Proposition 4.3] as required. $\square$

LEMMA 2.4
Let t and k be nonnegative integers. If $\dim\operatorname{Supp}(H_I^j(M,N))\le k$ for all $j<t$, then so is $H_I^j(M,N/\Gamma_{I_M}(N))$.

Proof
From the short exact sequence $0\to\Gamma_{I_M}(N)\to N\to N/\Gamma_{I_M}(N)\to 0$, we get the long exact sequence

$$\begin{aligned}\cdots &\to \operatorname{Ext}_R^j\big(M,\Gamma_{I_M}(N)\big)\to H_I^j(M,N)\to H_I^j(M,\overline{N})\\ &\to \operatorname{Ext}_R^{j+1}\big(M,\Gamma_{I_M}(N)\big)\to\cdots,\end{aligned}$$

for all j, where $\overline{N}=N/\Gamma_{I_M}(N)$. We assume that there exists an integer $i<t$ and $\mathfrak{p}\in\operatorname{Supp}(H_I^i(M,\overline{N}))$ such that $\dim(R/\mathfrak{p})>k$ and $\mathfrak{p}\notin\operatorname{Supp}(H_I^j(M,\overline{N}))$ for all $j<i$. Then by the long exact sequence as above we obtain the exact sequence

$$\begin{aligned}\cdots &\to \operatorname{Ext}_R^j\big(M,\Gamma_{I_M}(N)\big)_{\mathfrak{p}}\to H_I^j(M,N)_{\mathfrak{p}}\to H_I^j(M,\overline{N})_{\mathfrak{p}}\\ &\to \operatorname{Ext}_R^{j+1}\big(M,\Gamma_{I_M}(N)\big)_{\mathfrak{p}}\to\cdots.\end{aligned}$$

Note that $H_I^j(M,N)_{\mathfrak{p}}=0$ for all $j\le i$, while $H_I^j(M,\overline{N})_{\mathfrak{p}}=0$ for all $j<i$, and $H_I^i(M,\overline{N})_{\mathfrak{p}}\neq 0$. So, by the above exact sequence, we have $\operatorname{Ext}_R^j(M,\Gamma_{I_M}(N))_{\mathfrak{p}}=0$ for all $j\le i$, and $\operatorname{Ext}_R^{i+1}(M,\Gamma_{I_M}(N))_{\mathfrak{p}}\neq 0$. It implies that $\Gamma_{I_M}(N)_{\mathfrak{p}}\neq 0$ and

$$\operatorname{depth}\big(\operatorname{ann}(M)_{\mathfrak{p}},\Gamma_{I_M}(N)_{\mathfrak{p}}\big)=i+1\ge 1.$$

Hence $\operatorname{ann}(M)_{\mathfrak{p}} \nsubseteq \mathfrak{q}R_{\mathfrak{p}}$ for all $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}}(\Gamma_{I_M}(N)_{\mathfrak{p}})$. This contradicts with the fact that $\operatorname{Ass}_{R_{\mathfrak{p}}}(\Gamma_{I_M}(N)_{\mathfrak{p}}) = \operatorname{Ass}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) \cap V((I_M)_{\mathfrak{p}})$ and $I_M \supseteq \operatorname{ann}(M)$. $\square$

LEMMA 2.5

Let t be a nonnegative integer such that $\operatorname{Supp}(H_I^j(M,N)) \subseteq \operatorname{Max}(R)$ *for all $j < t$. Then $H_I^j(M,N)$ is Artinian for all $j < t$.*

Proof

We now prove the lemma by induction on t. If $t = 1$, then it is clear that $H_I^0(M,N)$ is Artinian. Assume that $t \geq 2$ and the lemma holds true for $t-1$. By the inductive hypothesis, the R-modules $H_I^j(M,N)$ are Artinian for all $j < t-1$. Therefore, by Lemma 2.2, $\operatorname{Hom}(R/I, H_I^{t-1}(M,N))$ is finitely generated. Thus, since $\operatorname{Supp}(\operatorname{Hom}(R/I, H_I^{t-1}(M,N))) \subseteq \operatorname{Max}(R)$, we obtain that $\operatorname{Hom}(R/I, H_I^{t-1}(M,N))$ is Artinian. On the other hand, as $H_I^{t-1}(M,N)$ is I-torsion, it follows by [27, Theorem 1.3] that $H_I^{t-1}(M,N)$ is Artinian. $\square$

3. Proof of Theorem 1.1

We first need the following lemma which has been proved in [30, Corollary 3.4] by L. Melkersson. We give here an another proof for this result with elementary arguments.

LEMMA 3.1

Let K be an R-module. Suppose $x \in I$ and $\operatorname{Supp}(K) \subset V(I)$*. If $(0:x)_K$ and K/xK are both I-cofinite, then K must be I-cofinite.*

Proof

Let t be a nonnegative integer. We need only to claim that $\operatorname{Ext}_R^t(R/I,K)$ is finitely generated. By the commutative diagram

$$0 \to (0 :_K x) \to K \xrightarrow{x} xK \to 0$$

$$x \downarrow \quad \searrow x$$

$$0 \to xK \to K \to K/xK \to 0$$

we obtain the following commutative diagram of long exact sequences

$$\cdots \to \operatorname{Ext}_R^t\big(R/I, (0 :_K x)\big) \to \operatorname{Ext}_R^t(R/I,K) \xrightarrow{x^{(t)}} \operatorname{Ext}_R^t(R/I,xK) \to \cdots$$

$$x^{(t)} \downarrow \qquad\qquad \searrow x$$

$$\cdots \to \operatorname{Ext}_R^{t-1}(R/I,K/xK) \to \operatorname{Ext}_R^t(R/I,xK) \xrightarrow{f_t} \operatorname{Ext}_R^t(R/I,K) \to \cdots,$$

where $x^{(t)} = \operatorname{Ext}_R^t(R/I,x)$. Note that K/xK is I-cofinite by the hypothesis; it implies that $\operatorname{Ext}_R^{t-1}(R/I,K/xK)$ is finitely generated. Thus $\operatorname{Ker}(f_t)$ is finitely generated. Moreover, the triangle is commutative, so that $x^{(t)}((0: x)_{\operatorname{Ext}_R^t(R/I,K)}) \subseteq \operatorname{Ker}(f_t)$. It follows that $x^{(t)}((0:x)_{\operatorname{Ext}_R^t(R/I,K)})$ is finitely generated. On the other hand, $(0:x)_K$ is I-cofinite by the hypothesis, so we obtain that $\operatorname{Ext}^t_R(R/I,(0:x)_K)$ is finitely generated. It implies that $\operatorname{Ker}(x^{(t)})$ is finitely generated. Therefore, by the following exact sequence,

$$\begin{aligned} 0 &\to \operatorname{Ker}(x^{(t)}) \cap (0:x)_{\operatorname{Ext}^t_R(R/I,K)} \to (0:x)_{\operatorname{Ext}^t_R(R/I,K)} \\ &\to x^{(t)}\big((0:x)_{\operatorname{Ext}^t_R(R/I,K)}\big) \to 0, \end{aligned}$$

we obtain that $(0:x)_{\operatorname{Ext}^t_R(R/I,K)}$ is finitely generated. Finally, note that for $x \in I$, it yields that $\operatorname{Ext}^t_R(R/I,K) = (0:x)_{\operatorname{Ext}^t_R(R/I,K)}$ is finitely generated as required. □

We now are ready to prove Theorem 1.1.

Proof of Theorem 1.1
Assume that $I = Rx$ is a principal ideal. From the short exact sequence

$$0 \to \Gamma_I(M) \to M \to \overline{M} \to 0,$$

where $\overline{M} = M/\Gamma_I(M)$, we get by [18] the following exact sequence:

$$H^{i-1}_I\big(\Gamma_I(M),N\big) \to H^i_I(\overline{M},N) \to H^i_I(M,N) \to H^i_I\big(\Gamma_I(M),N\big)$$

for all i. Since $\Gamma_I(M) = (0:I^k)_M$ for some positive integer k, we get by Lemma 2.1 that

$$H^i_I\big(\Gamma_I(M),N\big) = H^i_{I^k}\big((0:I^k)_M,N\big) \cong \operatorname{Ext}^i_R\big(\Gamma_I(M),N\big)$$

for all i. Hence $H^i_I(\Gamma_I(M),N)$ is finitely generated for all i, and it follows by the above exact sequence that $H^i_I(M,N)$ is I-cofinite if and only if so is $H^i_I(\overline{M},N)$. Hence we may assume that $\Gamma_I(M) = 0$, so that $I \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. It implies that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$. Thus we obtain an exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0.$$

From this we have the following exact sequence:

$$0 \to H^{i-1}_I(M,N)/xH^{i-1}_I(M,N) \to H^i_I(M/xM,N) \to (0:x)_{H^i_I(M,N)} \to 0$$

for all i. As $I = Rx$, we obtain by Lemma 2.1 that

$$H^i_I(M/xM,N) \cong \operatorname{Ext}^i_R(M/xM,N)$$

for all i. Hence $H^i_I(M/xM,N)$ is finitely generated for all i. Thus by the above exact sequence we obtain that

$$(0:x)_{H^i_I(M,N)} \qquad \text{and} \qquad H^i_I(M,N)/xH^i_I(M,N)$$

are finitely generated for all i. Therefore we get by Lemma 3.1 that $H^i_I(M,N)$ is I-cofinite for all i. □

By replacing M by R in Theorem 1.1 we obtain a theorem of K. I. Kawasaki on the cofiniteness of local cohomology modules as follows.

COROLLARY 3.2 ([22, THEOREM 1])

If I is a principal ideal, then $H_I^j(N)$ is I-cofinite for all finitely generated R-modules N and all j.

4. Proof of Theorem 1.2

Before proving Theorem 1.2, we need to recall some known facts on the theory of secondary representation.

In [24], I. G. Macdonald has developed the theory of attached prime ideals and secondary representation of a module, which is (in a certain sense) a dual to the theory of associated prime ideals and primary decompositions. A nonzero R-module K is called secondary if for each $a \in R$ multiplication by a on K is either surjective or nilpotent. Then $\mathfrak{p} = \sqrt{\operatorname{ann}(K)}$ is a prime ideal, and K is called $\mathfrak{p}$-secondary. We say that K has a secondary representation if there is a finite number of secondary submodules $K_1, K_2, \ldots, K_n$ such that $K = K_1 + K_2 + \cdots + K_n$. One may assume that the prime ideals $\mathfrak{p}_i = \sqrt{\operatorname{ann}(K_i)}$, $i = 1, 2, \ldots, n$ are all distinct and, by omitting redundant summands, that the representation is minimal. Then the set of prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ does not depend on the representation, and it is called the set of attached prime ideals of K and denoted by $\operatorname{Att}(K)$. Note that if A is an Artinian R-module, then A has a secondary representation; moreover the set of minimal prime ideals of $\operatorname{ann}_R(A)$ is just the set of minimal elements of $\operatorname{Att}_R(A)$ (see [24]). The basic properties on the set $\operatorname{Att}(A)$ of attached primes of A are referred in a paper by I. G. Macdonald [24]. If $0 \to A_1 \to A_2 \to A_3 \to 0$ is an exact sequence of Artinian R-modules, then

$$\operatorname{Att}(A_3) \subseteq \operatorname{Att}(A_2) \subseteq \operatorname{Att}(A_1) \cup \operatorname{Att}(A_3).$$

LEMMA 4.1

Let x be an element of R, let I be an ideal of R, and let A be an Artinian R-module. Then the following statements are true.

(i) *If $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}(A) \setminus \operatorname{Max}(R)$, then $\ell(A/xA) < \infty$.*
(ii) *If $(0 : I)_A$ is finitely generated, then $I \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}(A) \setminus \operatorname{Max}(R)$.*

Proof

(i) Suppose that $\operatorname{Att}(A) \setminus \operatorname{Max}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Let $A = A_1 + \cdots + A_n + B_1 + \cdots + B_t$ be a minimal secondary representation of A, where A_i is $\mathfrak{p}_i$-secondary and B_j is $\mathfrak{m}_i$-secondary for all $i = 1, \ldots, n$ and all $j = 1, \ldots, t$ (with $\mathfrak{m}_j \in \operatorname{Max}(R)$ for all $j = 1, \ldots, t$). Set $B = B_1 + \cdots + B_t$. Since $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, n$, we have $xA_i = A_i$ for all $i = 1, \ldots, n$. It follows that $xA = A_1 + \cdots + A_n + xB$. Thus $A/xA \cong B/(B \cap xA)$. Hence $\operatorname{Att}(A/xA) \subseteq \operatorname{Att}(B) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\} \subseteq \operatorname{Max}(R)$. Following [24], the set of minimal prime ideals of $\operatorname{ann}_R(A/xA)$ is just the set of minimal elements of $\operatorname{Att}_R(A/xA)$. Hence $\dim(R/\operatorname{ann}(A/xA)) = 0$. Then we get by [11, Proposition 2.4] that $\ell(A/xA) < \infty$ as required.

(ii) We first claim that $\sqrt{\operatorname{ann}(0 :_A I)} = \sqrt{\operatorname{ann}(0 :_A I^n)}$ for all $n \geq 2$. Consider $n = 2$, it is clear that $\sqrt{\operatorname{ann}(0 :_A I)} \supseteq \sqrt{\operatorname{ann}(0 :_A I^2)}$. Conversely, for any $a \in$

$\sqrt{\mathrm{ann}(0:_A I)}$ there is an integer $t>0$ such that $a^t(0:_A I)=0$. We now prove that $a^{2t}(0:_A I^2)=0$ (and therefore $a\in\sqrt{\mathrm{ann}(0:_A I^2)}$). Indeed, for any $y\in(0:_A I^2)$, then $I^2y=0$. So that $Iy\subseteq(0:_A I)$, thus $a^t(Iy)=0$. Hence $a^ty\in(0:_A I)$, and thus $a^t(a^ty)=0$. Therefore $a^{2t}y=0$. We now assume that $n>2$ and the claim is true for $n-1$. Let $a\in\sqrt{\mathrm{ann}(0:_A I)}$, then by induction assumption $a\in\sqrt{(0:_A I^{n-1})}$. Thus $a^t(0:_A I^{n-1})=0$ for some $t>0$. For any $y\in(0:_A I^n)$, then $I^{n-1}Iy=I^ny=0$. Hence $Iy\subseteq(0:_A I^{n-1})$, so that $I(a^ty)=a^t(Iy)=0$. It implies that $a^ty\in(0:_A I)$. On the other hand, since $a\in\sqrt{\mathrm{ann}(0:_A I)}$, so does $a^l(0:_A I)=0$ for some $l>0$. Therefore $a^{t+l}y=0$, and it yields that $a\in\sqrt{(0:_A I^n)}$. So we get the claim. Finally for any $\mathfrak{p}\in\mathrm{Att}(A)\setminus\mathrm{Max}(R)$ we obtain that $I\nsubseteq\mathfrak{p}$. Indeed, assume that $I\subseteq\mathfrak{p}$ for some $\mathfrak{p}\in\mathrm{Att}(A)\setminus\mathrm{Max}(R)$. Then there exists a submodule U of A such that U is $\mathfrak{p}$-secondary. Thus there is an integer n such that $\mathfrak{p}^nU=0$. Hence, $I\subseteq\mathfrak{p}$, so that $I^nU=0$. Therefore $U=(0:_U I^n)\subseteq(0:_A I^n)$. Hence by combining $\ell(0:_A I)<\infty$ with the above claim we get that $(0:_A I^n)$ is of finite length. It implies that $\ell(U)<\infty$, so $\mathfrak{p}\in\mathrm{Max}(R)$; this is a contradiction. □

LEMMA 4.2
Let t be a nonnegative integer. Then

(i) $H_I^t(M,N)$ *is I-cofinite if and only if* $H_I^t(M,N)$ *is I_M-cofinite, where* $I_M=\mathrm{ann}_R(M/IM)$.

(ii) $\mathrm{Hom}(R/I,H_I^t(M,N))$ *is finitely generated if and only if so is* $\mathrm{Hom}(R/I_M, H_I^t(M,N))$.

Proof
Set $K=H_I^t(M,N)$. Note that $\mathrm{Supp}(K)\subseteq\mathrm{Supp}(R/I_M)\subseteq\mathrm{Supp}(R/I)$.

(i) If K is I-cofinite, since $I\subseteq I_M$, then we get that K is I_M-cofinite by [12, Proposition 1]. Conversely, assume that K is I_M-cofinite. Thus, as $\sqrt{I_M}=\sqrt{I+\mathrm{ann}(M)}$, K is $(I+\mathrm{ann}(M))$-cofinite by [12, Proposition 1]. Let $x_1,\dots,x_t$, $y_1,\dots,y_s$ be generators of $I+\mathrm{ann}(M)$ such that $I=(x_1,\dots,x_t)$ and $\mathrm{ann}(M)=(y_1,\dots,y_s)$. Then Koszul cohomology modules $H^j(\underline{x},y_1,\dots,y_s;K)$ are finitely generated R-modules for all j by [29, Theorem 1.1]. (Here we set $\underline{x}=x_1,\dots,x_t$ for short.) We now claim by descending induction on l (with $0\le l\le s$) that $H^j(\underline{x},y_1,\dots,y_l;K)$ are finitely generated R-modules for all j, where we use the convention that $H^j(\underline{x};K)=H^j(\underline{x},y_1,\dots,y_l;K)$ if $l=0$. If $l=s$, then the claim is clear. Suppose that $l<s$ and $H^j(\underline{x},y_1,\dots,y_{l+1};K)$ are finitely generated R-modules for all j. We first consider the case $j=0$. As $y_{l+1}\in\mathrm{ann}(K)$, so we get that

$$\begin{aligned}H^0(\underline{x},y_1,\dots,y_l;K)&\cong\big(0:_K(\underline{x},y_1,\dots,y_l)R\big)\\&\cong\big(0:_K(\underline{x},y_1,\dots,y_l,y_{l+1})R\big)\\&\cong H^0(\underline{x},y_1,\dots,y_l,y_{l+1};K).\end{aligned}$$

Thus $H^0(\underline{x}, y_1, \ldots, y_l; K)$ is a finitely generated R-module. Assume that $j \geq 1$. We consider the following exact sequences (cf. [28, Section 5]):

$$H^{j-1}(\underline{x}, y_1, \ldots, y_l, y_{l+1}; K) \to H^j(\underline{x}, y_1, \ldots, y_l; K) \xrightarrow{y_{l+1}} H^j(\underline{x}, y_1, \ldots, y_l; K)$$

for all $j \geq 1$. Here $y_{l+1} \in \operatorname{ann}(K)$, so that $y_{l+1} H^j(\underline{x}, y_1, \ldots, y_l; K) = 0$. Hence, the above exact sequence implies that the sequence

$$H^{j-1}(\underline{x}, y_1, \ldots, y_l, y_{l+1}; K) \to H^j(\underline{x}, y_1, \ldots, y_l; K) \to 0$$

is exact for all $j \geq 1$. From this we get by induction assumption that $H^j(\underline{x}, y_1, \ldots, y_l; K)$ are finitely generated R-modules for all $j \geq 1$. Thus the claim is proved. In particular, $H^j(\underline{x}; K)$ are finitely generated R-modules for all j. Therefore, we get by [29, Theorem 1.1] again that K is I-cofinite.

(ii) We note that $\operatorname{Hom}(R/I + \operatorname{ann}(M), K) \cong \operatorname{Hom}(R/I, K)$, as $\operatorname{ann}(M) \subseteq \operatorname{ann}(K)$. Hence, since $\sqrt{I + \operatorname{ann}(M)} = \sqrt{I_M}$, the result follows by [12, Proposition 1]. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2

By Lemma 4.2, we need only to claim that $H^j_I(M, N)$ is I_M-cofinite for all $j < t$ and $\operatorname{Hom}(R/I_M, H^t_I(M, N))$ is finitely generated, provided $\dim \operatorname{Supp}(H^j_I(M, N)) \leq 1$ for all $j < t$ (where t is a given integer).

We prove the claim by induction on $t \geq 0$. The case of $t = 0$ is trivial. If $t = 1$, then it is clear that $H^0_I(M, N)$ is I_M-cofinite; moreover we get by Lemma 2.2 that $\operatorname{Hom}(R/I_M, H^1_I(M, N))$ is finitely generated. Assume that $t > 1$, and the result holds true for the case $t - 1$. From the short exact sequence $0 \to \Gamma_{I_M}(N) \to N \to \overline{N} \to 0$, we get the long exact sequence

$$\operatorname{Ext}^j_R\big(M, \Gamma_{I_M}(N)\big) \xrightarrow{f_j} H^j_I(M, N) \xrightarrow{g_j} H^j_I(M, \overline{N}) \xrightarrow{h_j} \operatorname{Ext}^{j+1}_R\big(M, \Gamma_{I_M}(N)\big),$$

where $\overline{N} = N/\Gamma_{I_M}(N)$. For each $j \geq 0$ we split the above exact sequence into the following two exact sequences:

$$0 \to \operatorname{Im} f_j \to H^j_I(M, N) \to \operatorname{Im} g_j \to 0 \qquad \text{and}$$
$$0 \to \operatorname{Im} g_j \to H^j_I(M, \overline{N}) \to \operatorname{Im} h_j \to 0.$$

Note that $\operatorname{Im} f_j$ and $\operatorname{Im} h_j$ are finitely generated for all $j \geq 0$. Then, for each $j < t$, we obtain that $H^j_I(M, N)$ is I_M-cofinite if and only if so is $H^j_I(M, \overline{N})$. On the other hand, we get by Lemma 2.4 that $\dim \operatorname{Supp}(H^j_I(M, \overline{N})) \leq 1$ for all $j < t$. Therefore, in order to prove the theorem for the case of $t > 1$, we may assume that $\Gamma_{I_M}(N) = 0$. Hence $I_M \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}$. Set

$$X = \bigcup_{j=0}^{t-1} \operatorname{Supp}\big(H^j_I(M, N)\big) \qquad \text{and} \qquad S = \big\{\mathfrak{p} \in X \bigm| \dim(R/\mathfrak{p}) = 1\big\}.$$

Then $S \subseteq \bigcup_{j=0}^{t-1} \operatorname{Ass}(H^j_I(M, N))$. Note that $H^j_I(M, N)$ is I_M-cofinite for all $j < t - 1$ and $\operatorname{Hom}(R/I_M, H^{t-1}_I(M, N))$ is finitely generated by the inductive hypothesis.

It implies that $\bigcup_{j=0}^{t-1} \operatorname{Ass}(H_I^j(M,N))$ is a finite set, and so S is a finite set. Assume that $S = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. Then it is clear that

$$\operatorname{Supp}_{R_{\mathfrak{p}_k}}\big(H^j_{IR_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})\big) \subseteq \operatorname{Max}(R_{\mathfrak{p}_k})$$

for all $j < t$ and all $k = 1, \ldots, n$. From this, we get by Lemma 2.5 that $H^j_{IR_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})$ is Artinian for all $j < t$ and all $k = 1, \ldots, n$. Note that $V(I_M) \subseteq V(I)$. Hence, it implies by Lemma 2.2 and [21, Lemma 1] that $\operatorname{Hom}(R_{\mathfrak{p}_k}/(I_M)R_{\mathfrak{p}_k}, H^j_{IR_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k}))$ is finitely generated for all $j < t$ and all $k = 1, \ldots, n$. Therefore it yields by Lemma 4.1(ii) that

$$V\big((I_M)R_{\mathfrak{p}_k}\big) \cap \operatorname{Att}_{R_{\mathfrak{p}_k}}\big(H^j_{IR_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})\big) \subseteq \operatorname{Max}(R_{\mathfrak{p}_k})$$

for all $j < t$ and all $k = 1, \ldots, n$. Let

$$T = \bigcup_{j=0}^{t-1} \bigcup_{k=1}^{n} \big\{\mathfrak{q} \in \operatorname{Spec} R \;\big|\; \mathfrak{q}R_{\mathfrak{p}_k} \in \operatorname{Att}_{R_{\mathfrak{p}_k}}\big(H^j_{IR_{\mathfrak{p}_k}}(M_{\mathfrak{p}_k}, N_{\mathfrak{p}_k})\big)\big\}.$$

Then we have $T \cap V(I_M) \subseteq S$. We now choose an element $x \in I_M$ such that

$$x \notin \Big(\bigcup_{\mathfrak{p} \in T \setminus V(I_M)} \mathfrak{p}\Big) \cup \Big(\bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(N)} \mathfrak{p}\Big).$$

Thus, we have the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$. It implies the following exact sequence:

$$H_I^j(M,N) \xrightarrow{x} H_I^j(M,N) \to H_I^j(M,N/xN) \to H_I^{j+1}(M,N)$$

for all $j \geq 0$. Thus, we have an exact sequence

$$(1) \qquad 0 \to H_I^j(M,N)/xH_I^j(M,N) \xrightarrow{\alpha_j} H_I^j(M,N/xN) \xrightarrow{\beta_j} (0:x)_{H_I^{j+1}(M,N)} \to 0$$

for all $j \geq 0$. Note that $\dim \operatorname{Supp}(H_I^j(M,N/xN)) \leq 1$ for all $j < t-1$ by the above exact sequence and by the hypothesis. So that, we get by the induction assumption that $H_I^0(M,N/xN), H_I^1(M,N/xN), \ldots, H_I^{t-2}(M,N/xN)$ are I_M-cofinite and $\operatorname{Hom}(R/I_M, H_I^{t-1}(M,N/xN))$ is finitely generated. Moreover, also by the induction assumption, we have that $H_I^0(M,N), H_I^1(M,N), \ldots, H_I^{t-2}(M,N)$ are I_M-cofinite and $\operatorname{Hom}(R/I_M, H_I^{t-1}(M,N))$ is finitely generated. For each $j < t$, we set $L_j = H_I^j(M,N)/xH_I^j(M,N)$. By the choice of x and by Lemma 4.1, we obtain that $(L_j)_{\mathfrak{p}_k}$ has finite length for all $j < t$ and all $k = 1, \ldots, n$. From this by the Noetherianness of $(L_j)_{\mathfrak{p}_k}$, there exists a finitely generated submodule L_{jk} of L_j such that $(L_j)_{\mathfrak{p}_k} = (L_{jk})_{\mathfrak{p}_k}$ for any $j < t$ and any $k = 1, \ldots, n$. Let $L'_j = L_{j1} + L_{j2} + \cdots + L_{jn}$. Then L'_j is a finitely generated submodule of L_j satisfying the following inclusion:

$$\operatorname{Supp}(L_j/L'_j) \subseteq X \setminus \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\} \subseteq \operatorname{Max}(R)$$

for all $j < t$. For each $j < t$, we set $N_j = H_I^j(M,N/xN)$ and $N'_j = \alpha_j(L'_j)$. Then N'_j is a finitely generated submodule of N_j and the sequence

$$(2) \qquad 0 \to L_j/L'_j \xrightarrow{\alpha_j^*} N_j/N'_j \xrightarrow{\beta_j^*} (0:x)_{H_I^{j+1}(M,N)} \to 0$$

is exact. We now prove that L_j is minimax for all $j<t$. Look at the exact sequence

$$\operatorname{Hom}(R/I_M,N_j)\to \operatorname{Hom}(R/I_M,N_j/N'_j)\to \operatorname{Ext}^1_R(R/I_M,N'_j).$$

For any $j<t$, since N'_j is finitely generated and $\operatorname{Hom}(R/I_M,N_j)$ is finitely generated, $\operatorname{Hom}(R/I_M,N_j/N'_j)$ is finitely generated. Hence we obtain by the sequence (2) that $\operatorname{Hom}(R/I_M,L_j/L'_j)$ is finitely generated for all $j<t$. While $\operatorname{Supp}(L_j/L'_j)\subseteq \operatorname{Max}(R)$ and L_j/L'_j is I_M-torsion, so L_j/L'_j is Artinian by [27, Theorem 1.3] for all $j<t$. Thus L_j is minimax for all $j<t$. Consider again the exact sequence (1), that is, the following sequence:

$$(1') \qquad 0\to L_j \xrightarrow{\alpha_j} N_j \xrightarrow{\beta_j} (0:x)_{H_I^{j+1}(M,N)}\to 0.$$

As $\operatorname{Hom}(R/I_M,N_j)$ is finitely generated for all $j<t$, so is $\operatorname{Hom}(R/I_M,L_j)$ for all $j<t$. From this, we obtain by [30, Proposition 4.3] that L_j is I_M-cofinite for all $j<t$. Keep in mind that N_j is I_M-cofinite for all $j<t-1$. Thus, from the sequence ($1'$), we have that $(0:x)_{H_I^j(M,N)}$ is I_M-cofinite for all $j<t$. In particular, $(0:x)_{H_I^{t-1}(M,N)}$ and $H_I^{t-1}(M,N)/xH_I^{t-1}(M,N)=L_{t-1}$ are I_M-cofinite. It implies that $H_I^{t-1}(M,N)$ is I_M-cofinite by Lemma 3.1. Thus $H_I^j(M,N)$ is I_M-cofinite for all $j<t$. On the other hand, by the sequence ($1'$) when $j=t-1$, we have the following exact sequence:

$$\operatorname{Hom}(R/I_M,N_{t-1})\to \operatorname{Hom}\big(R/I_M,(0:x)_{H_I^t(M,N)}\big)\to \operatorname{Ext}^1_R(R/I_M,L_{t-1}).$$

Thus, since $\operatorname{Hom}(R/I_M,N_{t-1})$ is finitely generated and L_{t-1} is I_M-cofinite, so it yields that $\operatorname{Hom}(R/I_M,H_I^t(M,N))=\operatorname{Hom}(R/I_M,(0:x)_{H_I^t(M,N)})$ is finitely generated. Hence the claim is proved, and the proof of Theorem 1.2 is completed. □

Note that in [14, Theorem 2.9], K. Divaani-Aazar and R. Sazeedeh showed that if $\mathfrak{p}$ is a prime ideal in a complete local ring $(R,\mathfrak{m})$ with $\dim(R/\mathfrak{p})=1$, then $H_{\mathfrak{p}}^j(M,N)$ is $\mathfrak{p}$-cofinite for all $j\geq 0$ whenever M has finite projective dimension. After that in [23, Corollary 3], K. I. Kawasaki proved that if $(R,\mathfrak{m})$ is a local ring and I an ideal of R with $\dim(R/I)=1$ then, $H_I^j(M,N)$ is I-cofinite for all $j\geq 0$ provided that M has finite projective dimension. Here, as an immediate consequence of Theorem 1.2, we get the following corollary, which is better than the above results.

COROLLARY 4.3

If $\dim\operatorname{Supp}(H_I^j(M,N))\leq 1$ *for all* j *(this is the case, e.g., if* $\dim(N/I_MN)\leq 1$*), then* $H_I^j(M,N)$ *is* I*-cofinite for all* $j\geq 0$.

We now recall the notion of Bass numbers: let K be an R-module, let i be an integer, and let $\mathfrak{p}$ be a prime ideal; then the ith Bass number $\mu^i(\mathfrak{p},K)$ of K with respect to $\mathfrak{p}$ was defined by $\mu^i(\mathfrak{p},K)=\dim_{k(\mathfrak{p})}(\operatorname{Ext}^i_R(R/\mathfrak{p},K)_{\mathfrak{p}})$. In [20], S. Kawakami and K. I. Kawasaki proved that if M has finite projective dimension and $\dim(R/I)=1$, then $\mu^i(\mathfrak{p},H^j_I(M,N))$ is finite for all $i,j\geq 0$ and all $\mathfrak{p}\in \operatorname{Spec}(R)$. The next corollary is a generalization of this result.

COROLLARY 4.4
Assume that $\dim \operatorname{Supp}(H^j_I(M,N))\leq 1$ *for all* j *(this is the case, e.g., if* $\dim(N/I_MN)\leq 1$*). Then* $\mu^i(\mathfrak{p},H^j_I(M,N))$ *is finite for all* $i,j\geq 0$ *and all* $\mathfrak{p}\in \operatorname{Spec}(R)$.

Proof
If $I\nsubseteq \mathfrak{p}$, then $\mu^i(\mathfrak{p},H^j_I(M,N))=0$. If $I\subseteq \mathfrak{p}$, then $\operatorname{Supp}(R/\mathfrak{p})\subseteq \operatorname{Supp}(R/I)$, so that $\operatorname{Ext}^i_R(R/\mathfrak{p},H^j_I(M,N))$ is finitely generated for all i, j by Corollary 4.3 and [12, Proposition 1]. Therefore $\mu^i(\mathfrak{p},H^j_I(M,N))$ is finite for all i, j, as required. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3
We first consider the case of $\dim(M)\leq 2$. By the short exact sequence $0\to \Gamma_I(M)\to M\to \overline{M}\to 0$ where $\overline{M}=M/\Gamma_I(M)$, we get the following exact sequence:

$$H^{j-1}_I\big(\Gamma_I(M),N\big)\xrightarrow{f_j} H^j_I(\overline{M},N)\xrightarrow{g_j} H^j_I(M,N)\xrightarrow{h_j} H^j_I\big(\Gamma_I(M),N\big)$$

(following [18]). It implies the exact sequences

$$0\to \operatorname{Im} f_j\to H^j_I(\overline{M},N)\to \operatorname{Im} g_j\to 0$$

and

$$0\to \operatorname{Im} g_j\to H^j_I(M,N)\to \operatorname{Im} h_j\to 0.$$

Since $\Gamma_I(M)=(0:I^k)_M$ for some integer k, so that

$$H^j_I\big(\Gamma_I(M),N\big)=H^j_{I^k}\big((0:I^k)_M,N\big)=\operatorname{Ext}^j_R\big(\Gamma_I(M),N\big)$$

for all j by Lemma 2.1. Thus $\operatorname{Im} f_j$ and $\operatorname{Im} h_j$ are finitely generated for all j. So by the above exact sequences we obtain that $H^j_I(\overline{M},N)$ is I-cofinite if and only if so is $H^j_I(M,N)$. Therefore we may assume that $\Gamma_I(M)=0$. Then there exists $x\in I$ such that x is an M-regular element. From the short exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$ we get the following exact sequence:

$$H^j_I(M/xM,N)\to (0:x)_{H^j_I(M,N)}\to 0$$

since $\dim(M/xM)\leq 1$, so that $\dim \operatorname{Supp}((0:x)_{H^j_I(M,N)})\leq 1$. Note that $H^j_I(M,N)$ is I-torsion and $x\in I$. Thus

$$\dim \operatorname{Supp}\big(H^j_I(M,N)\big)=\dim \operatorname{Supp}\big((0:x)_{H^j_I(M,N)}\big)\leq 1$$

for all j. From this we obtain by Corollary 4.3 that $H^j_I(M,N)$ is I-cofinite for all j.

For the rest of this proof, we consider the case of $\dim(N) \leq 2$. By the short exact sequence $0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$ where $\overline{N} = N/\Gamma_I(N)$, we get the following exact sequence:

$$\operatorname{Ext}^j_R\big(M, \Gamma_I(N)\big) \xrightarrow{u_j} H^j_I(M,N) \xrightarrow{v_j} H^j_I(M,\overline{N}) \xrightarrow{w_j} \operatorname{Ext}^{j+1}_R\big(M, \Gamma_I(N)\big).$$

It implies the exact sequences

$$0 \to \operatorname{Im} u_j \to H^j_I(M,N) \to \operatorname{Im} v_j \to 0$$

and

$$0 \to \operatorname{Im} v_j \to H^j_I(M,\overline{N}) \to \operatorname{Im} w_j \to 0.$$

Thus $\operatorname{Im} u_j$ and $\operatorname{Im} w_j$ are finitely generated for all j. So by the above exact sequences we obtain that $H^j_I(M,\overline{N})$ is I-cofinite if and only if so is $H^j_I(M,N)$. Hence we may assume that $\Gamma_I(N) = 0$. So we can take $y \in I$ such that y is an N-regular element. From the exact sequence $0 \to N \xrightarrow{y} N \to N/yN \to 0$ we have an exact sequence as follows:

$$H^j_I(M, N/yN) \to (0:y)_{H^{j+1}_I(M,N)} \to 0$$

for all j. So that $\dim \operatorname{Supp}(H^j_I(M,N)) = \dim \operatorname{Supp}((0:y)_{H^j_I(M,N)}) \leq 1$ for all $j \geq 1$. Note that $H^0_I(M,N) = \operatorname{Hom}(M, \Gamma_I(N)) = \operatorname{Hom}(M,0) = 0$. Thus $\dim \operatorname{Supp}(H^j_I(M,N)) \leq 1$ for all j. From this we get by Corollary 4.3 that $H^j_I(M,N)$ is I-cofinite for all j, and this finishes the proof of Theorem 1.3. □

As an immediate consequence of Theorem 1.3 we obtain the following results.

COROLLARY 5.1

If $\dim(R) \leq 2$, the $H^j_I(M,N)$ is I-cofinite for all j and all finitely generated R-modules M, N.

COROLLARY 5.2

If $\dim(N) \leq 2$, the $H^j_I(N)$ is I-cofinite for all j.

We next consider further a consequence of Theorems 1.2 and 1.3 on the finiteness of associated primes of generalized local cohomology modules. We first recall the notion of weakly Laskerian modules which was introduced in [13]: an R-module K is called *weakly Laskerian* if any quotient module of K has finitely many associated primes. Note that all Artinian modules, all finitely generated modules, and all modules with finite support are weakly Laskerian. Moreover, if $0 \to K_1 \to K_2 \to K_3 \to 0$ is an exact sequence, then K_2 is weakly Laskerian if and only if K_1 and K_3 are both weakly Laskerian. Note that if R is a Noetherian local ring and $\dim(N) \leq 3$ then the third author proved in [19, Theorem 1.1] that the modules $H^j_I(M,N)$ have only finitely many associated prime ideals for all j. In the following, we obtain a stronger result.

COROLLARY 5.3

Assume that $(R,\mathfrak{m})$ *is a Noetherian local ring. If* $\dim(M)\leq 3$ *or* $\dim(N)\leq 3$, *then* $\operatorname{Ext}^i_R(R/I,H^j_I(M,N))$ *is weakly Laskerian for all* $i,j\geq 0$. *In particular,* $\operatorname{Ass}_R(H^j_I(M,N))$ *is a finite set for all* $j\geq 0$.

Proof

Assume that $\dim(M)\leq 3$. By arguments similar to those in the proof of Theorem 1.3, we obtain the following exact sequences:

$$0\to \operatorname{Im} f_j\to H^j_I(\overline{M},N)\to \operatorname{Im} g_j\to 0$$

and

$$0\to \operatorname{Im} g_j\to H^j_I(M,N)\to \operatorname{Im} h_j\to 0,$$

where $\overline{M}=M/\Gamma_I(M)$. Thus we get the following exact sequences:

$$\cdots\to \operatorname{Ext}^i_R(R/I,\operatorname{Im} f_j)\to \operatorname{Ext}^i_R\big(R/I,H^j_I(\overline{M},N)\big)\to \operatorname{Ext}^i_R(R/I,\operatorname{Im} g_j)\to\cdots$$

and

$$\cdots\to \operatorname{Ext}^i_R(R/I,\operatorname{Im} g_j)\to \operatorname{Ext}^i_R\big(R/I,H^j_I(M,N)\big)\to \operatorname{Ext}^i_R(R/I,\operatorname{Im} h_j)\to\cdots.$$

Moreover, note that $\operatorname{Im} f_j$ and $\operatorname{Im} h_j$ are finitely generated for all j. It follows that $\operatorname{Ext}^i_R(R/I,H^j_I(M,N))$ is weakly Laskerian if and only if so is the module $\operatorname{Ext}^i_R(R/I,H^j_I(\overline{M},N))$. Therefore we may assume that $\Gamma_I(M)=0$. Thus we get an exact sequence $0\to M\xrightarrow{x} M\to M/xM\to 0$ where $x\in I$ is a regular element of M. It implies that $H^j_I(M/xM,N)\to (0:x)_{H^j_I(M,N)}\to 0$ is an exact sequence. Hence, as $\dim(M/xM)\leq 2$, we obtain

$$\dim\operatorname{Supp}\big(H^j_I(M,N)\big)\leq 2\quad \text{for all } j\geq 0.$$

For the case $\dim(N)\leq 3$, by similar arguments as in the proof of Theorem 1.3 we may reduce to the hypothesis that $\Gamma_I(N)=0$. Then by the exact sequence

$$H^j_I(M,N/yN)\to (0:y)_{H^{j+1}_I(M,N)}\to 0$$

with $y\in I$ is an $N-$regular element, we get that $\dim\operatorname{Supp}(H^j_I(M,N))\leq \dim\operatorname{Supp}(N/yN)\leq 2$ for all $j\geq 0$.

Therefore, for the rest of this proof, we need only to claim the weakly Laskerianness of $\operatorname{Ext}^u_R(R/I,H^v_I(M,N))$ for all $u,v\geq 0$ provided that

$$\dim\operatorname{Supp}\big(H^j_I(M,N)\big)\leq 2\quad \text{for all } j\geq 0.$$

Note that $H^j_I(M,N)\otimes_R\widehat{R}\cong H^j_{\widehat{I}}(\widehat{M},\widehat{N})$. Therefore, in view of [25, Lemma 2.1], we can assume that R is complete with $\mathfrak{m}$-adic topology. We now claim the weak Laskerianness of $\operatorname{Ext}^u_R(R/I,H^v_I(M,N))$ by way of contradiction. For any integers u, v, we set $K=\operatorname{Ext}^u_R(R/I,H^v_I(M,N))$. Assume that there exists a submodule T of K such that $\operatorname{Ass}(K/T)$ is an infinite set. Then there is a countably infinite subset $\{\mathfrak{p}_l\}_{l\in\mathbb{N}}$ of $\operatorname{Ass}(K/T)$ such that $\mathfrak{p}_l\neq\mathfrak{m}$ for all $l\in\mathbb{N}$. Let $S=R\setminus\bigcup_{l\in\mathbb{N}}\mathfrak{p}_l$. Then S is a multiplicative closed subset of R. Since $\{\mathfrak{p}_l\}_{l\in\mathbb{N}}\subseteq\operatorname{Ass}(K/T)$, we have $\{S^{-1}\mathfrak{p}_l\}_{l\in\mathbb{N}}\subseteq\operatorname{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$. Thus $\operatorname{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$ is an infinite set. On the other hand, as $\mathfrak{m} \not\subseteq \mathfrak{p}_l$ for all $l \in \mathbb{N}$, we get by [26, Lemma 3.2] that $\mathfrak{m} \not\subseteq \bigcup_{l\in\mathbb{N}} \mathfrak{p}_l$ and so that $\mathfrak{m} \cap S \neq \emptyset$. It implies that $\dim \operatorname{Supp}(H^j_{S^{-1}I}(S^{-1}M, S^{-1}N)) \leq 1$ for all $j \geq 0$. From this, we obtain by Corollary 4.3 that

$$S^{-1}K = \operatorname{Ext}^u_{S^{-1}R}\big(S^{-1}R/S^{-1}I, H^v_{S^{-1}I}(S^{-1}M, S^{-1}N)\big)$$

is finitely generated. It implies that $S^{-1}K/S^{-1}T$ is finitely generated. Hence $\operatorname{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$ is a finite set. On the other hand, by the hypothesis of T, the set $\operatorname{Ass}_{S^{-1}R}(S^{-1}K/S^{-1}T)$ is infinite. Hence we obtain a contradiction, and the claim follows. The last conclusion is clear. □

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