Modules that detect finite homological dimensions

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Abstract We study homological properties of test modules that are, in principle, modules that detect finite homological dimensions. The main outcome of our results is a generalization of a classical theorem of Auslander and Bridger: we prove that if a commutative Noetherian complete local ring R admits a test module of finite Gorenstein dimension, then R is Gorenstein.

1. Introduction

Throughout this paper, we assume that all rings are commutative Noetherian rings and all modules are finitely generated. Unless otherwise specified, R denotes a local ring with maximal ideal $\mathfrak{m}$ and residue field k. The aim of this paper is to study test modules.

DEFINITION 1.1

An R-module M is called a *test module* if all R-modules N with $\operatorname{Tor}^R_{\gg 0}(M,N)=0$ have finite projective dimension.

There are many examples of test modules with interesting consequences in the literature. For instance, it is well known that the residue field k of R is a test module (see [9, Section 1.3]). In general it requires highly nontrivial results to characterize all test modules, even over specific rings. One such result is due to Huneke and Wiegand [20, Theorem 1.9]: a test module over a singular hypersurface is nothing but a module of infinite projective dimension. This result was later obtained by Miller (see proof of [28, Theorem 1.1] in Section 1) and the third author [31, Corollary 7.2] by using different techniques.

In this paper we consider test modules discussed above in a broader context by studying their homological properties. We investigate when a module-finite algebra is a test module, and we prove the following.

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THEOREM 1.2

Let $R \to S$ be a finite local homomorphism of local rings. Assume that either

(1) *the ring S is regular or*
(2) *there exists a test S-module that has finite projective dimension over R.*

Then S is a test R-module.

It is remarkable that the existence of a test module of finite homological dimension characterizes the ring itself: if there exists a test module of finite projective dimension, then R is regular (see [9, Theorem 2.2.7]). Auslander and Bridger [3] proved that if the residue field k has finite *G-dimension* (see Definition 3.1), then R is Gorenstein. Corso–Huneke–Katz–Vasconcelos and Goto–Hayasaka, under mild technical conditions, generalized this classical theorem: Let I be an integrally closed $\mathfrak{m}$-primary ideal of a local ring R. Then I is a test module (see [14, Corollary 3.3]). If I has finite G-dimension and contains a non-zero-divisor of R, then R is Gorenstein (see [17, Theorem 1.1(2)]). We will obtain a generalization in this direction.

THEOREM 1.3

Let R be a homomorphic image of a Gorenstein local ring. If there exists a test module of finite G-dimension, then R is Gorenstein.

We also study the structure of test and nontest modules over complete intersections. Recall that R is called a *complete intersection* (resp., *hypersurface*) if its completion is a quotient of a regular local ring by a regular sequence (resp., regular element). Auslander and Bridger [3] introduced the notion of a *resolving subcategory* of the category $\operatorname{mod} R$ of finitely generated R-modules, which is a full subcategory containing free modules and closed under direct summands, extensions, and syzygies. For each R-module M the smallest resolving subcategory containing M is called the *resolving closure* of M. We will prove the following.

THEOREM 1.4

Let R be a complete intersection.

(1) *The test R-modules are precisely the R-modules of maximal complexity.*

(2) *The nontest R-modules form a resolving subcategory of* $\operatorname{mod} R$. *If it is written as the resolving closure of some module, then R is a hypersurface.*

The first assertion extends the result of Huneke and Wiegand stated above. The second says that nontest modules form a good subcategory but its structure is not simple in general.

The organization of this paper is as follows. In Section 2, we analyze basic properties of test modules. Theorem 1.2(1) and an extended version of Theorem 1.2(2) are shown in this section (Proposition 2.4 and Theorem 2.5). We also prove Theorem 1.4(1) in this section (Proposition 2.7). In Section 3, we study the existence of test modules of finite homological dimensions. We characterize test modules in terms of the vanishing of Ext, which yields a generalized version of Theorem 1.3 (Theorem 3.2). In Section 4, we develop categorical approaches for nontest modules. Theorem 1.4(2) is proved in this section (Corollaries 4.4 and 4.10).

2. Basic properties of test modules

In this section we analyze basic properties of test modules. We should note that modules akin to test modules were studied in the literature before (see, e.g., [21] and [27]).

First of all, we remark that our definition of a test module is different from the one defined by Ramras [29, Lemma 1.1] (see also [1] and [25]). He defined and studied test modules for projectivity in terms of the vanishing of a single Ext module. More precisely, an R-module M is called an *Ext-test module* (a test module in the sense of [25] and [29]) if every R-module P with $\mathrm{Ext}_R^1(P,M)=0$ is free. We record a few observations concerned with the test and Ext-test modules.

REMARK 2.1

(1) Nontrivial examples of test modules over arbitrary local rings are abundant: if $(R,\mathfrak{m})$ is a local ring and $M\in \mathsf{mod}\,R$, then there exists an integer $n>0$ such that $\mathfrak{m}^n M$ is a test module (see [2, 1.5(1), Definition 2.1, Propositions 2.2, and 2.3(5)] and [33, Lemma 2.4(b)].

(2) Test and Ext-test modules are different in general: by definition an Ext-test module has depth at most one, but a test module does not necessarily have this depth restriction.

(3) Ext-test modules are indeed test modules over complete intersections: this follows from the fact that the vanishing of $\mathrm{Ext}^{\gg 0}(-,-)$ is equivalent to the vanishing of $\mathrm{Tor}_{\gg 0}(-,-)$ over complete intersections by [5, Theorem 6.1].

(4) Test modules are indeed Ext-test modules over hypersurfaces that are either Artinian rings or one-dimensional domains: Assume that R is such a ring, assume that M is a test R-module, and assume that $\mathrm{Ext}_R^1(P,M)=0$ for some $P\in \mathsf{mod}\,R$. We may assume by [25, Theorem 1] that R is nonregular, whence $\mathrm{pd}_R\, M=\infty$. We see that $\mathrm{Ext}_R^{>0}(P,M)=0$ from [8, Corollary 3.5] and [10, Corollary 4.14]. This implies that P is free by [5, 5.12].

Next we prove that test modules behave well modulo non-zero-divisors. We denote by $\mathsf{T}(R)$ the full subcategory of $\mathsf{mod}\,R$ consisting of test modules, and by $\Omega^n M$ (or $\Omega_R^n M$ when necessary) the nth syzygy of an R-module M.

PROPOSITION 2.2

Let $(R,\mathfrak{m})$ be a local ring, and let M be an R-module. Let $x\in\mathfrak{m}$ be a non-zero-divisor on M. Then:

(i) *$M\in\mathsf{T}(R)$ if and only if $M/xM\in\mathsf{T}(R)$.*
(ii) *Assume further that x is a non-zero-divisor on R.*

(a) *If* $M/xM \in \mathsf{T}(R/xR)$*, then* $M \in \mathsf{T}(R)$*.*
(b) *If* $x \notin \mathfrak{m}^2$ *and* $M \in \mathsf{T}(R)$*, then* $M/xM \in \mathsf{T}(R/xR)$*.*

Proof
There is an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. Hence, $\mathrm{Tor}^R_{\gg 0}(M,N) = 0$ if and only if $\mathrm{Tor}^R_{\gg 0}(M/xM,N) = 0$ for all R-modules N. This proves (i). For the rest of the proof, we assume that x is a non-zero-divisor on both R and M. Assume that $M/xM \in \mathsf{T}(R/xR)$, and assume that $\mathrm{Tor}^R_{\gg 0}(M,N) = 0$ for some R-module N. Then, by the above exact sequence, $\mathrm{Tor}^R_{\gg 0}(M/xM,N') = 0$, where $N' := \Omega N$. Then, since x is a non-zero-divisor on both R and N', it follows that $\mathrm{Tor}^{R/xR}_{\gg 0}(M/xM,N'/xN') = 0$. Therefore, $\mathrm{pd}_{R/xR}(N'/xN') < \infty$. As $\mathrm{pd}_R(N') = \mathrm{pd}_{R/xR}(N'/xN')$ by [9, Lemma 1.3.5], this proves (ii)(a). Finally, assume that $x \notin \mathfrak{m}^2$ and $M \in \mathsf{T}(R)$. Suppose that $\mathrm{Tor}^{R/xR}_{\gg 0}(M/xM,T) = 0$ for some R/xR-module T. Then, as $\mathrm{Tor}^{R/xR}_i(M/xM,T) \cong \mathrm{Tor}^R_i(M,T)$ for all $i \geq 0$, we have $\mathrm{pd}_R T < \infty$. Since $x \notin \mathfrak{m}^2$, we have $\mathrm{pd}_{R/xR} T < \infty$ by [4, Proposition 3.3.5(1)]. This proves that $M/xM \in \mathsf{T}(R/xR)$ and hence finishes the proof. □

REMARK 2.3
The assumption that $x \notin \mathfrak{m}^2$ in Proposition 2.2(ii)(b) is necessary: Assume that $(R,\mathfrak{m})$ is a regular local ring, and assume that $0 \neq x \in \mathfrak{m}^2$. Set $S = R/xR$, and set $M = R$. Then $M \in \mathsf{mod}\, R = \mathsf{T}(R)$. However, since S is not regular, $M/xM \notin \mathsf{T}(R/xR)$.

Recall that a local homomorphism $f : R \to S$ of local rings is called a *finite local ring homomorphism* if S is a finitely generated R-module via f. Now we study the behavior of test modules under finite local ring homomorphisms. First, we point out that regular extensions of local rings are test modules.

PROPOSITION 2.4
Let $(R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ *be a finite local ring homomorphism. If* S *is regular, then* $S \in \mathsf{T}(R)$*.*

Proof
Suppose that $\mathrm{Tor}^R_{\gg 0}(M,S) = 0$ for some R-module M. There is an exact sequence $0 \to G_d \to G_{d-1} \to \cdots \to G_0 \to S/\mathfrak{m}S \to 0$ of S-modules with G_i being free. We see from this that $\mathrm{Tor}^R_{\gg 0}(M,S/\mathfrak{m}S) = 0$. Since $S/\mathfrak{m}S = k^{(n)}$ for some $n > 0$, we get $\mathrm{Tor}^R_{\gg 0}(M,k) = 0$. This implies that M has finite projective dimension, and hence $S \in \mathsf{T}(R)$. □

THEOREM 2.5
Let $R \to S$ *be a finite local ring homomorphism, and let* $M \in \mathsf{T}(S)$*. Assume that any* R*-module* X *with* $\mathrm{Tor}^R_{\gg 0}(S,X) = 0$ *satisfies* $\mathrm{Tor}^R_{\gg 0}(M,X) = 0$*. Then* $S \in \mathsf{T}(R)$*.*

Proof
Assume on the contrary that $S \notin \mathsf{T}(R)$. Then there exists an R-module L such that $\operatorname{Tor}^R_{\gg 0}(S,L)=0$ and $\operatorname{pd}_R L=\infty$. Hence, $\operatorname{Tor}^R_{>0}(S,T)=0$ for some $T=\Omega^n_R L$. Set $d=\operatorname{depth} S$, and let $N=\Omega^d_R T$. There is an exact sequence $0\to N\to F_{d-1}\to \cdots\to F_0\to T\to 0$ of R-modules, where each F_i is a free R-module. We deduce that

$$0\to N\otimes_R S\to F_{d-1}\otimes_R S\to\cdots\to F_0\otimes_R S\to T\otimes_R S\to 0$$

is exact. This implies that $\operatorname{depth}_S(N\otimes_R S)\geq d$. Using [11, A.4.20], we have isomorphisms

$$N\otimes^{\mathbf{L}}_R M\simeq N\otimes^{\mathbf{L}}_R (S\otimes^{\mathbf{L}}_S M)\simeq (N\otimes^{\mathbf{L}}_R S)\otimes^{\mathbf{L}}_S M\simeq (N\otimes_R S)\otimes^{\mathbf{L}}_S M$$

in the derived category of R, whose last isomorphism holds by $\operatorname{Tor}^R_{>0}(S,N)=0$. Thus, $\operatorname{Tor}^R_i(M,N)\cong\operatorname{Tor}^S_i(M,N\otimes_R S)$ for all $i>0$. By assumption, we have that $\operatorname{Tor}^R_{\gg 0}(M,N)=0$, whence $\operatorname{Tor}^S_{\gg 0}(M,N\otimes_R S)=0$. As $M\in\mathsf{T}(S)$, it holds that $\operatorname{pd}_S(N\otimes_R S)<\infty$. The Auslander–Buchsbaum formula shows that $N\otimes_R S$ is a free S-module. Let G be a minimal free resolution of N over R. Then $G\otimes_R S$ is a minimal free resolution of $N\otimes_R S$ over S. The uniqueness of minimal free resolutions implies that $G_i\otimes_R S=0$, and hence $G_i=0$, for all $i>0$. Therefore, N is a free R-module, and $\operatorname{pd}_R L<\infty$. This is a contradiction. □

We record a direct consequence of Theorem 2.5.

COROLLARY 2.6
Let $R\to S$ be a finite local ring homomorphism. If there is $M\in\mathsf{T}(S)$ with $\operatorname{pd}_R M<\infty$, then $S\in\mathsf{T}(R)$. In particular, if $S=R/(\underline{x})$ where $\underline{x}$ is a regular sequence on R, then the ring R is regular.

The category of test modules over complete intersection rings is determined in terms of *complexity*. Recall that the complexity $\operatorname{cx}_R M$ of an R-module M is the dimension of the *support variety* $V(M)$ associated to M (see [4] and [5] for details). In an earlier version of this article, we made use of Corollary 2.6 and [23, Theorem 1.3] and proved Proposition 2.7 below for complete intersection local rings which are *complete*. The authors are grateful to Petter Andreas Bergh for explaining a *completion-free* proof of this fact.

PROPOSITION 2.7
Let R be a local complete intersection. Then

$$\mathsf{T}(R)=\{M\in\operatorname{\mathsf{mod}} R\mid \operatorname{cx}_R M=\operatorname{codim} R\}.$$

Proof
($\supseteq$): This follows from [22, Corollary 1.2].

($\subseteq$): Put $c=\operatorname{codim} R$. Let $M\in\operatorname{\mathsf{mod}} R$ with $\operatorname{cx}_R(M)<c$. Then $\dim V(M)<c=\dim\overline{k}^c$, where $\overline{k}$ is the algebraic closure of k. This implies that there exists a

closed homogeneous variety W in $\overline{k}^c$ with $\dim W > 0$ and $W \cap V(M) = \{0\}$. Now, by [7, Corollary 2.3], there exists $N \in \mathsf{mod}\, R$ with $V(N) = W$. Since $\mathrm{cx}_R N = \dim V(N) = \dim W > 0$, the R-module N has infinite projective dimension. Recall that $V(N) \cap V(M) = \{0\}$; thus, we deduce from [5, Theorem IV] that $\mathrm{Tor}^R_{\gg 0}(M, N) = 0$. Consequently, $M \notin \mathsf{T}(R)$. □

Here are some consequences of Proposition 2.7; the first one is the result of Huneke and Wiegand [20, Theorem 1.9] discussed in the introduction.

COROLLARY 2.8

(i) (Huneke–Wiegand) *Let R be a hypersurface. Then an R-module M is in $\mathsf{T}(R)$ if and only if M has infinite projective dimension.*

(ii) *Let $R \to S$ be a finite local ring homomorphism of local complete intersections. If there exists an S-module M such that $\mathrm{cx}_R M = 0$ and $\mathrm{cx}_S M = \mathrm{codim}\, S$, then $\mathrm{cx}_R S = \mathrm{codim}\, R$. In other words, if M has minimal complexity over R and has maximal complexity over S, then S has maximal complexity over R.*

Proof

Only the second claim in (ii) requires a proof. By Proposition 2.7, $M \in \mathsf{T}(S)$. Corollary 2.6 implies that $S \in \mathsf{T}(R)$. Again by Proposition 2.7, we have $\mathrm{cx}_R S = \mathrm{codim}\, R$. □

REMARK 2.9

(1) Local rings over which all modules of infinite projective dimension are test modules are *not* necessarily hypersurfaces. A natural example of such a ring is a Golod ring that is not Gorenstein (e.g., $\mathbb{C}[[t^3, t^4, t^5]]$).

(2) Test modules do not behave well under localization. Let $(R, \mathfrak{m})$ be a 2-dimensional local hypersurface such that R_p is not regular for some $p \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$ (e.g., $R = \mathbb{C}[[x, y, z]]/(xy)$ and $p = (x, y)R$). Let $M = \Omega^2_R k$. Then $M \in \mathsf{T}(R)$ by Corollary 2.8(i). However, M_p is free over R_p, and hence $M_p \notin \mathsf{T}(R_p)$.

(3) Nontest modules do not behave well under localization. Let $(R, \mathfrak{m})$ be a complete intersection of codimension 2 such that R_p is regular for all $p \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$ (e.g., $R = \mathbb{C}[[x, y, z, v]]/(x^2 + y^2 + z^2 + v^2, x^3 + y^3 + z^3 + v^3)$). Let $M \in \mathsf{mod}\, R$ with $\mathrm{cx}_R M = 1$ (see [6, Example 5.7]). Then $M \notin \mathsf{T}(R)$ by Proposition 2.7. However, $M_p \in \mathsf{T}(R_p)$ for all $p \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$.

3. Homological dimensions of test modules

In this section we study the existence of test modules of finite homological dimensions. We start by recalling G_C-dimension; it is a homological invariant for modules, originally introduced by Golod [16], associated to a fixed semidualizing module C.

DEFINITION 3.1

Let C be a *semidualizing* R-module, that is, an R-module C such that the natural homomorphism $R \to \mathrm{Hom}_R(C,C)$ is an isomorphism and $\mathrm{Ext}_R^{>0}(C,C)=0$. An R-module X is called *totally C-reflexive* if the natural map $X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X,C),C)$ is an isomorphism and $\mathrm{Ext}_R^{>0}(X,C)=\mathrm{Ext}_R^{>0}(\mathrm{Hom}_R(X,C),C)=0$. The G_C*-dimension* of an R-module M, denoted by $\mathrm{G}_C\text{-}\mathrm{dim}_R\, M$, is defined as the infimum of the integers $n \geq 0$ such that there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$, where each X_i is totally C-reflexive.

A totally R-reflexive module is simply called *totally reflexive*. The G_R-dimension of M is nothing but the *Gorenstein dimension* (*G-dimension* for short) introduced by Auslander and Bridger [3] and simply denoted by $\mathrm{G}\text{-}\mathrm{dim}_R\, M$. A lot of studies on G-dimension have been done so far. The details are stated in the book [11] and the survey article [12].

Corso, Huneke, Katz, and Vasconcelos [14, Corollary 3.3] proved that integrally closed $\mathfrak{m}$-primary ideals can be used to test for finite projective dimension. More precisely, they proved that if $(R,\mathfrak{m})$ is a local ring and I is an integrally closed $\mathfrak{m}$-primary ideal of R, then $\mathrm{Tor}_n^R(R/I,N)=0$ if and only if $\mathrm{pd}_R\, N < n$. Goto and Hayasaka [17, Theorem 1.1] proved that if such an ideal I contains a non-zero-divisor of R and $\mathrm{G}\text{-}\mathrm{dim}_R\, I < \infty$, then R is Gorenstein. Thus, integrally closed $\mathfrak{m}$-primary ideals are test modules, and the existence of such ideals having finite G-dimension and positive grade forces the ring to be Gorenstein.

The main purpose of this section is to generalize this. More precisely, we would like to replace the ideal I considered with an arbitrary test module of finite G-dimension and deduce that R is Gorenstein. For this purpose, we introduce the following category:

$$\mathsf{EI}(R) = \big\{ M \in \mathsf{mod}\, R \mid \text{all } R\text{-modules } N \text{ with } \mathrm{Ext}_R^{\gg 0}(M,N)=0 \\ \text{satisfy } \mathrm{id}_R\, N < \infty \big\}.$$

The theorem below is the main result of this section. We refer the reader to [18, V], [11, A.8], and [24, Section 1] for details of dualizing complexes.

THEOREM 3.2

Let R be a commutative Noetherian ring (not necessarily local) admitting a dualizing complex. Then one has $\mathsf{T}(R)=\mathsf{EI}(R)$.

Proof

Let D be a dualizing complex of R. Let $M \in \mathsf{T}(R)$, and let $X \in \mathsf{mod}\, R$ such that $\mathrm{Ext}_R^{\gg 0}(M,X)=0$. By [11, A.4.24] we have an isomorphism in the derived category of R:

$$M \otimes_R^{\mathbf{L}} Y \simeq \mathbf{R}\mathrm{Hom}\big(\mathbf{R}\mathrm{Hom}(M,X),D\big),$$

where $Y := \mathbf{R}\mathrm{Hom}(X, D)$ is a homologically bounded complex. Since $\mathbf{R}\mathrm{Hom}(M, X)$ is homologically bounded, so is $M \otimes_R^{\mathbf{L}} Y$. Take a projective resolution of Y:

$$(F, \partial) = (\cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_{t+1} \xrightarrow{\partial_{t+1}} F_t \to 0).$$

As $H_{\gg 0}(F) = 0 = H_{\gg 0}(M \otimes_R F)$, we can choose an integer n such that the truncation $Q = (\cdots \to F_{n+1} \xrightarrow{\partial_{n+1}} F_n \to 0)$ of F is a projective resolution of $N := \operatorname{coker} \partial_{n+1}$ and such that $\mathrm{Tor}^R_{>0}(M, N) = 0$. Since $M \in \mathsf{T}(R)$, this implies that $\mathrm{pd}_R N < \infty$, and hence $\mathrm{pd}_R Q < \infty$ as $Q \simeq \Sigma^n N$. There is a short exact sequence of complexes (see [11, A.1.17])

$$0 \to P \to F \to Q \to 0,$$

where $P := (0 \to F_{n-1} \to \cdots \to F_t \to 0)$. Since $\mathrm{pd}_R P < \infty$, we have $\mathrm{pd}_R F < \infty$, and hence $\mathrm{pd}_R Y < \infty$. Thus, $\mathrm{id}_R \mathbf{R}\mathrm{Hom}(Y, D) < \infty$. We have an isomorphism $X \simeq \mathbf{R}\mathrm{Hom}(Y, D)$ by [18, V.2.1], which yields $\mathrm{id}_R X < \infty$. Therefore, $M \in \mathsf{EI}(R)$.

Conversely, let $M \in \mathsf{EI}(R)$. Let $X \in \operatorname{\mathsf{mod}} R$ such that $\mathrm{Tor}^R_{\gg 0}(M, X) = 0$. Similarly to the above, we can prove that $\mathrm{fd}_R X < \infty$; use the isomorphism $\mathbf{R}\mathrm{Hom}(M \otimes_R^{\mathbf{L}} X, D) \simeq \mathbf{R}\mathrm{Hom}(M, \mathbf{R}\mathrm{Hom}(X, D))$ (see [11, A.4.21]). We obtain $\mathrm{pd}_R X < \infty$ (see [24, Remark 1.6]). □

COROLLARY 3.3

Let R be a commutative Noetherian ring with a dualizing complex. Assume that there exist $M \in \mathsf{T}(R)$ and $N \in \operatorname{\mathsf{mod}} R$ with $\operatorname{Supp} N = \operatorname{Spec} R$ and $\mathrm{Ext}_R^{\gg 0}(M, N) = 0$. Then R is Cohen–Macaulay. If, moreover, $\mathrm{pd}_R N < \infty$, then R is Gorenstein.

Proof

Since $M \in \mathsf{EI}(R)$ by Theorem 3.2, it holds that $\mathrm{id}_R N < \infty$. For all $p \in \operatorname{Spec} R$, we have $N_p \neq 0$ and $\mathrm{id}_{R_p} N_p < \infty$. The theorem called Bass's conjecture [9, Corollaries 9.6.2 and 9.6.4, Remark 9.6.4(iii)] yields that R_p is Cohen–Macaulay, and so is R. Now assume that $\mathrm{pd}_R N < \infty$. Then N_p is a nonzero R_p-module of finite projective and injective dimensions for all $p \in \operatorname{Spec} R$. Hence, R_p is Gorenstein by [9, Exercise 3.1.25], and so is R. □

In the next corollary, under the hypothesis that the ring considered has a dualizing complex, we obtain a generalization of the result due to Corso–Huneke–Katz–Vasconcelos and Goto–Hayasaka, which accomplishes our main purpose of this section.

COROLLARY 3.4

Let R be a ring with a dualizing complex, and let M be a test module.

(i) *If $\mathrm{G}_C\text{-}\dim_R M < \infty$ for some semidualizing module C, then R is Cohen–Macaulay.*

(ii) *If $\mathrm{G}\text{-}\dim_R M < \infty$, then R is Gorenstein.*

Proof

(i) Since $\mathrm{Hom}_R(C,C) \cong R$, we have $\mathrm{Supp}\, C = \mathrm{Spec}\, R$. It is easy to see that $\mathrm{Ext}_R^{\gg 0}(M,C) = 0$. By Corollary 3.3, R is Cohen–Macaulay.

(ii) This follows from Corollary 3.3. □

The conclusion of Corollary 3.4 naturally raises the following question. We refer to [6] for details of the *complete intersection dimension* CI-dim_R.

QUESTION 3.5

Let R be a local ring. Let M be a test module with $\text{CI-dim}_R M < \infty$. Then must R be a complete intersection?

Corollary 2.6 more or less supports an affirmative answer to this question, but we do not know entirely. The difficulty we face here is that we do not know whether the property of being a test module is preserved under local flat extensions, even under completion.

Next we investigate the category $\mathsf{EI}(R)$ when R is Cohen–Macaulay. Set

$$\mathsf{EP}(R) = \{M \in \mathsf{mod}\, R \mid \text{all } R\text{-modules } N \text{ with } \mathrm{Ext}_R^{\gg 0}(N,M) = 0 \\ \text{satisfy } \mathrm{pd}_R N < \infty\}.$$

PROPOSITION 3.6

Let R be a Cohen–Macaulay local ring with a canonical module ω. Put $(-)^\dagger = \mathrm{Hom}_R(-,\omega)$. For each maximal Cohen–Macaulay R-module M one has

$$M \in \mathsf{T}(R) \Longleftrightarrow M^\dagger \in \mathsf{EP}(R).$$

Proof

There are isomorphisms in the derived category of R: $\mathbf{R}\mathrm{Hom}_R(X \otimes_R^{\mathbf{L}} M, \omega) \cong \mathbf{R}\mathrm{Hom}_R(X, M^\dagger)$ and $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, M^\dagger), \omega) \cong X \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M^\dagger, \omega) \cong X \otimes_R^{\mathbf{L}} M$ (see [11, A.4.21 and A.4.24]). These give rise to spectral sequences

$$ {}^1E_2^{p,q} = \mathrm{Ext}_R^p\big(\mathrm{Tor}_q^R(X,M),\omega\big) \Rightarrow H^{p+q} = \mathrm{Ext}_R^{p+q}(X,M^\dagger) \qquad \text{and}$$

$$ {}^2E_2^{p,q} = \mathrm{Ext}_R^p\big(\mathrm{Ext}_R^{-q}(X,M^\dagger),\omega\big) \Rightarrow H^{p+q} = \mathrm{Tor}_{-p-q}^R(X,M).$$

By using these sequences one can easily deduce the equivalence. □

Recall that a local ring R is called *G-regular* (see [30]) if $\text{G-dim}_R M = \mathrm{pd}_R M$ for all R-modules M. The above proposition gives a sufficient condition for a local ring to be G-regular in terms of test modules.

COROLLARY 3.7

Let R be a Cohen–Macaulay local ring with a canonical module ω. If ω is a test module, then R is G-regular.

Proof
Proposition 3.6 implies that $R \in \mathsf{EP}(R)$. Let M be an R-module. If $\text{G-dim}_R M < \infty$, then $\text{Ext}_R^{\gg 0}(M,R) = 0$, and hence $\text{pd}_R M < \infty$. This shows that $\text{G-dim}_R M = \text{pd}_R M$. □

4. Categorical approaches for nontest modules

In this section we continue studying the homological properties of test modules, with a special attention to the full subcategory $\mathsf{NT}(R)$ of $\mathsf{mod}\, R$ consisting of nontest modules:

$$\begin{aligned} \mathsf{NT}(R) &= \mathsf{T}(R)^{\mathrm{c}} \\ &= \bigl\{ M \in \mathsf{mod}\, R \bigm| \mathrm{Tor}^R_{\gg 0}(M,N) = 0 \text{ for some } R\text{-module } N \notin \mathsf{fpd}(R) \bigr\}, \end{aligned}$$

where $\mathsf{fpd}(R)$ denotes the full subcategory of $\mathsf{mod}\, R$ consisting of modules of finite projective dimension. Note that unless R is regular one has

$$\mathsf{NT}(R) \supsetneq \mathsf{fpd}(R).$$

We begin by considering the closedness of $\mathsf{NT}(R)$ under (finite) direct sums. First we confirm that it does not always hold.

EXAMPLE 4.1
Let k be a field, and put $R = k[x,y,z]/(x^2,y^2,z^2,yz)$. Then R is a non-Gorenstein local ring such that the cube of the maximal ideal is zero. Let $M = R/(x)$, and let $N = R/(y,z)$. Then $M,N \notin \mathsf{fpd}(R)$ and $\mathrm{Tor}^R_{>0}(M,N) = 0$; hence, $M,N \in \mathsf{NT}(R)$. Suppose that $\mathrm{Tor}^R_{\gg 0}(M \oplus N, L) = 0$ for some $L \in \mathsf{mod}\, R$. There are exact sequences $0 \to k^2 \to M \to k \to 0$ and $0 \to k \to N \to k \to 0$. The first sequence implies that $\mathrm{Tor}^R_{i+1}(k,L) \cong \mathrm{Tor}^R_i(k^2,L)$, and hence $\beta^R_{i+1}(L) = 2\beta^R_i(L)$ for $i \gg 0$. Similarly, it follows from the second sequence that $\beta^R_{i+1}(L) = \beta^R_i(L)$ for $i \gg 0$. Such equalities of Betti numbers of L can occur only when $\text{pd}_R L < \infty$. This proves that $M \oplus N$ is a test module, that is, that $M \oplus N \notin \mathsf{NT}(R)$.

In general we have the following result.

PROPOSITION 4.2
Let $(R,\mathfrak{m},k)$ be a non-Gorenstein local ring with $\mathfrak{m}^3 = 0$. Let M be a nonfree totally reflexive R-module. Then $M, E_R(k) \in \mathsf{NT}(R)$ and $M \oplus E_R(k) \notin \mathsf{NT}(R)$.

Proof
Note that M and $E := E_R(k)$ have infinite projective dimension. Corollary 3.7 implies that $E \in \mathsf{NT}(R)$. Setting $(-)^* = \mathrm{Hom}(-,R)$ and $(-)^\vee = \mathrm{Hom}(-,E)$, we deduce that

$$M \otimes^{\mathbf{L}}_R E \cong M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R,E) \cong \mathbf{R}\mathrm{Hom}_R\bigl(\mathbf{R}\mathrm{Hom}_R(M,R),E\bigr) \cong M^{*\vee}.$$

Here the second isomorphism follows from [11, A.4.24], and the total reflexivity of M yields the third isomorphism. Hence, we have $\mathrm{Tor}^R_{>0}(M,E)=0$. In particular, $M,E\in\mathsf{NT}(R)$.

Let $L\in\mathsf{mod}\,R$, and assume that $\mathrm{Tor}^R_{\gg 0}(M\oplus E,L)=0$. Then $\mathrm{Tor}^R_{\gg 0}(M,L)=\mathrm{Tor}^R_{\gg 0}(E,L)=0$. It follows from [19, Proposition 2.9] that $\mathrm{cx}_R\,L\leq 1$ and $\mathrm{cx}_R\,M=1$. One also sees that $\mathrm{cx}_R(M\otimes_R L)=\mathrm{cx}_R\,M+\mathrm{cx}_R\,L$. The complexity of an R-module can only be 0, 1, or ∞ by [26] (see also [13, 1.1]). Hence, $\mathrm{cx}_R\,L=0$; that is, $\mathrm{pd}_R\,L<\infty$. Thus, $M\oplus E\in\mathsf{T}(R)$. □

In fact, closure under direct sums is equivalent to closure under extensions.

PROPOSITION 4.3

Let R be a local ring. The following are equivalent:

(i) $\mathsf{NT}(R)$ *is closed under extensions.*
(ii) $\mathsf{NT}(R)$ *is closed under direct sums.*

Proof

Clearly, (i) implies (ii). Assume that (ii) holds. Let $0\to M\to U\to N\to 0$ be an exact sequence with $M,N\in\mathsf{NT}(R)$. Then there is an R-module X with $\mathrm{pd}_R\,X=\infty$ such that $\mathrm{Tor}^R_{\gg 0}(M\oplus N,X)=0$. Hence, $\mathrm{Tor}^R_{\gg 0}(U,X)=0$, and thus $U\in\mathsf{NT}(R)$. □

If R is a nonregular complete intersection, then $\mathsf{NT}(R)$ is closed under direct sums by Proposition 2.7. Hence, we deduce the following result from Proposition 4.3.

COROLLARY 4.4

If $\mathsf{NT}(R)$ is closed under direct sums, then $\mathsf{NT}(R)$ is a resolving subcategory of $\mathsf{mod}\,R$. Thus, $\mathsf{NT}(R)$ is resolving when R is a nonregular complete intersection.

We state here a conjecture of David A. Jorgensen (personal communication, 2011) (cf. Remark 4.8 below).

CONJECTURE 4.5

Let R be a local ring. Assume that $\mathsf{NT}(R)\neq\mathsf{fpd}(R)$. If $\mathsf{NT}(R)$ is closed under direct sums, then R is a complete intersection of codimension at least 2.

Next we investigate nontest modules in resolving subcategories. For $M\in\mathsf{mod}\,R$ we denote by $\mathsf{res}\,M$ the resolving closure of M. The full subcategory of $\mathsf{mod}\,R$ consisting of maximal Cohen–Macaulay modules is denoted by $\mathsf{CM}(R)$.

PROPOSITION 4.6

Let R be a Henselian local ring, and let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. Suppose that there are only finitely many nonisomorphic indecomposable modules *in* $\mathcal{X} \cap \mathsf{NT}(R)$*. Then* $\operatorname{Ext}_R^{\gg 0}(M,R) = 0$ *if and only if* $\operatorname{pd}_R M < \infty$ *for all* $M \in \mathcal{X} \cap \mathsf{NT}(R)$*. In particular,* $\operatorname{G-dim}_R M = \operatorname{pd}_R M$*.*

Proof
Take a module $M \in \mathcal{X} \cap \mathsf{NT}(R)$ with $\operatorname{Ext}_R^{\gg 0}(M,R) = 0$. Assume that $\operatorname{pd}_R M = \infty$. Given $X \in \operatorname{mod} R$, we set $\mathsf{NT}_X(R) = \{M \in \operatorname{mod} R \mid \operatorname{Tor}^R_{\gg 0}(M,X) = 0\}$. It is easy to see that $\mathsf{NT}_X(R)$ is a resolving subcategory of $\operatorname{mod} R$, and

$$\mathsf{NT}(R) = \bigcup_{\operatorname{pd} X = \infty} \mathsf{NT}_X(R).$$

There is an R-module X with $\operatorname{pd}_R X = \infty$ and $M \in \mathsf{NT}_X(R)$. Hence, $\operatorname{res} M \subseteq \mathsf{NT}_X(R) \subseteq \mathsf{NT}(R)$, and we have $\operatorname{res} M \subseteq \mathcal{X} \cap \mathsf{NT}(R)$. By assumption, there are only finitely many nonisomorphic indecomposable modules in $\operatorname{res} M$ (so $\operatorname{res} M$ is contravariantly finite). As $\Omega^d k \in \mathsf{T}(R)$, we have $\Omega^d k \notin \operatorname{res} M$. Hence, by [32, Theorem 1.4], R is Cohen–Macaulay and $\operatorname{res} M = \mathsf{CM}(R)$. This is a contradiction since $\Omega^d k \in \mathsf{CM}(R)$. Consequently, $\operatorname{pd}_R M < \infty$. □

COROLLARY 4.7
Let R be a Henselian local ring. Assume that there are only finitely many indecomposable R-modules (up to isomorphism) in $\mathsf{CM}(R) \cap \mathsf{NT}(R)$*.*

(i) *If* $M \in \mathsf{CM}(R) \cap \mathsf{NT}(R)$*, then either M is free or* $\operatorname{G-dim}_R M = \infty$*.*
(ii) *If R is Gorenstein and nonregular, then* $\mathsf{NT}(R) = \mathsf{fpd}(R)$*.*

Proof
The assertion (i) is immediate from Proposition 4.6: take $\mathcal{X} = \mathsf{CM}(R)$. As to (ii), let $Z \in \mathsf{NT}(R)$. Then $\Omega^d Z \in \mathsf{CM}(R) \cap \mathsf{NT}(R)$, where $d = \dim R$. By (i), $\Omega^d Z$ is free. □

REMARK 4.8
Recall that $\mathsf{NT}(R) = \mathsf{fpd}(R)$ if R is a hypersurface (Corollary 2.8(i)). In view of this fact, it is worth noting that a Henselian Gorenstein ring satisfying the hypotheses of Corollary 4.7 is not necessarily a hypersurface: Let k be a field, and let $R = k[x,y,z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$. Then R is an Artinian (hence, Henselian) Gorenstein local ring that is not a hypersurface. By [19, Theorem 3.1(2)] we see that $\mathsf{NT}(R) = \mathsf{fpd}(R)$ and that R is the unique indecomposable module in $\mathsf{CM}(R) \cap \mathsf{NT}(R)$.

It is not known whether there exist modules M over arbitrary local rings R such that $\operatorname{pd} M = \infty$ and $\operatorname{Tor}^R_{\gg 0}(M,M) = 0$. Our next result determines certain conditions, in terms of the category $\mathsf{NT}(R)$, for the existence of such modules.

PROPOSITION 4.9
Let R be a local ring. Assume that $\mathsf{NT}(R) \neq \mathsf{fpd}(R)$*, and assume that* $\mathsf{NT}(R) =$

$\mathsf{res}\, M$ *for some* $M \in \mathsf{mod}\, R$*. Then there is* $X \in \mathsf{NT}(R)$ *with* $\mathrm{pd}_R X = \infty$ *and* $\mathrm{Tor}^R_{\gg 0}(X,X) = 0$.

Proof
If $\mathrm{pd}_R M < \infty$, then $\mathsf{res}\, M \subseteq \mathsf{fpd}(R)$; hence, $\mathsf{NT}(R) = \mathsf{fpd}(R)$. This contradiction shows $\mathrm{pd}_R M = \infty$. Since $M \in \mathsf{NT}(R)$, we have $M \in \mathsf{NT}_X(R)$ for some module X with $\mathrm{pd}_R X = \infty$. As $\mathrm{pd}_R M = \infty$, we see that $X \in \mathsf{NT}(R)$. It holds that $\mathsf{NT}(R) = \mathsf{res}\, M \subseteq \mathsf{NT}_X(R)$, whence $\mathsf{NT}(R) = \mathsf{NT}_X(R)$. Thus, $X \in \mathsf{NT}_X(R)$, so that $\mathrm{Tor}^R_{\gg 0}(X,X) = 0$. □

Using the above proposition, we obtain an interesting property of $\mathsf{NT}(R)$.

COROLLARY 4.10
Let R *be a local complete intersection of codimension at least* 2*. Then* $\mathsf{NT}(R) \neq \mathsf{res}\, M$ *for all* R*-modules* M*.*

Proof
It follows from [6, Example 5.7] that there is an R-module N of complexity 1. By Proposition 2.7, N is a nontest module. Therefore, if $\mathsf{NT}(R) = \mathsf{res}\, M$ for some $M \in \mathsf{mod}\, R$, then Proposition 4.9 implies that there exists $X \in \mathsf{NT}(R)$ with $\mathrm{pd}_R X = \infty$ and $\mathrm{Tor}^R_{\gg 0}(X,X) = 0$. Since R is a complete intersection, such an R-module X cannot exist by [22, Corollary 1.2]. □

REMARK 4.11
The assumption of Corollary 4.10 on the codimension cannot be weakened. Indeed, let R be a hypersurface (i.e., $\mathrm{codim}\, R \le 1$). Then $\mathsf{NT}(R) = \mathsf{fpd}(R)$ by Corollary 2.8(i). So, if R is reduced of dimension 1, then $\mathsf{NT}(R)$ coincides with the resolving closure of the Auslander transpose of k by [15, Theorem 2.1].

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