# The cancellation problem over Noetherian one-dimensional domains

M'hammed El Kahoui and Mustapha Ouali

**Abstract** Let $R$ be a commutative Noetherian one-dimensional domain containing $\mathbb{Q}$. In this paper we prove that if an $R$-algebra $A$ is such that $A^{[n]} \cong_R R^{[n+2]}$, for some $n \geq 1$, then $A \cong_R R^{[2]}$. In terms of affine fibrations this means that every stably trivial $\mathbb{A}^2$-fibration over $R$ is actually trivial. On the other hand, it is known that this result does not hold in general if $R$ has dimension at least two or if $R$ does not contain $\mathbb{Q}$.

## 1. Introduction

Throughout, all rings are assumed to be commutative with unity. Given a ring $R$ and a positive integer $m$ we denote by $R^{[m]}$ the polynomial $R$-algebra in $m$ variables (by convention we let $R^{[0]} = R$). By a *coordinate system* of $R^{[m]}$ we mean a list $x = x_1, \ldots, x_m$ of $m$ polynomials which generates $R^{[m]}$ as an $R$-algebra.

Let us recall the following fundamental problem of affine algebraic geometry, known as the Zariski cancellation problem (see, e.g., [23], [21]).

PROBLEM 1 (CANCELLATION PROBLEM)

Let $K$ be a field, and let $(m, n)$ be a pair of positive integers. Given a $K$-algebra $A$ such that $A^{[n]} \cong_K K^{[m+n]}$, does it follow that $A \cong_K K^{[m]}$?

We will say that the cancellation property holds for $(m, n)$ if the above problem has a positive answer for every field $K$.

The fact that the cancellation property holds for $(1, n)$ follows essentially from the results of Abhyankar, Heinzer, and Eakin [1]. More generally, it follows from the results of Hamann [17] that the $(1, n)$-cancellation property still holds if instead of fields one considers Noetherian rings containing $\mathbb{Q}$. On the other hand, from the results of Miyanishi and Sugie [24], Fujita [16], and Kambayashi [19] it follows that the cancellation property holds for $(2, n)$ in the case of fields of charateristic zero. The case of algebraically closed fields of positive characteristic was proved by Russell in [25]. It was also proved by Derksen, van den Essen, and van Rossum [8] that the $(2, n)$-cancellation property holds true when fields are

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replaced by Dedekind domains containing $\mathbb{Q}$. For $m \geq 3$, the problem is to our knowledge still open, but a candidate counterexample in positive characteristic was given by Asanuma in [2].

The main result of this paper is that the $(2,n)$-cancellation property holds over an arbitrary Noetherian one-dimensional domain $R$ containing $\mathbb{Q}$. The assumptions that $R$ is one-dimensional and contains $\mathbb{Q}$ cannot be dropped. Indeed, a classical example of Hochster [18] shows that this result does not hold in general if $R$ has dimension at least two. Another classical example of Asanuma [2, Theorem 5.1] shows that the result does not hold in general if $R$ is a one-dimensional domain which does not contain $\mathbb{Q}$.

The paper is organized as follows. In Section 2 we recall the results of affine fibrations theory to be used in this paper. Section 3 is devoted to the proof of the main result of this paper. As a consequence of the main result we show that if $A$ is an $\mathbb{A}^2$-fibration over $R = K^{[2]}$, where $K$ is a field of characteristic zero, then $A/pA \cong_{R/pR} (R/pR)^{[2]}$ for every prime polynomial $p \in R$. This answers in particular a question raised by Vénéreau (see [27] and [15, Problem 13]) concerning the polynomial $v_1 = y + x[xz + y(yu + z^2)]$ in $\mathbb{C}[x,y,z,u]$, a candidate counterexample to several open problems in affine algebraic geometry.

## 2. Affine fibrations

In this section we recall the results of affine fibrations theory to be used in this paper. Given a ring $R$ and $\mathfrak{p} \in \operatorname{Spec} R$, the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is denoted by $K(\mathfrak{p})$. Given an $R$-module $M$ we let $\operatorname{Sym}_R(M)$ be the symmetric algebra of $M$. For an $R$-algebra $A$ we let $\Omega_{A/R}$ (resp., $\operatorname{Der}_R(A)$) be the $A$-module of Kähler differentials of $A$ over $R$ (resp., $R$-derivations of $A$).

DEFINITION 2.1

Given $m \geq 0$, an $R$-algebra $A$ is said to be an $\mathbb{A}^m$-fibration over $R$ if it satisfies the following properties.

i. $A$ is finitely generated as an $R$-algebra.
ii. $A$ is flat as an $R$-module.
iii. For every $\mathfrak{p} \in \operatorname{Spec} R$ we have $K(\mathfrak{p}) \otimes_R A \cong_{K(\mathfrak{p})} K(\mathfrak{p})^{[m]}$.

From the property (iii) one easily deduces that the morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} R$, induced by the homomorphism $R \longrightarrow A$, is surjective. This property together with the flatness assumption implies that $A$ is faithfully flat over $R$. In particular, the homomorphism $R \longrightarrow A$ is injective, and hence we can view $R$ as a subring of $A$.

An $\mathbb{A}^m$-fibration $A$ over $R$ is said to be *trivial* if $A \cong_R R^{[m]}$. The fibration is said to be *stably trivial* if $A^{[n]} \cong_R R^{[m+n]}$ for some $n \geq 0$.

The following fundamental result due to Asanuma concerns the stable structure of $\mathbb{A}^m$-fibrations (see [2, Theorem 3.4]).

THEOREM 2.2

*Let $R$ be a Noetherian ring, and let $A$ be an $\mathbb{A}^m$-fibration over $R$. Then $\Omega_{A/R}$ is a finitely generated projective $A$-module of rank $m$. Moreover, $A$ is up to isomorphism an $R$-subalgebra of $R^{[n]}$ for some $n$ such that*

$$A^{[n]} \cong \mathrm{Sym}_{R^{[n]}}(R^{[n]} \otimes_A \Omega_{A/R})$$

*as $R$-algebras.*

As a direct consequence of Theorem 2.2, if an $\mathbb{A}^m$-fibration $A$ over a Noetherian ring $R$ is such that $\Omega_{A/R}$ is a free $A$-module, then $A$ is stably trivial. Another consequence of Theorem 2.2 (see [2, Corollary 3.5]) is that if $A$ is an $\mathbb{A}^m$-fibration over a regular ring $R$, then there exists $n \geq 0$ and a rank $m$ finitely generated projective $R$-module $M$ such that $A^{[n]} \cong_R \mathrm{Sym}_R(M)^{[n]}$. In particular, if $R$ is a polynomial ring over a field, then by the Quillen–Suslin theorem $A$ is stably trivial.

## 2.1. A criterion for an $\mathbb{A}^1$-fibration to be trivial

In this subsection we give a criterion for an $\mathbb{A}^1$-fibration, over an arbitrary Noetherian domain containing $\mathbb{Q}$, to be trivial. For this, we need to recall the following cancellation result due to Hamann [17].

THEOREM 2.3

*Let $R$ be a Noetherian ring containing $\mathbb{Q}$. Then for every $R$-algebra $A$ such that $A^{[n]} \cong_R R^{[n+1]}$, for some $n \geq 1$, we have $A \cong_R R^{[1]}$.*

Combining Theorems 2.2 and 2.3 yields the following result (see [5, Theorem 3.4]).

THEOREM 2.4

*Let $R$ be a Noetherian ring containing $\mathbb{Q}$, and let $A$ be an $\mathbb{A}^1$-fibration over $R$. Then $A$ is trivial over $R$ if and only if $\Omega_{A/R}$ is a free $A$-module.*

Recall that an $R$-derivation $\xi \in \mathrm{Der}_R(A)$ is said to be *fixed point free* if its image generates the unit ideal of $A$.

Let $A$ be an $\mathbb{A}^m$-fibration over $R$. By Theorem 2.2 the $A$-module $\Omega_{A/R}$ is finitely generated and projective. From the well-known fact that finitely generated projective modules are reflexive it follows that the freeness of $\Omega_{A/R}$ is equivalent to the freeness of its dual $\mathrm{Der}_R(A)$. As a consequence, we have the following result.

COROLLARY 2.5

*Let $R$ be a Noetherian domain containing $\mathbb{Q}$, and let $A$ be an $\mathbb{A}^1$-fibration over $R$. Then $A$ is trivial over $R$ if and only if there exists $\xi \in \mathrm{Der}_R(A)$ which is fixed point free.*

*Proof*
Clearly, if $A$ is trivial over $R$, say, $A = R[v] = R^{[1]}$, then the $R$-derivation of $A$ defined by $\xi(v) = 1$ is fixed point free. Conversely, let $\xi \in \mathrm{Der}_R(A)$ be fixed point free, and let $\xi_1 \in \mathrm{Der}_R(A)$. Let $K$ be the quotient field of $R$, and let $S = R \setminus \{0\}$. Since $A$ is an $\mathbb{A}^1$-fibration over $R$ and $R$ is a domain we have $K \otimes_R A \cong_K K^{[1]}$. Since, moreover, $K \otimes_R A \cong_K A_S$ we have $A_S \cong_K K^{[1]}$, and hence we can find $v \in A$ transcendental over $R$ such that $A_S = K[v]$. Thus, if we let $\xi(v) = \alpha$ and $\xi_1(v) = \beta$, then $\alpha, \beta \in A$ and we have $\alpha\xi_1 = \beta\xi$. The assumption that $\xi$ is fixed point free implies that there exist $a_1, \ldots, a_r \in A$ and $u_1, \ldots, u_r \in A$ such that $\sum u_i \xi(a_i) = 1$. This yields $\alpha \sum u_i \xi_1(a_i) = \beta$ and hence $\xi_1 = \beta_1 \xi$, where $\beta_1 = \sum u_i \xi_1(a_1)$. Thus, $\mathrm{Der}_R(A) = A\xi$, and so it is free. Since on the other hand $\Omega_{A/R}$ is reflexive and its dual $\mathrm{Der}_R(A)$ is free the $A$-module $\Omega_{A/R}$ is free as well. By Theorem 2.4, $A$ is trivial over $R$. □

We will also need the well-known fact that every $\mathbb{A}^1$-fibration over a principal ideal domain (PID) is trivial. In fact, much more general results can be found in the literature (see, e.g., [20], [9], [6], [10]), but they will not be needed for our purpose.

## 2.2. Some results on $\mathbb{A}^2$-fibrations

A well known result due to Sathaye [26, Theorem 1] asserts that every $\mathbb{A}^2$-fibration over a discrete valuation ring containing $\mathbb{Q}$ is trivial. This result together with the results of Bass, Connell, and Wright [4] implies that every $\mathbb{A}^2$-fibration over a PID containing $\mathbb{Q}$ is trivial (see [5, Corollary 4.8]). For an arbitrary Noetherian one-dimensional domain containing $\mathbb{Q}$, Asanuma and Bhatwadekar proved in [3, Theorem 3.8] the following generalization of this result.

THEOREM 2.6
*Let $R$ be a Noetherian one-dimensional domain containing $\mathbb{Q}$, and let $A$ be an $\mathbb{A}^2$-fibration over $R$. Then there exists $u \in A$ transcendental over $R$ such that $A$ is an $\mathbb{A}^1$-fibration over $R[u]$.*

If in addition to the assumptions of Theorem 2.6 the ring $R$ is seminormal, then $A \cong_R \mathrm{Sym}_R(M)^{[1]}$, where $M$ is a finitely generated projective $R$-module of rank one (see [3, Corollary 3.9]).

Based on the fact that the cancellation property holds for $(2, n)$ in characteristic zero, Freudenburg proved in [14, Corollary 2.2] the following result.

THEOREM 2.7
*Let $R$ be a ring containing $\mathbb{Q}$, and let $A$ be an $R$-algebra such that $A^{[n]} \cong_R R^{[n+2]}$ for some $n \geq 1$. Then $A$ is an $\mathbb{A}^2$-fibration over $R$.*

REMARK 2.8
In [14, Theorem 3.1], Freudenburg proved a result for $\mathbb{A}^2$-fibrations over polynomial rings similar to Corollary 2.5. Given a field $K$ of characteristic zero, the

result in question states that an $\mathbb{A}^2$-fibration $A$ over $R = K^{[n]}$ is trivial if and only if there exists a locally nilpotent $R$-derivation $\xi$ of $A$ with a slice. Then in [14, Question 2], the author asks whether the condition that $\xi$ has a slice can be weakened to the condition that $\xi$ is fixed point free. In a recent paper [11], we proved that this question has an affirmative answer in the more general setting where $R$ is a factorial regular ring containing $\mathbb{Q}$.

## 3. The $(2, n)$-cancellation problem over Noetherian one-dimensional domains containing $\mathbb{Q}$

Let $R$ be a ring containing $\mathbb{Q}$, and let $A$ be an $R$-algebra. It is proved in [8] that if $R$ is a Dedekind domain and $A^{[n]} \cong_R R^{[n+2]}$, for some $n \geq 1$, then $A \cong_R R^{[2]}$ (see also [12, Theorem 4.5]). In this section, we show that the same result holds true over an arbitrary Noetherian one-dimensional domain $R$ containing $\mathbb{Q}$.

THEOREM 3.1

*Let $R$ be a Noetherian one-dimensional domain containing $\mathbb{Q}$. Then for every $R$-algebra $A$ such that $A^{[n]} \cong_R R^{[n+2]}$, for some $n \geq 1$, we have $A \cong_R R^{[2]}$.*

*Proof*

Let $x = x_1, \dots, x_n$ be a list of indeterminates over $A$, and let $y = y_1, \dots, y_{n+2}$ be a list of algebraically independent elements of $A[x]$ over $R$ such that $A[x] = R[y]$.

From Theorem 2.7 we deduce that $A$ is an $\mathbb{A}^2$-fibration over $R$. Then by Theorem 2.6 there exists $u \in A$ such that $A$ is an $\mathbb{A}^1$-fibration over $R[u]$. To prove that $A$ is trivial over $R[u]$ it suffices by Corollary 2.5 to find a fixed point free $R[u]$-derivation of $A$.

Let us consider the $R[u, x]$-derivation of $A[x] = R[y]$ defined for every $f \in R[y]$ by

$$\xi(f) = \det \mathrm{Jac}_y(u, x, f).$$

Let us first prove that $\xi(A) \subseteq A$. Let $K$ be the quotient field of $R$, and let $S = R \setminus \{0\}$. Since $A$ is an $\mathbb{A}^1$-fibration over $R[u]$ the localization $A_S$ is an $\mathbb{A}^1$-fibration over $R[u]_S = K[u]$. Since, moreover, $K[u]$ is a PID, $A_S$ is trivial over $K[u]$, and so there exists $v \in A$ such that $A_S = K[u][v] = K[u]^{[1]}$. This gives $A[x]_S = K[u, x, v] = K[y]$, and hence $u, x, v$ is a coordinate system of $K[y]$. From this fact it follows that $\xi(v) \in K \setminus \{0\}$. On the other hand, since $v \in A$ we have $\xi(v) \in A[x]$, and so $\xi(v) \in A[x] \cap (K \setminus \{0\}) = R \setminus \{0\}$ by the faithful flatness of $A$ over $R$. Now, if $a \in A$ we can write $a = p(u, v)$ in $A_S = K[u, v]$, and applying $\xi$ to $a$ we get $\xi(a) = \partial_v p(u, v)\xi(v) \in A_S \cap A[x] = A$. Thus, if we let $\xi_0$ be the restriction of $\xi$ to $A$, then $\xi_0 \in \mathrm{Der}_{R[u]}(A)$ and $\xi$ is nothing but the extension of $\xi_0$ to $A[x]$ obtained by letting $\xi(x_i) = 0$. It follows that $\xi(A[x])$ and $\xi_0(A)$ generate the same ideal of $A[x]$, and since $A[x]$ is faithfully flat over $A$, the derivation $\xi_0$ is fixed point free if and only if $\xi$ is fixed point free.

Assume towards contradiction that $\xi$ is not fixed point free. Let $\mathfrak{m}$ be a maximal ideal of $A[x]$ that contains $\xi(A[x])$, and let $\mathfrak{m}_0 = R \cap \mathfrak{m}$. Since $\xi(v) \in \xi(A[x]) \subset \mathfrak{m}$ and $\xi(v) \in R \setminus \{0\}$ we have $\mathfrak{m}_0 \neq (0)$, and hence $\mathfrak{m}_0$ is a maximal ideal since $R$ is assumed to be a one-dimensional domain.

Notice that the inclusion homomorphisms $R \hookrightarrow R[u] \hookrightarrow A \hookrightarrow A[x]$ are faithfully flat. In particular, we have the following commutative diagram, where the $\pi_i$'s and $\pi$ stand for the canonical projections:

$$\begin{array}{ccccccc}
R & \hookrightarrow & R[u] & \hookrightarrow & A & \hookrightarrow & A[x] \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi} \\
R/\mathfrak{m}_0 & \hookrightarrow & R[u]/\mathfrak{m}_0R[u] & \hookrightarrow & A/\mathfrak{m}_0A & \hookrightarrow & A[x]/\mathfrak{m}_0A[x]
\end{array}$$

Let $\overline{\xi}$ be the derivation of $A[x]/\mathfrak{m}_0A[x]$ induced by $\xi$. Then we have $\overline{\xi} \circ \pi = \pi \circ \xi$. On the other hand, we have

$$A[x]/\mathfrak{m}_0A[x] = R[y]/\mathfrak{m}_0R[y] \cong_{R/\mathfrak{m}_0} (R/\mathfrak{m}_0)^{[n+2]},$$

and $\pi(y) = \pi(y_1), \ldots, \pi(y_{n+2})$ is a coordinate system of $A[x]/\mathfrak{m}_0A[x]$ over $R/\mathfrak{m}_0$. Since, moreover, $\xi = \det \operatorname{Jac}_y(u, x, -)$ the derivation $\overline{\xi}$ is nothing but the Jacobian derivation $\det \operatorname{Jac}_{\pi(y)}(\pi(u), \pi(x), -)$.

Now we show that $\pi(u), \pi(x)$ can be extended to a coordinate system of $A[x]/\mathfrak{m}_0A[x]$ over $R/\mathfrak{m}_0$. Since $A$ is an $\mathbb{A}^1$-fibration over $R[u]$, it follows that $A/\mathfrak{m}_0A$ is an $\mathbb{A}^1$-fibration over $R[u]/\mathfrak{m}_0R[u] \cong_{R/\mathfrak{m}_0} (R/\mathfrak{m}_0)^{[1]}$. The fact that $R/\mathfrak{m}_0$ is a field then implies that $A/\mathfrak{m}_0A$ is trivial over $R[u]/\mathfrak{m}_0R[u]$, and so we can find $w \in A$ such that $\pi_3(w)$ generates $A/\mathfrak{m}_0A$ as an $(R[u]/\mathfrak{m}_0R[u])$-algebra. As a consequence, $\pi_3(u), \pi_3(w)$ generate $A/\mathfrak{m}_0A \cong_{R/\mathfrak{m}_0} (R/\mathfrak{m}_0)^{[2]}$ as an $R/\mathfrak{m}_0$-algebra. On the other hand, we have $A[x]/\mathfrak{m}_0A[x] \cong_{A/\mathfrak{m}_0A} (A/\mathfrak{m}_0A)^{[n]}$, and the system $\pi(x)$ generates $A[x]/\mathfrak{m}_0A[x]$ over $A/\mathfrak{m}_0A$. It then follows that $\pi(u), \pi(w), \pi(x)$ is a generating, and hence a coordinate, system of $A[x]/\mathfrak{m}_0A[x]$ over $R/\mathfrak{m}_0$. This shows that $\overline{\xi}(\pi(w)) = \pi(\xi(w))$ is a unit in $A[x]/\mathfrak{m}_0A[x]$, and hence there exist $\alpha \in A[x]$ and $f \in \mathfrak{m}_0A[x]$ such that $\alpha\xi(w) = 1 + f$. Since $\alpha\xi(w) \in \mathfrak{m}$ and $f \in \mathfrak{m}_0A[x] \subseteq \mathfrak{m}$ we get $1 \in \mathfrak{m}$, which contradicts the fact that $\mathfrak{m}$ is a proper ideal of $A[x]$. □

REMARK 3.2

If in addition to the assumptions of Theorem 3.1 the ring $R$ is seminormal we can supply a much shorter proof. Indeed, by Theorem 2.7, $A$ is an $\mathbb{A}^2$-fibration over $R$. On the other hand, from a corollary of Theorem 2.6, see [3, Corollary 3.9], we have $A \cong_R \operatorname{Sym}_R(M)^{[1]}$ for some finitely generated projective $R$-module $M$ of rank one. This gives $\operatorname{Sym}_R(M)^{[n+1]} \cong_R R^{[n+2]}$ and then $\operatorname{Sym}_R(M) \cong_R R^{[1]}$ by Theorem 2.3. Since $A \cong_R \operatorname{Sym}_R(M)^{[1]}$ we finally get $A \cong_R R^{[2]}$.

REMARK 3.3

From Theorem 2.7 it follows that Theorem 3.1 is equivalent to saying that every stably trivial $\mathbb{A}^2$-fibration over a Noetherian one-dimensional domain containing $\mathbb{Q}$ is trivial.

As a direct consequence of Theorem 3.1 we have the following result.

COROLLARY 3.4

*Let $R$ be a Noetherian domain containing $\mathbb{Q}$, and let $A$ be a stably trivial $\mathbb{A}^2$-fibration over $R$. Then for every prime ideal $\mathfrak{p}$ of $R$ such that $R/\mathfrak{p}$ is one-dimensional we have*

$$A/\mathfrak{p}A \cong_{R/\mathfrak{p}} (R/\mathfrak{p})^{[2]}.$$

*Proof*

Since $A$ is a stably trivial $\mathbb{A}^2$-fibration over $R$, it follows that $A/\mathfrak{p}A$ is a stably trivial $\mathbb{A}^2$-fibration over $R/\mathfrak{p}$. The claimed result then follows from Theorem 3.1 since $R/\mathfrak{p}$ is assumed to be one-dimensional. □

Given a field $K$ of characteristic zero, it is still an open problem whether every $\mathbb{A}^2$-fibration over the polynomial ring $K^{[2]}$ is trivial. The following result gives a property of such fibrations.

COROLLARY 3.5

*Let $K$ be a field of characteristic zero, and let $A$ be an $\mathbb{A}^2$-fibration over $R = K^{[2]}$. Then for every prime polynomial $p$ of $R$ we have*

$$A/pA \cong_{R/pR} (R/pR)^{[2]}.$$

*Proof*

As noticed in the paragraph after Theorem 2.2, it follows from [2, Corollary 3.5] and the Quillen–Suslin Theorem that every $\mathbb{A}^m$-fibration over a polynomial ring over a field is stably trivial. Thus, $A$ is a stably trivial $\mathbb{A}^2$-fibration over $K^{[2]}$, and the claimed result follows from Corollary 3.4. □

The above corollary answers in particular a question raised by Vénéreau (see [15, Problem 13]) concerning the polynomial $v_1 = y + x[xz + y(yu + z^2)]$, a candidate counterexample to several open problems in affine algebraic geometry. More general methods to construct Vénéreau-type polynomials can be found in [7] and [22].

Let $R = \mathbb{C}[x, v_1] = \mathbb{C}^{[2]}$, and let $A = \mathbb{C}[x, y, z, u] = \mathbb{C}^{[4]}$. It is proved in [27] that $A$ is an $\mathbb{A}^2$-fibration over $R$. But to our knowledge it is still an open question whether $A$ is trivial over $R$. Clearly, if for some prime polynomial $p \in R$ the fibration $A/pA$ is not trivial over $R/pR$, then $A$ is not a trivial fibration over $R$. Vénéreau's question [15, Problem 13] was then whether this is the case for some prime polynomial $p \in R$. He also proposed $p = x^2 - v_1^3$ as an example for which no answer was known. In [13] van den Essen, Maubach and Vénéreau obtained that $A/(x^2 - v_1^3)A$ is trivial over $R/(x^2 - v_1^3)R$ as a consequence of their main result. But in fact, Corollary 3.5 shows that this holds for every prime polynomial $p \in \mathbb{C}[x, v_1]$.

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*El Kahoui*: Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, P.O. Box 2390, Marrakesh, Morocco; elkahoui@uca.ma

*Ouali*: Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, P.O. Box 2390, Marrakesh, Morocco; mustouali@gmail.com