# Quantum unipotent subgroup and dual canonical basis

Yoshiyuki Kimura

**Abstract** In a series of works, Geiss, Leclerc, and Schröer defined the cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup, associated with a Weyl group element $w$. And they proved that cluster monomials are contained in Lusztig's *dual semicanonical basis* $\mathcal{S}^*$. We give a setup for the quantization of their results and propose a conjecture that relates the quantum cluster algebras in Berenstein and Zelevinsky's work to the *dual canonical basis* $\mathbf{B}^{\mathrm{up}}$. In particular, we prove that the quantum analogue $\mathcal{O}_q[N(w)]$ of $\mathbb{C}[N(w)]$ has the induced basis from $\mathbf{B}^{\mathrm{up}}$, which contains quantum flag minors and satisfies a factorization property with respect to the "$q$-center" of $\mathcal{O}_q[N(w)]$. This generalizes Caldero's results from finite type to an arbitrary symmetrizable Kac–Moody Lie algebra.

**Contents**



## 1. Introduction

### 1.1. The canonical basis $\mathbf{B}$ and the dual canonical basis $\mathbf{B}^{\mathrm{up}}$

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody Lie algebra, let $\mathbf{U}_q(\mathfrak{g})$ be its associated quantized enveloping algebra, and let $\mathbf{U}_q^-(\mathfrak{g})$ be its negative part. In [40], Lusztig constructed the canonical basis $\mathbf{B}$ of $\mathbf{U}_q^-(\mathfrak{g})$ by a geometric method when $\mathfrak{g}$ is symmetric. In [27], Kashiwara constructed the (lower) global basis $G^{\mathrm{low}}(\mathscr{B}(\infty))$ by a purely algebraic method. Grojnowski and Lusztig [25] showed that the two bases coincide when $\mathfrak{g}$ is symmetric. We call the basis the *canonical basis*. There are two remarkable properties of the canonical basis: one is the positivity of

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structure constants of multiplication and comultiplication, and another is Kashiwara's crystal structure $\mathscr{B}(\infty)$, which is a combinatorial machinery useful for applications to representation theory, such as tensor product decomposition.

Since $\mathbf{U}_q^-(\mathfrak{g})$ has a natural pairing which makes it into a (twisted) self-dual bialgebra, we consider the dual basis $\mathbf{B}^{\mathrm{up}}$ of the canonical basis in $\mathbf{U}_q^-(\mathfrak{g})$. We call it the *dual canonical basis.*

### 1.2. Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky [16] and intensively studied also with Berenstein (see [17], [2], [18]) with an aim of providing a concrete and combinatorial setting for the study of Lusztig's (dual) canonical basis and total positivity. Quantum cluster algebras were also introduced by Berenstein and Zelevinsky [4] and Fock and Goncharov [13]–[15] independently. The definition of (quantum) cluster algebras was motivated by Berenstein and Zelevinsky's earlier work [3] where combinatorial and multiplicative structures of the dual canonical basis were studied for $\mathfrak{g} = \mathfrak{sl}_n$ $(2 \le n \le 4)$. Let us quote from [16]:

> We conjecture that the above examples can be extensively generalized: for any simply-connected connected semisimple group $G$, the coordinate rings $\mathbb{C}[G]$ and $\mathbb{C}[G/N]$, as well as coordinate rings of many other interesting varieties related to $G$, have a natural structure of a cluster algebra. This structure should serve as an algebraic framework for the study of "dual canonical basis" in these coordinate rings and their $q$-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the cluster monomials) belong to this dual canonical basis.

In [2], it was shown that the coordinate ring of the double Bruhat cell contains a cluster algebra as a subalgebra, which is conjecturally equal to the whole algebra.

A cluster algebra $\mathscr{A}$ is a subalgebra of rational function field $\mathbb{Q}(x_1, x_2, \ldots, x_r)$ of $r$ indeterminates which is equipped with a distinguished set of generators (*cluster variables*) which is grouped into overlapping subsets (*clusters*) consisting of precisely $r$ elements. Each subset is defined inductively by a sequence of certain combinatorial operations (*seed mutations*) from the initial seed. The monomials in the variables of a given single cluster are called *cluster monomials.* However, it is not known whether cluster algebras have a basis, related to the dual canonical basis, which includes all cluster monomials in general.

### 1.3. Cluster algebra and the dual semicanonical basis

In a series of works [19]–[24], Geiss, Leclerc, and Schröer introduced a cluster algebra structure on the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent subgroup associated with a Weyl group element $w$. Furthermore, they show that the *dual semicanonical basis* $\mathcal{S}^*$ is compatible with the inclusion $\mathbb{C}[N(w)] \subset U(\mathfrak{n})^*_{\mathrm{gr}}$ and contains all cluster monomials. Here the dual semicanonical basis is the dual basis of the semicanonical basis of $U(\mathfrak{n})$, introduced by Lusztig [41], [44], and *compatible* means that $\mathcal{S}^* \cap \mathbb{C}[N(w)]$ forms a $\mathbb{C}$-basis of $\mathbb{C}[N(w)]$.

It is known that canonical and semicanonical bases share similar combinatorial properties (crystal structure), but they are different. (Examples can be found in [34].*)

### 1.4. Cluster algebra and the dual canonical basis

Our main result is to give a first step towards a quantum analogue of Geiss, Leclerc, and Schröer's results.

(1) The dual canonical basis is compatible with the quantum unipotent subgroup $\mathcal{O}_q[N(w)]$ which is a quantum analogue of $\mathbb{C}[N(w)]$; that is, $\mathbf{B}^{\mathrm{up}}(w) := \mathbf{B}^{\mathrm{up}} \cap \mathcal{O}_q[N(w)]$ forms a $\mathbb{Q}(q)$-basis of $\mathcal{O}_q[N(w)]$ (see Theorem 4.25).

(2) Quantum flag minors are mutually $q$-commuting, and their monomials are contained in the dual canonical basis up to some $q$-shifts. Here quantum flag minors are defined as certain matrix coefficients with respect to extremal vectors in integrable highest weight modules (see Theorem 6.26).

(3) The "$q$-center" of $\mathcal{O}_q[N(w)]$ is generated by some of the quantum flag minors. Moreover, any dual canonical basis element in $\mathbf{B}^{\mathrm{up}}(w)$ can be factored into the product of an element in the "$q$-center" of $\mathcal{O}_q[N(w)]$ and an "interval-free" element (see Theorem 6.27).

When $\mathfrak{g}$ is of finite type, Caldero [7]–[9] proved the above results in a series of works (see also [6, 6.3]). ($\mathcal{O}_q[N(w)]$ is denoted by $\mathbf{U}_q(\mathfrak{n}_w)$ in [8].) We generalize them to an arbitrary symmetrizable Kac–Moody Lie algebra. Key tools are the Poincaré–Birkhoff–Witt basis of $\mathcal{O}_q[N(w)]$ and the crystal structures. They are already used by Caldero, but the author cannot follow the proofs of several claims. We use the quantum closed unipotent cell (see Theorem 5.13) as a new tool and give self-contained proofs in this paper.

### 1.5. Quantization conjectures and their consequences

The above properties (1), (2), and (3) can be thought of as initial steps toward construction of structures of a quantum cluster algebra. The corresponding properties of the "classical limit" $\mathbb{C}[N(w)]$ were shown in [22] if the dual canonical basis is replaced by the dual semicanoncial basis. We conjecture that the remaining structures of a quantum cluster algebra exist on $\mathcal{O}_q[N(w)]$ as in [22]. Let $\mathcal{O}_q[N(w)]_{\mathcal{A}}$ be the integral form defined by the dual canonical basis $\mathbf{B}^{\mathrm{up}}(w)$ where $\mathcal{A} = \mathbb{Q}[q^{\pm 1}]$.

CONJECTURE 1.1 (QUANTIZATION CONJECTURE)

(1) *We take a reduced expression $\widetilde{w} = (i_1, \ldots, i_\ell)$ of the Weyl group element $w$; then we have an isomorphism of $\mathcal{A}$-algebras*

$$\Phi^q_{\widetilde{w}} : \mathscr{A}^q(\Gamma_{\widetilde{w}}, \Lambda_{\widetilde{w}}) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Q}[q^{\pm}] \simeq \mathcal{O}_q[N(w)]_{\mathcal{A}},$$

*In [34], $\underline{\mathcal{S}}$ is the specialization of the dual canonical basis, while $\underline{\Sigma}$ is the dual semicanonical basis thanks to [23].

*which sends the initial seed to the quantum flag minors* $\{\Delta^q_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}\}_{1\le k\le\ell}$ *(see Definition 6.1), where* $\Gamma_{\widetilde{w}}$ *is the frozen quiver in [2] and [22] and* $\Lambda_{\widetilde{w}}$ *is the compatible pair in [4, Section 10.3].*

(2) *Under this isomorphism, the quantum cluster monomials of* $\mathscr{A}^q(\Gamma_{\widetilde{w}},\Lambda_{\widetilde{w}})$ *are contained in the dual canonical basis* $\mathbf{B}^{\mathrm{up}}(w)$ *up to some q-shifts.*

By Geiss, Leclerc, and Schröer's result, we have an isomorphism of $\mathbb{C}$-algebras

$$\Phi_{\widetilde{w}} : \mathscr{A}(\Gamma_{\widetilde{w}}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[N(w)],$$

which sends the initial seed to the specialized quantum flag minors $\{\Delta_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}\}_{1\le k\le\ell}$, where $\Gamma_{\widetilde{w}}$ is the frozen quiver as above. Let $\mathcal{A}\to\mathbb{C}$ be the algebra homomorphism defined by $q\mapsto 1$. If we specialize Conjecture 1.1 to $q=1$, we obtain the following "weak" conjecture.

CONJECTURE 1.2 (WEAK QUANTIZATION CONJECTURE)
*Under the isomorphism* $\Phi_{\widetilde{w}}$*, the cluster monomials of* $\mathbb{C}[N(w)]$ *are contained in the specialized dual canonical basis* $\mathbf{B}^{\mathrm{up}}(w)$ *at* $q=1$*.*

Some parts of Conjecture 1.1 were shown for the $A_2$, $A_3$, $A_4$-cases with $w=w_0$ in [3] and [19, Section 12] and for $A_1^{(1)}$ with $w=c^2$ in [34].

The definition of the quantum cluster algebra $\mathscr{A}^q(\Gamma_{\widetilde{w}},\Lambda_{\widetilde{w}})$ will not be explained. So we explain the meaning of this conjecture as properties of the dual canonical basis without referring to the axiom of a quantum cluster algebra (see [4]).

An element $x\in\mathbf{B}^{\mathrm{up}}\setminus\{1\}$ is called *prime* if it does not have a nontrivial factorization $x=q^N x_1x_2$ with $x_1,x_2\in\mathbf{B}^{\mathrm{up}}$ and $N\in\mathbb{Z}$. A subset $\mathbf{x}=\{x_1,\dots,x_l\}\subset\mathbf{B}^{\mathrm{up}}$ is called *strongly compatible* if for all $m_1,\dots,m_l\in\mathbb{Z}_{\ge 0}$, the monomial $x_1^{m_1}\cdots x_l^{m_l}\in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$, that is, $x_1^{m_1}\cdots x_l^{m_l}$ is contained in the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ up to some $q$-shifts. In particular, $x$ is contained in a compatible family; then it satisfies $x^m\in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$ for all $m\ge 1$. A strongly compatible subset $\mathbf{x}=\{x_1,\dots,x_l\}$ is called *maximal* in $\mathbf{B}^{\mathrm{up}}(w)$ if whenever $y\in\mathbf{B}^{\mathrm{up}}(w)$ satisfies $yx_i\in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}(w)$ for any $x_i$, then there exist $m_1,\dots,m_l$ and $N$ such that $y=q^N x_1^{m_1}\cdots x_l^{m_l}$.

Our quantization conjecture means that there are lots of maximal strongly compatible subsets of $\mathbf{B}^{\mathrm{up}}(w)$, which are constructed recursively from $\{\Delta^q_{s_{i_1}\cdots s_{i_k}\varpi_{i_k},\varpi_{i_k}}\}_{1\le k\le\ell}$. For example, for finite-type $\mathfrak{g}$ with $w=c^2$ for a (bipartite) Coxeter element $c$, it is expected that the dual canonical basis $\mathbf{B}^{\mathrm{up}}(w)$ is covered by the (finite) union of the maximal compatible families. But the union is not the whole $\mathbf{B}^{\mathrm{up}}(w)$; either $w$ is longer than $c^2$ or $\mathfrak{g}$ is not of finite type.

Our quantization conjecture implies several conjectures on (quantum) cluster algebras. Let us spell out a few.

If $\mathfrak{g}$ is symmetric, we have the positivity of structure constants with respect to the dual canonical basis by the construction of [40]. This implies the *positivity conjecture* for the cluster algebras $\mathscr{A}(\Gamma_{\widetilde{w}})$ (resp., the quantum cluster algebras

$\mathscr{A}^q(\Gamma_{\widetilde{w}}, \Lambda_{\widetilde{w}})$), stating that cluster monomials are Laurent polynomials with positive coefficients in cluster variables (resp., $q$ and cluster variables) of any seed. This conjecture is known in the following several cases:

- bipartite cluster algebras at arbitrary seed (see [48]) using monoidal categorification (see [26]),
- acyclic cluster algebras at the initial seed (see [50], [48, Appendix]),
- cluster algebras coming from triangulated surfaces at arbitrary seed (see [46]),
- $T$-system cluster algebras of type A at special seeds (see [12]).

In fact, these results apply only to cluster algebras, not quantum ones except for those in [50]. The fourth result applies only to special cluster variables. Thus we have much stronger positivity conjecturally.

The quantization conjecture also provides us a *monoidal categorification* of $\mathbb{C}[N(w)]$ in the sense of Hernandez and Leclerc [26]. It roughly says that there is a monoidal abelian category $\mathfrak{N}(w)$ whose complexified Grothendieck ring $K_0(\mathfrak{N}(w)) \otimes_{\mathbb{Z}} \mathbb{C}$ has the cluster algebra structure of $\mathbb{C}[N(w)]$, so that the cluster monomials are classes of simple objects. If the weak quantization conjecture is true (and $\mathfrak{g}$ is symmetric), the category $\mathfrak{N}(w)$ is given as the category of finite-dimensional modules of the (equivariant) Ext-algebras of the simple (equivariant) perverse sheaves belonging to $\mathbf{B}^{\mathrm{up}}(w)$. Thanks to [54], $\mathfrak{N}(w)$ is also considered as the extension-closed subcategory of the module category of the Khovanov–Lauda–Rouquier algebra (see [31], [32], [52]) consisting of finite-dimensional modules whose composition factors are contained in $\mathbf{B}^{\mathrm{up}}(w)$.

When $\mathfrak{g}$ is symmetric, Geiss, Leclerc, and Schröer conjecture that certain dual semicanonical basis elements are specializations of the corresponding dual canonical basis elements. This is called the *open orbit conjecture*. This class of the dual semicanonical basis element contains all the cluster monomials. (Conjecturally it consists exactly of the cluster monomials; see [5, Conjecture II 5.3].) The open orbit conjecture for the cluster monomials is equivalent to the weak quantization conjecture.

This paper is organized as follows. In Section 2, we give a review of the quantized enveloping algebra and its canonical basis. In Section 3, we give a review of the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ and its multiplicative properties. In Section 4, we define the quantum unipotent subgroup and prove its compatibility with the dual canonical basis in Theorem 4.25. In Section 5, we define the quantum closed unipotent cell and study its relationship with the quantum unipotent subgroup in Theorem 5.13. In Section 6, we give quantum flag minors and prove their multiplicative properties in Theorems 6.26 and 6.27.

## 2. Preliminaries: Quantized enveloping algebras and the canonical bases

We briefly recall the definition of the quantized enveloping algebra and its canonical basis in this section.

### 2.1. Definition of $\mathbf{U}_q(\mathfrak{g})$

*2.1.1*

A *root datum* consists of

(1) $\mathfrak{h}$: a finite-dimensional $\mathbb{Q}$-vector space,
(2) a finite index set $I$,
(3) $P \subset \mathfrak{h}^*$: a lattice (weight lattice),
(4) $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ with natural pairing $\langle\ ,\ \rangle : P^\vee \otimes P \to \mathbb{Z}$,
(5) $\alpha_i \in P$ for $i \in I$ (simple roots),
(6) $h_i \in P^\vee$ for $i \in I$ (simple coroots),
(7) $(\cdot,\cdot)$ a $\mathbb{Q}$-valued symmetric bilinear form on $\mathfrak{h}^*$

satisfying the following conditions:

(a) $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in P$;
(b) $a_{ij} = \langle h_i, \alpha_j\rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ gives a symmetrizable generalized Cartan matrix; that is, $\langle h_i, \alpha_i\rangle = 2$, and $\langle h_i, \alpha_j\rangle \in \mathbb{Z}_{\le 0}$ and $\langle h_i, \alpha_j\rangle = 0 \Leftrightarrow \langle h_j, \alpha_i\rangle = 0$ for $i \neq j$;
(c) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$; that is, $d_i := (\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$;
(d) $\{\alpha_i\}_{i\in I}$ are linearly independent.

We call $(I, \mathfrak{h}, (\ ,\ ))$ a *Cartan datum*. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset P$ be the root lattice. Let $Q_\pm = \pm\sum_{i\in I} \mathbf{Z}_{\ge 0}\alpha_i$. For $\xi = \sum_{i\in I} \xi_i\alpha_i \in Q$, we define $\mathrm{tr}(\xi) = \sum_{i\in I} \xi_i$. And we assume that there exists $\varpi_i \in P$ such that $\langle h_i, \varpi_j\rangle = \delta_{i,j}$ for any $i, j \in I$. We call $\varpi_i$ the *fundamental weight* corresponding to $i \in I$. We say that $\lambda \in P$ is *dominant* if $\langle h_i, \lambda\rangle \ge 0$ for all $i \in I$ and denote by $P_+$ the set of dominant integral weights. We denote $\overline{P} := \bigoplus_{i\in I} \mathbb{Z}\varpi_i$ and $\overline{P}_+ := \overline{P} \cap P_+ = \bigoplus_{i\in I} \mathbb{Z}_{\ge 0}\varpi_i$.

*2.1.2*

Let $(I, \mathfrak{h}, (\ ,\ ))$ be a Cartan datum. Let $\mathfrak{g}$ be the symmetrizable Kac–Moody Lie algebra corresponding to the generalized Cartan matrix $A = (a_{ij})$ with the Cartan subalgebra $\mathfrak{h}$; that is, $\mathfrak{g}$ is the Lie algebra generated by $\{h; h \in \mathfrak{h}\}$, $e_i$, and $f_i$ $(i \in I)$ with the following relations:

(i) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$,
(ii) $[h, e_i] = \langle h, \alpha_i\rangle e_i$, $[h, f_i] = -\langle h, \alpha_i\rangle f_i$,
(iii) $[e_i, f_j] = \delta_{ij} h_i$, and
(iv) $(\mathrm{ad} e_i)^{1-\langle h_i, \alpha_j\rangle} e_j = (\mathrm{ad} f_i)^{1-\langle h_i, \alpha_j\rangle} f_j = 0$ for $i \neq j$.

We denote the Lie subalgebra generated by $\{f_i\}_{i\in I}$ by $\mathfrak{n}$.

*2.1.3*

Suppose that a root datum is given. We introduce an indeterminate $q$. For $i \in I$, we set $q_i = q^{(\alpha_i,\alpha_i)/2}$. For $\xi = \sum_{i\in I} \xi_i\alpha_i \in Q$, we set $q_\xi := \prod_{i\in I} (q_i)^{\xi_i} = q^{(\xi,\rho)}$, where $\rho$ is the sum of all fundamental weights. We define $\mathbb{Q}$-subalgebras $\mathcal{A}_0$, $\mathcal{A}_\infty$, and $\mathcal{A}$ of $\mathbb{Q}(q)$ by

$$\mathcal{A}_0 := \bigl\{ f \in \mathbb{Q}(q); f \text{ is regular at } q = 0 \bigr\},$$

$$\mathcal{A}_\infty := \{ f \in \mathbb{Q}(q); f \text{ is regular at } q = \infty \},$$
$$\mathcal{A} := \mathbb{Q}[q^{\pm 1}].$$

*2.1.4*

The *quantized enveloping algebra* $\mathbf{U}_q(\mathfrak{g})$ associated with a root datum is the $\mathbb{Q}(q)$-algebra generated by $e_i, f_i$ $(i \in I)$, $q^h$ $(h \in P^\vee)$ with the following relations:

(i) $q^0 = 1, q^h q^{h'} = q^{h+h'}$,
(ii) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$,
(iii) $e_i f_j - f_j e_j = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1})$,
(iv) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0$ ($q$-Serre relations),

where $t_i = q^{((\alpha_i, \alpha_i)/2) h_i}$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = [n]_i [n-1]_i \cdots [1]_i$ for $n > 0$, and $[0]! = 1$, $e_i^{(k)} = e_i^k / [k]_i!, f_i^{(k)} = f_i^k / [k]_i!$ for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$.

*2.1.5*

Let $\mathbf{U}_q^+(\mathfrak{g})$ (resp., $\mathbf{U}_q^-(\mathfrak{g})$) be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{g})$ generated by $e_i$ (resp., $f_i$) for $i \in I$. Then we have the triangular decomposition

$$\mathbf{U}_q(\mathfrak{g}) \simeq \mathbf{U}_q^-(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q)[P^\vee] \otimes_{\mathbb{Q}(q)} \mathbf{U}_q^+(\mathfrak{g}),$$

where $\mathbb{Q}(q)[P^\vee]$ is the group algebra over $\mathbb{Q}(q)$, that is, $\bigoplus_{h \in P^\vee} \mathbb{Q}(q) q^h$.

*2.1.6*

For $\xi \in Q$, we define its *root space* $\mathbf{U}_q(\mathfrak{g})_\xi$ by

$$\mathbf{U}_q(\mathfrak{g})_\xi = \{ x \in \mathbf{U}_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h, \xi \rangle} x \quad \text{for all } h \in P^\vee \}.$$

Then we have the root space decomposition

$$\mathbf{U}_q^\pm(\mathfrak{g}) = \bigoplus_{\xi \in Q_\pm} \mathbf{U}_q(\mathfrak{g})_\xi.$$

An element $x \in \mathbf{U}_q(\mathfrak{g})$ is *homogeneous* if $x \in \mathbf{U}_q(\mathfrak{g})_\xi$ for some $\xi \in Q$, and we set $\mathrm{wt}(x) = \xi$.

*2.1.7*

Let $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbf{U}_q^-(\mathfrak{g})$ generated by $f_i^{(k)}$ for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$. Let $(\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})_\xi := \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} \cap \mathbf{U}_q^-(\mathfrak{g})_\xi$. We have

$$\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\xi \in Q_-} (\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})_\xi.$$

*2.1.8*

We define a $\mathbb{Q}(q)$-algebra anti-involution $* : \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$(2.1) \qquad *(e_i) = e_i, \qquad *(f_i) = f_i, \qquad *(q^h) = q^{-h}.$$

We call this the $*$-*involution.*

We define a $\mathbb{Q}$-algebra automorphism $^{-}: \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$\overline{e_i} = e_i, \qquad \overline{f_i} = f_i, \qquad \overline{q} = q^{-1}, \qquad \overline{q^h} = q^{-h}. \tag{2.2}$$

We call this the *bar involution.*

We remark that these two involutions preserve $\mathbf{U}_q^+(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g})$, and we have $^{-} \circ * = * \circ ^{-}$.

*2.1.9*

In this article, we choose the coproduct on $\mathbf{U}_q(\mathfrak{g})$ following (see [27]):

$$\Delta(q^h) = q^h \otimes q^h, \tag{2.3a}$$

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \tag{2.3b}$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i. \tag{2.3c}$$

*2.1.10*

We introduce Lusztig's $\mathbb{Q}(q)$-valued symmetric nondegenerate bilinear form $(\ ,\ )_L$ on $\mathbf{U}_q^-(\mathfrak{g})$. We first define a $\mathbb{Q}(q)$-algebra structure on $\mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = q^{-(\mathrm{wt}(x_2), \mathrm{wt}(y_1))} x_1 x_2 \otimes y_1 y_2,$$

where $x_i, y_i$ $(i = 1, 2)$ are homogeneous elements.

Let $r : \mathbf{U}_q^-(\mathfrak{g}) \to \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ be a $\mathbb{Q}(q)$-algebra homomorphism defined by

$$r(f_i) = f_i \otimes 1 + 1 \otimes f_i \ (i \in I).$$

We call this the *twisted coproduct.*

By [45, Section 1.2.5], the algebra $\mathbf{U}_q^-(\mathfrak{g})$ has a unique nondegenerate $\mathbb{Q}(q)$-valued symmetric bilinear form $(\ ,\ )_L : \mathbf{U}_q^-(\mathfrak{g}) \times \mathbf{U}_q^-(\mathfrak{g}) \to \mathbb{Q}(q)$ which satisfies

$$(1, 1)_L = 1, \tag{2.4a}$$

$$(f_i, f_j)_L = \frac{\delta_{i,j}}{1 - q_i^2}, \tag{2.4b}$$

$$(x, yy')_L = (r(x), y \otimes y')_L, \tag{2.4c}$$

$$(xx', y)_L = (x \otimes x', r(y))_L, \tag{2.4d}$$

where the form on $\mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ is defined by $(x_1 \otimes y_1, x_2 \otimes y_2)_L = (x_1, x_2)_L (y_1, y_2)_L$.

*2.1.11*

The relation between the coproduct $\Delta$ and the twisted coproduct $r$ is given as follows.

LEMMA 2.5

*For homogeneous* $x \in \mathbf{U}_q^-(\mathfrak{g})_\xi$*, we have*

$$\Delta(x) = \sum x_{(1)} t_{-\operatorname{wt}(x_{(2)})} \otimes x_{(2)}, \tag{2.6}$$

*where* $r(x) = \sum x_{(1)} \otimes x_{(2)}$*,* $t_\xi = q^{\nu(\xi)}$*, and* $\nu(\xi) = \sum_i \frac{(\alpha_i,\alpha_i)}{2} \xi_i h_i$ *for* $\xi = \sum \xi_i \alpha_i \in Q$*.*

*2.1.12*

For $i \in I$, we define the unique $\mathbb{Q}(q)$-linear map ${}_ir : \mathbf{U}_q^- \to \mathbf{U}_q^-$ (resp., $r_i : \mathbf{U}_q^- \to \mathbf{U}_q^-$) given by ${}_ir(1) = 0, {}_ir(f_j) = \delta_{i,j}$ (resp., $r_i(1) = 0, r_i(f_j) = \delta_{i,j}$) for all $i, j \in I$, and

$${}_ir(xy) = {}_ir(x)y + q^{-(\operatorname{wt} x, \alpha_i)} x \; {}_ir(y), \tag{2.7a}$$

$$r_i(xy) = q^{-(\operatorname{wt} y, \alpha_i)} r_i(x)y + x r_i(y) \tag{2.7b}$$

for homogeneous $x, y \in \mathbf{U}_q^-$. From the definition, we have

$$(f_i x, y)_L = \frac{1}{1-q_i^2} (x, {}_iry)_L, \tag{2.8a}$$

$$(x f_i, y)_L = \frac{1}{1-q_i^2} (x, r_i y)_L. \tag{2.8b}$$

## 2.2. Canonical basis of $\mathbf{U}_q^-(\mathfrak{g})$

In this subsection, we give a brief review of the theory of the canonical basis following Kashiwara [27], [30]. Note that Kashiwara called it the *lower global basis.*

*2.2.1*

LEMMA 2.9 ([27, LEMMA 3.4.1], [49])

*For* $x \in \mathbf{U}_q^-(\mathfrak{g})$ *and any* $i \in I$*, we have*

$$[e_i, x] = \frac{r_i(x) t_i - t_i^{-1} {}_ir(x)}{q_i - q_i^{-1}}.$$

*2.2.2*

Kashiwara [27, Section 3.4] has proved that there is a unique nondegenerate symmetric bilinear form $(\cdot,\cdot)_K$ on $\mathbf{U}_q^-(\mathfrak{g})$ such that

$$(f_i x, y)_K = \big(x, {}_ir(y)\big)_K, \tag{2.10a}$$

$$(1,1)_K = 1. \tag{2.10b}$$

LEMMA 2.11 ([27, LEMMA 3.4.7], [45, LEMMA 1.2.15])

*For* $x \in \mathbf{U}_q^-(\mathfrak{g})$ *with* ${}_ir(x) = 0$ *for all* $i \in I$ *and* $\operatorname{wt}(x) \neq 0$*, then we have* $x = 0$*.*

*2.2.3*

We have the following relation between Kashiwara's bilinear form $(\ ,\ )_K$ and Lusztig's one $(\ ,\ )_L$.

LEMMA 2.12 ([36, SECTION 2.2])

*For homogeneous $x, y \in \mathbf{U}_q^-(\mathfrak{g})_\xi$ with $\xi = -\sum n_i\alpha_i \in Q_-$, we have*

$$(x,y)_K = \prod_{i\in I}(1-q_i^2)^{n_i}(x,y)_L.$$

This can be proved by an induction on $\mathrm{wt}(x)$ by using Lemma 2.11, (2.10a), and (2.8a).

LEMMA 2.13 ([45, LEMMA 1.2.8(B)])

*For any homogeneous $x, y \in \mathbf{U}_q^-(\mathfrak{g})$, we have*

$$(x,y)_K = \big(*(x), *(y)\big)_K.$$

*2.2.4*

The *reduced $q$-analogue* $\mathscr{B}_q(\mathfrak{g})$ of a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ is the $\mathbb{Q}(q)$-algebra generated by ${}_ir$ and $f_i$ with the $q$-Boson relations ${}_irf_j = q^{-(\alpha_i,\alpha_j)}f_j\ {}_ir + \delta_{i,j}$ for $i, j \in I$ and the $q$-Serre relations for ${}_ir$ and $f_i$ for $i \in I$. Then $\mathbf{U}_q^-(\mathfrak{g})$ becomes a $\mathscr{B}_q(\mathfrak{g})$-module by Lemma 2.9.

By the $q$-Boson relation, any element $x \in \mathbf{U}_q^-(\mathfrak{g})$ can be uniquely written as $x = \sum_{n\geq 0} f_i^{(n)}x_n$ with ${}_ir(x_n) = 0$ for any $n \geq 0$. So we define Kashiwara's *modified root operators* $\widetilde{f}_i$ and $\widetilde{e}_i$ by

$$\widetilde{e}_i x = \sum_{n\geq 1} f_i^{(n-1)}x_n,$$

$$\widetilde{f}_i x = \sum_{n\geq 0} f_i^{(n+1)}x_n.$$

By using these operators, Kashiwara introduced the crystal basis $(\mathscr{L}(\infty), \mathscr{B}(\infty))$ of $\mathbf{U}_q^-(\mathfrak{g})$.

THEOREM 2.14 ([27, THEOREM 4])

*Let*

$$\mathscr{L}(\infty) := \sum_{l\geq 0, i_1, i_2, \ldots, i_l \in I} \mathcal{A}_0 \widetilde{f}_{i_1}\cdots\widetilde{f}_{i_l}1 \subset \mathbf{U}_q^-(\mathfrak{g}),$$

$$\mathscr{B}(\infty) := \big\{\widetilde{f}_{i_1}\cdots\widetilde{f}_{i_l}1 \bmod q\mathscr{L}(\infty); l \geq 0, i_1, i_2, \ldots, i_l \in I\big\} \subset \mathscr{L}(\infty)/q\mathscr{L}(\infty).$$

*Then we have the following:*

1. *$\mathscr{L}(\infty)$ is a free $\mathcal{A}_0$-module with $\mathbb{Q}(q)\otimes_{\mathcal{A}_0}\mathscr{L}(\infty) = \mathbf{U}_q^-(\mathfrak{g})$;*
2. *$\widetilde{e}_i\mathscr{L}(\infty) \subset \mathscr{L}(\infty)$ and $\widetilde{f}_i\mathscr{L}(\infty) \subset \mathscr{L}(\infty)$;*
3. *$\mathscr{B}(\infty)$ is a $\mathbb{Q}$-basis of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$;*

(4) $\widetilde{f}_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ *and* $\widetilde{e}_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty) \cup \{0\}$*;*
(5) *for* $b \in \mathscr{B}(\infty)$ *with* $\widetilde{e}_i(b) \neq 0$*, we have* $\widetilde{f}_i \widetilde{e}_i b = b$.

We call $(\mathscr{L}(\infty), \mathscr{B}(\infty))$ the *(lower) crystal basis* of $\mathbf{U}_q^-(\mathfrak{g})$ and call $\mathscr{L}(\infty)$ the *(lower) crystal lattice*. We denote $1 \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty)$ by $u_\infty$ hereafter. For $b \in \mathscr{B}(\infty)$, we set $\varepsilon_i(b) := \max\{n \in \mathbb{Z}_{\geq 0}; \widetilde{e}_i^n b \neq 0\} < \infty$ and $\widetilde{e}_i^{\max}(b) := \widetilde{e}_i^{\varepsilon_i(b)} b \in \mathscr{B}(\infty)$.

*2.2.5*
We have the following compatibility of the $*$-involution with the crystal lattice $\mathscr{L}(\infty)$.

THEOREM 2.15 ([27, PROPOSITION 5.2.4], [29, THEOREM 2.1.1])
*We have*

$$*\big(\mathscr{L}(\infty)\big) = \mathscr{L}(\infty), \tag{2.16a}$$

$$*\big(\mathscr{B}(\infty)\big) = \mathscr{B}(\infty). \tag{2.16b}$$

For $i \in I$ and $b \in \mathscr{B}(\infty)$, we set

$$\widetilde{f}_i^*(b) := (* \circ \widetilde{f}_i \circ *)(b), \tag{2.17a}$$

$$\widetilde{e}_i^*(b) := (* \circ \widetilde{e}_i \circ *)(b). \tag{2.17b}$$

For $b \in \mathscr{B}(\infty)$, we set $\varepsilon_i^*(b) := \max\{n \in \mathbb{Z}_{\geq 0}; \widetilde{e}_i^{*n} b \neq 0\} < \infty$ and $\widetilde{e}_i^{*\max}(b) := \widetilde{e}_i^{*\varepsilon_i^*(b)} b \in \mathscr{B}(\infty)$. We have $\varepsilon_i^*(b) = \varepsilon_i(*b)$.

*2.2.6*
We recall some results on the relationship between the crystal lattice $\mathscr{L}(\infty)$ and Kashiwara's form $(\cdot,\cdot)_K$.

PROPOSITION 2.18 ([27, PROPOSITION 5.1.2])
*We have*

$$\big(\mathscr{L}(\infty), \mathscr{L}(\infty)\big)_K \subset \mathcal{A}_0.$$

*Therefore, the* $\mathbb{Q}$*-valued inner product on* $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$ *given by* $(\cdot,\cdot)_K|_{q=0}$ *is well defined, which we denote by* $(\cdot,\cdot)_0$*. Then we have the following properties:*

(1) $(\widetilde{e}_i u, u')_0 = (u, \widetilde{f}_i u')_0$ *for* $u, u' \in \mathscr{L}(\infty)/q\mathscr{L}(\infty)$*,*
(2) $\mathscr{B}(\infty) \subset \mathscr{L}(\infty)/q\mathscr{L}(\infty)$ *is an orthonormal basis with respect to* $(\cdot,\cdot)_0$*.*

*Moreover, we have*

$$\mathscr{L}(\infty) = \big\{x \in \mathbf{U}_q^-(\mathfrak{g}); \big(x, \mathscr{L}(\infty)\big)_K \subset \mathcal{A}_0\big\}; \tag{2.19}$$

*that is, the crystal lattice* $\mathscr{L}(\infty)$ *is a self-dual lattice with respect to* $(\cdot,\cdot)_K$*.*

*2.2.7*
Let $\bar{\ }:\mathbb{Q}(q)\to\mathbb{Q}(q)$ be the $\mathbb{Q}$-algebra involution sending $q$ to $q^{-1}$. Let $V$ be a vector space over $\mathbb{Q}(q)$, let $\mathscr{L}_0$ be an $\mathcal{A}_0$-submodule of $V$, let $\mathscr{L}_\infty$ be an $\mathcal{A}_\infty$-submodule of $V$, and let $V_{\mathcal{A}}$ be an $\mathcal{A}$-submodule of $V$. We set $E:=\mathscr{L}_0\cap\mathscr{L}_\infty\cap V_{\mathcal{A}}$.

DEFINITION 2.20
We say that a triple $(\mathscr{L}_0,\mathscr{L}_\infty,V_{\mathcal{A}})$ is *balanced* if each $\mathscr{L}_0,\mathscr{L}_\infty$, and $V_{\mathcal{A}}$ generates $V$ as $\mathbb{Q}(q)$-vector space and if one of the following equivalent conditions is satisfied:

(1) $E\to\mathscr{L}_0/q\mathscr{L}_0$ is an isomorphism;
(2) $E\to\mathscr{L}_\infty/q^{-1}\mathscr{L}_\infty$ is an isomorphism;
(3) $(\mathscr{L}_0\cap V_{\mathcal{A}})\oplus(q^{-1}\mathscr{L}_\infty\cap V_{\mathcal{A}})\to V_{\mathcal{A}}$ is an isomorphism;
(4) $\mathcal{A}_0\otimes_{\mathbb{Q}}E\to\mathscr{L}_0$, $\mathcal{A}_\infty\otimes_{\mathbb{Q}}E\to\mathscr{L}_\infty$, $\mathcal{A}\otimes_{\mathbb{Q}}E\to V_{\mathcal{A}}$, and $\mathbb{Q}(q)\otimes_{\mathbb{Q}}E\to V$ are isomorphisms.

THEOREM 2.21 ([27, THEOREM 6])
*The triple $(\mathscr{L}(\infty),\overline{\mathscr{L}(\infty)},\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}})$ is balanced.*

Let $G^{\mathrm{low}}:\mathscr{L}(\infty)/q\mathscr{L}(\infty)\to E:=\mathscr{L}(\infty)\cap\overline{\mathscr{L}(\infty)}\cap\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ be the inverse of $E\xrightarrow{\sim}\mathscr{L}(\infty)/q\mathscr{L}(\infty)$. Then $\{G^{\mathrm{low}}(b);b\in\mathscr{B}(\infty)\}$ forms an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$. This basis is called the *canonical basis* of $\mathbf{U}_q^-(\mathfrak{g})$. Using this characterization, we obtain the following compatibility of the canonical basis and the $*$-involution.

PROPOSITION 2.22
*We have*

$$*G^{\mathrm{low}}(b)=G^{\mathrm{low}}(*b).$$

*2.2.8*
For integrable highest weight modules, we can define the (lower) crystal basis and the canonical basis of them as for $\mathbf{U}_q^-(\mathfrak{g})$ (see [27, Theorems 2, 6] for more details). Let $M$ be an integrable $\mathbf{U}_q(\mathfrak{g})$-module, and let $M=\bigoplus_{\lambda\in P}M_\lambda$ be its weight decomposition. By the theory of integrable representations of $\mathbf{U}_q(\mathfrak{sl}_2)$, we have

$$M=\bigoplus_{0\le n\le\langle h_i,\lambda\rangle}f_i^{(n)}\big(\mathrm{Ker}(e_i)\cap M_\lambda\big).$$

For $u\in\mathrm{Ker}(e_i)\cap M_\lambda$ and $0\le n\le\langle h_i,\lambda\rangle$, we define Kashiwara's modified operators or (lower) crystal operators $\widetilde{e}_i^{\mathrm{low}}$ and $\widetilde{f}_i^{\mathrm{low}}$ by

$$\widetilde{e}_i^{\mathrm{low}}(f_i^{(n)}u)=f_i^{(n-1)}u,$$
$$\widetilde{f}_i^{\mathrm{low}}(f_i^{(n)}u)=f_i^{(n+1)}u.$$

Here we understand $f_i^{(-1)}u$ and $f_i^{(\langle h_i,\lambda\rangle+1)}u$ as zero. Note that we denote the operators $\widetilde{f}_i$ and $\widetilde{e}_i$ in [27, 2.2] by $\widetilde{f}_i^{\mathrm{low}}$ and $\widetilde{e}_i^{\mathrm{low}}$ following [28, Section 3.1].

Let $\lambda \in P_+$ and $V(\lambda)$ be the integrable highest weight $\mathbf{U}_q(\mathfrak{g})$-module generated by a highest weight vector $u_\lambda$ of weight $\lambda$. Let $\mathscr{L}^{\mathrm{low}}(\lambda)$ be the $\mathcal{A}_0$-submodule spanned by $\widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_r}^{\mathrm{low}} u_\lambda$. Let $\mathscr{B}^{\mathrm{low}}(\lambda)$ be the subset of $\mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$ consisting of the nonzero vectors of the form $\widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_r}^{\mathrm{low}} u_\lambda$, that is

$$\mathscr{L}^{\mathrm{low}}(\lambda) := \sum \mathcal{A}_0 \widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_r}^{\mathrm{low}} u_\lambda \subset V(\lambda),$$

$$\mathscr{B}^{\mathrm{low}}(\lambda) := \{\widetilde{f}_{i_1}^{\mathrm{low}} \cdots \widetilde{f}_{i_r}^{\mathrm{low}} u_\lambda \bmod q\mathscr{L}^{\mathrm{low}}(\lambda)\} \setminus \{0\} \subset \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda).$$

THEOREM 2.23 ([27, THEOREM 2])

(1) *$\mathscr{L}^{\mathrm{low}}(\lambda)$ is a free $\mathcal{A}_0$-submodule with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}^{\mathrm{low}}(\lambda) \simeq V(\lambda)$ and $\mathscr{L}^{\mathrm{low}}(\lambda) = \bigoplus_{\mu \in P} \mathscr{L}^{\mathrm{low}}(\lambda)_\mu$ where $\mathscr{L}^{\mathrm{low}}(\lambda)_\mu = \mathscr{L}^{\mathrm{low}}(\lambda) \cap V(\lambda)_\mu$.*

(2) *$\widetilde{e}_i^{\mathrm{low}} \mathscr{L}^{\mathrm{low}}(\lambda) \subset \mathscr{L}^{\mathrm{low}}(\lambda)$ and $\widetilde{f}_i^{\mathrm{low}} \mathscr{L}^{\mathrm{low}}(\lambda) \subset \mathscr{L}^{\mathrm{low}}(\lambda)$.*

(3) *$\mathscr{B}^{\mathrm{low}}(\lambda)$ is a $\mathbb{Q}$-basis of $\mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$ and $\mathscr{B}^{\mathrm{low}}(\lambda) = \bigsqcup_{\mu \in P} \mathscr{B}^{\mathrm{low}}(\lambda)_\mu$ where $\mathscr{B}^{\mathrm{low}}(\lambda)_\mu = \mathscr{B}^{\mathrm{low}}(\lambda) \cap \mathscr{L}^{\mathrm{low}}(\lambda)_\mu / q\mathscr{L}^{\mathrm{low}}(\lambda)_\mu$.*

(4) *For $i \in I$, we have $\widetilde{e}_i \mathscr{B}(\lambda) \subset \mathscr{B}(\lambda) \cup \{0\}$ and $\widetilde{f}_i \mathscr{B}(\lambda) \subset \mathscr{B}(\lambda) \cup \{0\}$.*

(5) *For $b, b' \in \mathscr{B}^{\mathrm{low}}(\lambda)$, $b' = \widetilde{f}_i^{\mathrm{low}} b$ is equivalent to $b = \widetilde{e}_i^{\mathrm{low}} b'$.*

We call $(\mathscr{L}^{\mathrm{low}}(\lambda), \mathscr{B}^{\mathrm{low}}(\lambda))$ the *lower crystal basis* of $V(\lambda)$ and call $\mathscr{L}^{\mathrm{low}}(\lambda)$ the *lower crystal lattice*.

Let $^-$ be the bar involution defined by $\overline{xu_\lambda} = \overline{x} u_\lambda$ for $x \in \mathbf{U}_q(\mathfrak{g})$. Set $V(\lambda)_{\mathcal{A}} := \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} u_\lambda$.

THEOREM 2.24 ([27, THEOREM 6])

*The triple $(\mathscr{L}^{\mathrm{low}}(\lambda), \overline{\mathscr{L}^{\mathrm{low}}(\lambda)}, V(\lambda)_{\mathcal{A}})$ is balanced.*

Let $G_\lambda^{\mathrm{low}}$ be the inverse of $\mathscr{L}^{\mathrm{low}}(\lambda) \cap \overline{\mathscr{L}^{\mathrm{low}}(\lambda)} \cap V(\lambda)_{\mathcal{A}} \xrightarrow{\sim} \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$. We call $G_\lambda^{\mathrm{low}}(\mathscr{B}^{\mathrm{low}}(\lambda))$ the *canonical basis* of $V(\lambda)$.

*2.2.9*

We have a compatibility of the (lower) crystal basis of $\mathbf{U}_q^-(\mathfrak{g})$ and the integrable modules $V(\lambda)$. Let $\pi_\lambda : \mathbf{U}_q^-(\mathfrak{g}) \to V(\lambda)$ be the $\mathbf{U}_q^-(\mathfrak{g})$-module homomorphism defined by $x \mapsto xu_\lambda$.

THEOREM 2.25 ([27, THEOREM 5])

*We have the following properties:*

(1) *$\pi_\lambda \mathscr{L}(\infty) = \mathscr{L}(\lambda)$; hence $\pi_\lambda$ induces a surjection homomorphism $\pi_\lambda : \mathscr{L}(\infty)/q\mathscr{L}(\infty) \to \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$;*

(2) *$\pi_\lambda$ induces a bijection $\{b \in \mathscr{B}(\infty); \pi_\lambda(b) \neq 0\} \simeq \mathscr{B}^{\mathrm{low}}(\lambda)$;*

(3) *$\widetilde{f}_i^{\mathrm{low}} \circ \pi_\lambda(b) = \pi_\lambda \circ \widetilde{f}_i(b)$ if $\pi_\lambda(b) \neq 0$;*

(4) *$\widetilde{e}_i^{\mathrm{low}} \circ \pi_\lambda(b) = \pi_\lambda \circ \widetilde{e}_i(b)$ if $\widetilde{e}_i \circ \pi_\lambda(b) \neq 0$.*

We denote the inverse of the bijection $\pi_\lambda$ by $j_\lambda$.

*2.2.10*
We also have a compatibility of the canonical basis of $\mathbf{U}_q^-(\mathfrak{g})$ and the integrable modules $V(\lambda)$ via $\pi_\lambda$.

THEOREM 2.26 ([27, SECTION 7.3 LEMMA 7.3.2])
*For $\lambda \in P_+$ and $b \in \mathscr{B}(\infty)$ with $\pi_\lambda(b) \neq 0$, we have*

$$G^{\mathrm{low}}(b)u_\lambda = G_\lambda^{\mathrm{low}}\big(\pi_\lambda(b)\big).$$

*2.2.11*
For the canonical basis, we have the following expansion of left and right multiplication with respect to $f_i^{(m)}$.

THEOREM 2.27 ([29, SECTION 3.1 (3.1.2)])
*For $b \in \mathscr{B}(\infty)$, we have*

$$\begin{aligned} f_i^{(m)} G^{\mathrm{low}}(b) = & \begin{bmatrix} \varepsilon_i(b)+m \\ m \end{bmatrix} G^{\mathrm{low}}(\widetilde{f}_i^m b) \\ & + \sum_{\varepsilon_i(b')>\varepsilon_i(b)+m} f_{bb';i}^{(m)}(q) G^{\mathrm{low}}(b'), \end{aligned} \tag{2.28a}$$

$$\begin{aligned} G^{\mathrm{low}}(b) f_i^{(m)} = & \begin{bmatrix} \varepsilon_i^*(b)+m \\ m \end{bmatrix} G^{\mathrm{low}}(\widetilde{f}_i^{*m} b) \\ & + \sum_{\varepsilon_i^*(b')>\varepsilon_i^*(b)+m} f_{bb';i}^{*(m)}(q) G^{\mathrm{low}}(b'), \end{aligned} \tag{2.28b}$$

*where $f_{bb';i}^{(m)}(q) = \overline{f_{bb';i}^{(m)}(q)}, f_{bb';i}^{*(m)}(q) = \overline{f_{bb';i}^{*(m)}(q)} \in \mathcal{A}$.*

As a corollary of the above theorem, we have the following compatibilities of the right and left ideals $f_i^n \mathbf{U}_q^-(\mathfrak{g})$ and $\mathbf{U}_q^-(\mathfrak{g}) f_i^n$ with the canonical basis.

THEOREM 2.29 ([27, THEOREM 7])
*For $i \in I$ and $n \geq 1$, we have*

$$f_i^n \mathbf{U}_q^-(\mathfrak{g}) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{b \in \mathscr{B}(\infty), \varepsilon_i(b) \geq n} \mathcal{A} G^{\mathrm{low}}(b),$$

$$\mathbf{U}_q^-(\mathfrak{g}) f_i^n \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{b \in \mathscr{B}(\infty), \varepsilon_i^*(b) \geq n} \mathcal{A} G^{\mathrm{low}}(b).$$

## 2.3. Abstract crystal

The notion of a (abstract) crystal was introduced in [29] by abstracting the crystal basis of $\mathbf{U}_q^-(\mathfrak{g})$ and of irreducible highest weight representations which are constructed in [27]. We recall it briefly. For more detail, see [30].

*2.3.1*

DEFINITION 2.30

A *crystal* $\mathscr{B}$ associated with a root datum is a set $\mathscr{B}$ endowed with maps $\mathrm{wt} : \mathscr{B} \to P, \varepsilon_i, \varphi_i : \mathscr{B} \to \mathbb{Z} \sqcup \{-\infty\}$, $\widetilde{e}_i, \widetilde{f}_i : \mathscr{B} \to \mathscr{B} \sqcup \{0\}$ $(i \in I)$ satisfying the following conditions:

(a) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$,
(b) $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$, if $\widetilde{e}_i b \in \mathscr{B}$,
(c) $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$, if $\widetilde{f}_i b \in \mathscr{B}$,
(d) $b' = \widetilde{f}_i b \Leftrightarrow b = \widetilde{e}_i b'$ for $b, b' \in \mathscr{B}$,
(e) if $\varphi_i(b) = -\infty$ for $b \in \mathscr{B}$, then $\widetilde{e}_i b = \widetilde{f}_i b = 0$.

Let $\mathrm{wt}_i(b) = \langle h_i, \mathrm{wt}(b)\rangle$.

A crystal $\mathscr{B}$ is called *upper normal* (resp., *lower normal*) if, for all $b \in \mathscr{B}$, $\varepsilon_i(b) \in \mathbb{Z}$ and $\varepsilon_i(b) = \max\{k \geq 0; \widetilde{e}_i^k b \in \mathscr{B}\}$ (resp., $\varphi_i(b) = \max\{k \geq 0; \widetilde{f}_i^k b \in \mathscr{B}\}$). In such a case, we set $\widetilde{e}_i^{\max} b := \widetilde{e}_i^{\varepsilon_i(b)} b$ (resp., $\widetilde{f}_i^{\max} b := \widetilde{f}_i^{\varepsilon_i(b)} b$). If a crystal is upper normal and lower normal, it is called *seminormal*.

DEFINITION 2.31

For a given two crystals $\mathscr{B}_1, \mathscr{B}_2$ and for $h \in \mathbb{Z}_{\geq 1}$, a map $\psi : \mathscr{B}_1 \sqcup \{0\} \to \mathscr{B}_2 \sqcup \{0\}$ is called a *morphism of amplitude h* of crystals from $\mathscr{B}_1$ to $\mathscr{B}_2$ if it satisfies the following properties for $b \in \mathscr{B}_1$ and $i \in I$:

(a) $\psi(0) = 0$,
(b) $\mathrm{wt}(\psi(b)) = h\,\mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = h\varepsilon_i(b)$, $\varphi_i(\psi(b)) = h\varphi_i(b)$ if $\psi(b) \in \mathscr{B}_2$,
(c) $\widetilde{e}_i^h \psi(b) = \psi(\widetilde{e}_i b)$ if $\psi(b) \in \mathscr{B}_2$, $\widetilde{e}_i b \in \mathscr{B}_1$,
(d) $\widetilde{f}_i^h \psi(b) = \psi(\widetilde{f}_i b)$ if $\psi(b) \in \mathscr{B}_2$, $\widetilde{f}_i b \in \mathscr{B}_1$.

When $h = 1$, it is simply called a *morphism of crystals*. A morphism $\psi : \mathscr{B}_1 \to \mathscr{B}_2$ is *strict* if $\psi$ commutes with $\widetilde{e}_i, \widetilde{f}_i$ for all $i \in I$ without any restriction. A strict morphism $\psi : \mathscr{B}_1 \to \mathscr{B}_2$ is called a *strict embedding* if $\psi$ is an injective map from $\mathscr{B}_1 \sqcup \{0\}$ to $\mathscr{B}_2 \sqcup \{0\}$.

DEFINITION 2.32

The *tensor product* $\mathscr{B}_1 \otimes \mathscr{B}_2$ of crystals $\mathscr{B}_1$ and $\mathscr{B}_2$ is defined to be the set $\mathscr{B}_1 \times \mathscr{B}_2$ with maps given by

(2.33a) $$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

(2.33b) $$\varepsilon_i(b_1 \otimes b_2) = \max\bigl(\varepsilon_i(b_1), \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)\bigr),$$

(2.33c) $$\varphi_i(b_1 \otimes b_2) = \max\bigl(\varphi_i(b_2), \varphi_i(b_1) + \mathrm{wt}_i(b_2)\bigr),$$

(2.33d) $$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2 & \text{otherwise}, \end{cases}$$

$$\text{(2.33e)} \qquad \widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

Here $(b_1, b_2)$ is denoted by $b_1 \otimes b_2$ and $0 \otimes b_2$, $b_1 \otimes 0$ are identified with zero.

Iterating (2.33d) and (2.33e), we obtain the followings:

$$\text{(2.34a)} \quad \widetilde{e}_i^n(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i^n b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ \widetilde{e}_i^{n-\varepsilon_i(b_2)+\varphi_i(b_2)} b_1 & \\ \quad \otimes \widetilde{e}_i^{\varepsilon_i(b_2)-\varphi_i(b_1)} b_2 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1) \geq \varepsilon_i(b_2) - n, \\ b_1 \otimes \widetilde{e}_i^n b_2 & \text{if } \varepsilon_i(b_2) - n \geq \varphi_i(b_1), \end{cases}$$

$$\text{(2.34b)} \quad \widetilde{f}_i^n(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i^n b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) + n, \\ \widetilde{f}_i^{\varphi_i(b_1)-\varepsilon_i(b_2)} b_1 & \\ \quad \otimes \widetilde{f}_i^{n-\varphi_i(b_1)+\varepsilon_i(b_2)} b_2 & \text{if } \varepsilon_i(b_2) + n \geq \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i^n b_2 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1). \end{cases}$$

*2.3.2*
The (lower) crystal basis $\mathscr{B}(\infty)$ of $\mathbf{U}_q^-(\mathfrak{g})$ is an example of an abstract crystal which is upper normal but not lower normal. The lower crystal basis $\mathscr{B}^{\text{low}}(\lambda)$ of $V(\lambda)$ for $\lambda \in P_+$ is an example of seminormal crystal. We may also write $\mathscr{B}(\lambda)$ instead of $\mathscr{B}^{\text{low}}(\lambda)$, when it is considered as an abstract crystal.

EXAMPLE 2.35
For $i \in I$, let $\mathscr{B}_i = \{b_i(n); n \in \mathbb{Z}\}$. We can endow it with a structure of the abstract crystal by $\mathrm{wt}(b_i(n)) = n\alpha_i$, $\varepsilon_i(b_i(n)) = -n$, $\varphi_i(b_i(n)) = n$, $\varepsilon_j(b_i(n)) = \varphi_j(b_i(n)) = -\infty$, for $j \neq i$, and

$$\widetilde{f}_j b_i(n) = \begin{cases} b_i(n-1) & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

$$\widetilde{e}_j b_i(n) = \begin{cases} b_i(n+1) & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

*2.3.3*
For the crystal $\mathscr{B}(\infty)$, we have the following strict embedding.

THEOREM 2.36 ([29, THEOREM 2.2.1])
(1) *For each* $i \in I$, *there exists a strict embedding* $\Psi_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty) \otimes \mathscr{B}_i$ *which satisfies* $\Psi_i(u_\infty) = u_\infty \otimes b_i(0)$.
(2) *If* $\Psi_i(b) = b' \otimes \widetilde{f}_i^n b_i(0)$, *we have*

$$\Psi_i(\widetilde{e}_i^* b) = \begin{cases} b' \otimes \widetilde{f}_i^{n-1} b_i(0) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$$

$$\Psi_i(\widetilde{f}_i^* b) = b' \otimes \widetilde{f}_i^{n+1} b_i(0).$$

(3) *We have* $\operatorname{Im}\Psi_i = \{b' \otimes \widetilde{f}_i^n b_i(0); \varepsilon_i^*(b') = 0, n \geq 0\}$.

In particular, we have

$$\Psi_i(b) = \widetilde{e}_i^{*\varepsilon_i^*(b)} b \otimes \widetilde{f}_i^{\varepsilon_i^*(b)} b_i(0), \tag{2.37}$$

$$\varepsilon_i(b) = \max\bigl(\varepsilon_i(\widetilde{e}_i^{*\max} b), -\varphi_i(b)\bigr). \tag{2.38}$$

*2.3.4*
For $m \geq 1$, we have the following crystal morphism of amplitude $m$ which is called *inflation of order* $m$ in [30, Definition 8.1.4].

PROPOSITION 2.39 ([30, PROPOSITION 8.1.3], [47, PROPOSITION 3.2])

(1) *For* $m \in \mathbb{Z}_{\geq 1}$, *there exists a unique crystal morphism* $\mathcal{S}_m : \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ *of amplitude* $m$ *satisfying*

$$\operatorname{wt}(\mathcal{S}_m b) = m \operatorname{wt}(b), \qquad \varepsilon_i(\mathcal{S}_m b) = m\varepsilon_i(b), \qquad \varphi_i(\mathcal{S}_m b) = m\varphi_i(b),$$

$$\mathcal{S}_m(\widetilde{e}_i b) = \widetilde{e}_i^m \mathcal{S}_m(b), \qquad \mathcal{S}_m(\widetilde{f}_i b) = \widetilde{f}_i^m \mathcal{S}_m(b),$$

$$\mathcal{S}_m(u_\infty) = u_\infty.$$

(2) *Let* $b \in \mathscr{B}(\infty)$. *Then we have* $(* \circ \mathcal{S}_m)(b) = (\mathcal{S}_m \circ *)(b)$. *In particular, for any* $b \in \mathscr{B}(\infty)$, *we have*

$$\varepsilon_i^*(\mathcal{S}_m b) = m\varepsilon_i^*(b), \qquad \varphi_i^*(\mathcal{S}_m b) = m\varphi_i^*(b),$$

$$\mathcal{S}_m(\widetilde{e}_i^* b) = \widetilde{e}_i^{*m} \mathcal{S}_m(b), \qquad \mathcal{S}_m(\widetilde{f}_i^* b) = \widetilde{f}_i^{*m} \mathcal{S}_m(b).$$

## 3. The dual canonical basis

### 3.1

In this subsection, we recall the definition of the dual canonical basis and its characterization in terms of the *dual bar involution* $\sigma$ with a balanced triple. We define $\mathbf{B}^{\mathrm{up}} \subset \mathbf{U}_q^-(\mathfrak{g})$ as the dual basis of $\mathbf{B}$ under the Kashiwara's bilinear form $(\ ,\ )_K$. We define the *dual bar involution* $\sigma : \mathbf{U}_q^-(\mathfrak{g}) \to \mathbf{U}_q^-(\mathfrak{g})$ so that

$$\bigl(\sigma(x), y\bigr)_K = \overline{(x, \overline{y})_K}$$

holds for all $y$ (see [4, 10.2]). This is well defined since $(\cdot,\cdot)_K$ is nondegenerate. By its definition, we have $\sigma(x) = x$ for $x \in \mathbf{B}^{\mathrm{up}}$, and this is a $\mathbb{Q}$-linear involutive automorphism of $\mathbf{U}_q^-(\mathfrak{g})$ which satisfies $\sigma(fx) = \overline{f}\sigma(x)$ for all $f \in \mathbb{Q}(q)$ and $x \in \mathbf{U}_q^-(\mathfrak{g})$.

*3.1.1*
For $\xi = \sum \xi_i \alpha_i \in Q$, we define

$$N(\xi) := \frac{1}{2}\Big((\xi,\xi) + \sum \xi_i(\alpha_i,\alpha_i)\Big) = \frac{1}{2}\big((\xi,\xi) + 2(\xi,\rho)\big). \tag{3.1}$$

We have $N(-\alpha_i) = 0$ for all $i \in I$ and $N(\xi+\eta) = N(\xi) + N(\eta) + (\xi,\eta)$ for all $\xi,\eta \in Q$.

PROPOSITION 3.2

*We assume that $x, y \in \mathbf{U}_q^-(\mathfrak{g})$ are homogeneous.*

(1) *If $r(x) = \sum x_{(1)} \otimes x_{(2)}$, we have*

$$r(\overline{x}) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} \overline{x_{(2)}} \otimes \overline{x_{(1)}}.$$

(2) *We set $\{x,y\}_K := \overline{(\overline{x},\overline{y})_K}$; then we have*

$$\{x,y\}_K = q^{N(\mathrm{wt}\, x)}(x, *y)_K.$$

(3) *We have*

$$\sigma(x) = q^{N(\mathrm{wt}\, x)}(* \circ {}^-)(x).$$

*Proof*

For the convenience of the reader, we give a proof.

(1) We follow the argument in [45, Lemma 1.2.10]. For generators of $\mathbf{U}_q^-(\mathfrak{g})$, we have $r(f_i) = f_i \otimes 1 + 1 \otimes f_i = r(\overline{f_i})$. We prove the assertion by the induction on wt, so we assume that (1) holds for homogeneous $x', x''$ and show that it holds also for $x = x'x''$. First we write $r(x') = \sum x'_{(1)} \otimes x'_{(2)}$ and $r(x'') = \sum x''_{(1)} \otimes x''_{(2)}$. By assumption, we have $r(\overline{x'}) = \sum q^{-(\mathrm{wt}\, x'_{(1)}, \mathrm{wt}\, x'_{(2)})} \overline{x'_{(2)}} \otimes \overline{x'_{(1)}}$ and $r(\overline{x''}) = \sum q^{-(\mathrm{wt}\, x''_{(1)}, \mathrm{wt}\, x''_{(2)})} \overline{x''_{(2)}} \otimes \overline{x''_{(1)}}$. We have $r(x'x'') = r(x')r(x'') = \sum q^{-(\mathrm{wt}\, x'_{(2)}, \mathrm{wt}\, x''_{(1)})} x'_1 x''_1 \otimes x'_2 x''_2$ and

$$r(\overline{x'})r(\overline{x''}) = \sum q^{-(\mathrm{wt}\, x''_{(1)}, \mathrm{wt}\, x''_{(2)}) - (\mathrm{wt}\, x'_{(1)}, \mathrm{wt}\, x'_{(2)}) - (\mathrm{wt}\, x'_{(1)}, \mathrm{wt}\, x''_{(2)})} \overline{x'_{(2)} x''_{(2)}} \otimes \overline{x'_{(1)} x''_{(1)}}.$$

Then the assertion follows.

(2) We follow the argument in [45, Lemma 1.2.11(2)]. For the generators, we have $\{f_i, f_i\}_K = (f_i, f_i)_K = q^{N(\mathrm{wt}\, f_i)}(f_i, f_i)_K$.

We prove the assertion by induction on $\mathrm{tr}(\mathrm{wt}\, x) = \mathrm{tr}(\mathrm{wt}\, y)$. We prove that (2) holds for $y = y'y''$ and for any $x$ assuming it holds for $y', y''$. First we write $r(x) = \sum x_{(1)} \otimes x_{(2)}$ with $x_{(1)}$ and $x_{(2)}$ homogeneous. We have

$$\begin{aligned}
&(\overline{x}, \overline{y})_K \\
&= \big(r(\overline{x}), \overline{y'} \otimes \overline{y''}\big)_K = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} (\overline{x_{(2)}} \otimes \overline{x_{(1)}}, \overline{y'} \otimes \overline{y''})_K \\
&= \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} (\overline{x_{(2)}}, \overline{y'})_K (\overline{x_{(1)}}, \overline{y''})_K \\
&= \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)}) - N(\mathrm{wt}\, x_{(1)}) - N(\mathrm{wt}\, x_{(2)})} \overline{(x_{(2)}, *y')_K (x_{(1)}, *y'')_K} \\
&= \sum q^{-N(\mathrm{wt}\, x)} \overline{(x_{(2)}, *y')_K (x_{(1)}, *y'')_K},
\end{aligned}$$

where we have used the induction hypothesis in the fourth equality. On the other hand, we have

$$\begin{aligned}
&q^{-N(\operatorname{wt} x)}\overline{(x,*y)_K}\\
&\quad = q^{-N(\operatorname{wt} x)}\overline{\bigl(r(x),*y''\otimes *y'\bigr)_K}\\
&\quad = q^{-N(\operatorname{wt} x)}\sum \overline{(x_{(1)}\otimes x_{(2)},*y''\otimes *y')_K}.
\end{aligned}$$

Hence we obtain the assertion.

(3) We have $(\sigma(x),y)_K = \overline{(x,\overline{y})_K} = q^{N(\mathrm{wt}(x))}(\overline{x},*y)_K = q^{N(\mathrm{wt}(x))}((*\circ{}^{-})(x), y)_K$, where we used Lemma 2.13. Since this holds for any $y$, the assertion follows. □

*3.1.2*

By its construction, we have a characterization of the dual canonical basis $\mathbf{B}^{\mathrm{up}}$ in terms of the dual bar involution $\sigma$ and the crystal lattice $\mathscr{L}(\infty)$ of $\mathbf{U}_q^-(\mathfrak{g})$. We note that $\mathscr{L}(\infty)$ is a self-dual $\mathcal{A}_0$ lattice, see (2.19), and hence we do not need to introduce the dual lattice of $\mathscr{L}(\infty)$.

PROPOSITION 3.3

*We set*

$$\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} := \bigl\{x\in \mathbf{U}_q^-(\mathfrak{g}); \bigl(x,\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}\bigr)_K\subset \mathcal{A}\bigr\}.$$

*Then* $(\mathscr{L}(\infty),\sigma(\mathscr{L}(\infty)),\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}})$ *is a balanced triple for the dual canonical basis* $\mathbf{B}^{\mathrm{up}}$.

Here we have the following isomorphism of $\mathbb{Q}$-vector spaces:

$$\mathscr{L}(\infty)\cap\sigma\bigl(\mathscr{L}(\infty)\bigr)\cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \mathscr{L}(\infty)/q\mathscr{L}(\infty).$$

We denote its inverse by $G^{\mathrm{up}}$. Then we have $\mathbf{B}^{\mathrm{up}} = G^{\mathrm{up}}(\mathscr{B}(\infty))$.

*3.1.3*

The above proposition gives a characterization of the dual canonical basis elements.

COROLLARY 3.4 ([37, PROPOSITION 16])

*A homogeneous* $x\in \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}\cap\mathscr{L}(\infty)\cap\sigma(\mathscr{L}(\infty))$ *is an element of the dual canonical basis if and only if there exists* $b\in\mathscr{B}(\infty)$ *such that*

$$\begin{aligned}
\sigma(x) &= x,\\
x &\equiv b \mod q\mathscr{L}(\infty).
\end{aligned}$$

*3.1.4*

We have the following compatibility of the dual canonical basis and the $*$-involution from Proposition 2.22.

LEMMA 3.5

*For $b \in \mathscr{B}(\infty)$, we have*

$$G^{\mathrm{up}}(*b) = *G^{\mathrm{up}}(b).$$

### 3.2. Compatible subset

In this subsection, we introduce the concept of *compatible subsets* of $\mathscr{B}(\infty)$. Roughly speaking, they are closed under the multiplication up to $q$-shifts, considered as subsets of the dual canonical basis $\mathbf{B}^{\mathrm{up}}$.

*3.2.1*

By Proposition 3.2(3), we obtain the following.

PROPOSITION 3.6

*For homogeneous $x_1, x_2 \in \mathbf{U}_q^-(\mathfrak{g})$, we have*

$$\sigma(x_1 x_2) = q^{(\mathrm{wt}\, x_1, \mathrm{wt}\, x_2)} \sigma(x_2)\sigma(x_1). \tag{3.7}$$

Then we obtain the following property.

COROLLARY 3.8

*Let $b_1, b_2 \in \mathscr{B}(\infty)$, and consider the following expansion:*

$$G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = \sum_{\mathrm{wt}(b)=\mathrm{wt}(b_1)+\mathrm{wt}(b_2)} d^b_{b_1,b_2}(q) G^{\mathrm{up}}(b).$$

*Then we have $d^b_{b_1,b_2}(q^{-1}) = q^{(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} d^b_{b_2,b_1}(q)$. In particular, if we have $G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$, then we have*

$$G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = q^{-N-(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} G^{\mathrm{up}}(b_2)G^{\mathrm{up}}(b_1).$$

*Here we define $b_1 \circledast b_2 \in \mathscr{B}(\infty)$ for such a pair $b_1, b_2 \in \mathscr{B}(\infty)$; that is, $G^{\mathrm{up}}(b_1) \times G^{\mathrm{up}}(b_2) = q^N G^{\mathrm{up}}(b_1 \circledast b_2)$ for some $N \in \mathbf{Z}$,*

*Proof*

The first statement is clear from (3.7). Suppose that $G^{\mathrm{up}}(b_1) \times G^{\mathrm{up}}(b_2) = q^N G^{\mathrm{up}}(b_1 \circledast b_2)$ for $b_1 \circledast b_2 \in \mathscr{B}(\infty)$ and $N \in \mathbf{Z}$; that is, suppose that $d^b_{b_1,b_2}(q) = q^N \delta_{b, b_1 \circledast b_2}$ for $b_1 \circledast b_2 \in \mathscr{B}(\infty)$. Then we have

$$d^b_{b_2,b_1}(q) = q^{-(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} d^b_{b_1,b_2}(q^{-1}) = q^{-(\mathrm{wt}\, b_1, \mathrm{wt}\, b_2)} q^{-N} \delta_{b, b_1 \circledast b_2}.$$

This implies that if $G^{\mathrm{up}}(b_1)$ and $G^{\mathrm{up}}(b_2)$ satisfy $d^b_{b_1,b_2}(q) = q^N \delta_{b, b_1 \circledast b_2}$ for some $b_1 \circledast b_2 \in \mathscr{B}(\infty)$, then $G^{\mathrm{up}}(b_1)$ and $G^{\mathrm{up}}(b_2)$ $q$-commute. □

Motivated by this corollary, we introduce the following definition.

DEFINITION 3.9

(1) We denote $x \simeq y$ for $x, y \in \mathbf{U}_q^-(\mathfrak{g})$ if there exists $N \in \mathbb{Z}$ such that $x = q^N y$.

(2) For $b_1, b_2 \in \mathscr{B}(\infty)$, we call $b_1$ and $b_2$ *multiplicative* or *compatible* if there exists a unique $b_1 \circledast b_2 \in \mathscr{B}(\infty)$ such that

$$G^{\mathrm{up}}(b_1 \circledast b_2) \simeq G^{\mathrm{up}}(b_1) G^{\mathrm{up}}(b_2).$$

By Corollary 3.8 this condition is independent of the order on $b_1$ and $b_2$. We write $b_1 \perp b_2$ when this holds.

(3) Elements $b_1, \ldots, b_l \in \mathscr{B}(\infty)$ are called *compatible* if the following holds:

$$G^{\mathrm{up}}(b_1) \cdots G^{\mathrm{up}}(b_l) \simeq G^{\mathrm{up}}(b_1 \circledast \cdots \circledast b_l)$$

for a unique $b_1 \circledast \cdots \circledast b_l \in \mathscr{B}(\infty)$. This condition is also independent of the ordering on $b_1, \ldots, b_l$.

(4) An element $b \in \mathscr{B}(\infty)$ is called *real* if $G^{\mathrm{up}}(b) G^{\mathrm{up}}(b) \simeq G^{\mathrm{up}}(b^{[2]})$ for a unique $b^{[2]} \in \mathscr{B}(\infty)$, that is $b \perp b$.

(5) An element $b \in \mathscr{B}(\infty)$ is called *strongly real* if $G^{\mathrm{up}}(b)^m \simeq G^{\mathrm{up}}(b^{[m]})$ for a unique $b^{[m]} \in \mathscr{B}(\infty)$ for any $m$; that is, $\underbrace{b, \ldots, b}_{m \text{ times}}$ is compatible for any $m$.

(6) Elements $b_1, \ldots, b_l$ are called *strongly compatible* if for any $m_1, \ldots, m_l \in \mathbb{Z}_{\geq 0}$, the product $G^{\mathrm{up}}(b_1)^{m_1} \cdots G^{\mathrm{up}}(b_l)^{m_l} \simeq G^{\mathrm{up}}(b_1^{[m_1]} \circledast \cdots \circledast b_l^{[m_l]})$ for a unique $b_1^{[m_1]} \circledast \cdots \circledast b_l^{[m_l]} \in \mathscr{B}(\infty)$.

REMARK 3.10

For $b_1, b_2 \in \mathscr{B}(\infty)$, we say that a pair $(b_1, b_2)$ is *quasi-commutative* if we have $G^{\mathrm{up}}(b_1) G^{\mathrm{up}}(b_2) \simeq G^{\mathrm{up}}(b_2) G^{\mathrm{up}}(b_1)$ following [3] and [51]. In [3, Introduction], Berenstein and Zelevinsky conjectured that the quasi-commutativity and compatibility are equivalent. The above corollary proves Reineke's result that the compatibility for $b_1$ and $b_2$ implies the quasi-commutativity and also generalizes Reineke's result from when $\mathfrak{g}$ is symmetric to arbitrary symmetrizable $\mathfrak{g}$.

REMARK 3.11

The relation $b_1 \perp b_2$ is *not* an equivalence relation, as there exist $b$ which do not satisfy $b \perp b$. In particular, such elements are counterexamples for Berenstein and Zelevinsky's conjecture in [3]. In [35], Leclerc said that $b$ is *real* if $b \perp b$ and *imaginary* otherwise. He constructed examples of imaginary elements in [35]. Other examples closely related to this paper are given in [34, Corollary 4.4].

REMARK 3.12

Even if $b_1 \perp b_2$, we cannot determine $N$ in $d^b_{b_1, b_2} = q^N \delta_{b, b_1 \circledast b_2}$ in terms of weight of $b_1, b_2$. In Section 4, we have its formula in terms of the Lusztig data of $b$ and $b'$ associated with a reduced expression $\widetilde{w}$.

COROLLARY 3.13

(1) *If* $b_1 \perp b_2$*, then* $*b_1 \perp *b_2$*.*

(2) *If* $b$ *is real, then* $*b$ *is also real.*

*3.2.2*
Let ${}_ir^{(m)} := {}_ir^m/[m]!$ and ${r_i}^{(m)} := {r_i}^m/[m]!$. These operators are adjoint to the left and right multiplications of $f_i^{(m)}$ by (2.10a). From Theorem 2.27, we get the following expansions for the actions of ${}_ir^{(m)}$ and ${r_i}^{(m)}$.

THEOREM 3.14
*For $b \in \mathscr{B}(\infty)$, we have*

$$ {}_ir^{(m)} G^{\mathrm{up}}(b) = \begin{bmatrix} \varepsilon_i(b) \\ m \end{bmatrix} G^{\mathrm{up}}(\tilde{e}_i^m b) + \sum_{\varepsilon_i(b') < \varepsilon_i(b) - m} E^{(m)}_{bb';i}(q) G^{\mathrm{up}}(b'), \tag{3.15a} $$

$$ {r_i}^{(m)} G^{\mathrm{up}}(b) = \begin{bmatrix} \varepsilon_i^*(b) \\ m \end{bmatrix} G^{\mathrm{up}}(\tilde{e}_i^{*m} b) + \sum_{\varepsilon_i^*(b') < \varepsilon_i^*(b) - m} E^{*(m)}_{bb';i}(q) G^{\mathrm{up}}(b'), \tag{3.15b} $$

*where $E^{(m)}_{bb';i}(q) = \overline{E^{(m)}_{bb';i}(q)}, E^{*(m)}_{bb';i}(q) = \overline{E^{*(m)}_{bb';i}(q)} \in \mathcal{A}$.*

As a special case, we have the following result.

COROLLARY 3.16 ([28, LEMMA 5.1.1.])
*Let $b \in \mathscr{B}(\infty)$ with $\varepsilon_i(b) = c$ (resp., $\varepsilon_i^*(b) = c$). Then we have ${}_ir^{(c)} G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\tilde{e}_i^{\max} b)$ (resp., ${r_i}^{(c)} G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\tilde{e}_i^{*\max} b)$).*

By the above corollary and (2.7a), we obtain the following result.

COROLLARY 3.17 ([51, LEMMA 2.1])
*For $b_1, b_2 \in \mathscr{B}(\infty)$ with*

$$ G^{\mathrm{up}}(b_1) G^{\mathrm{up}}(b_2) = \sum d^b_{b_1,b_2}(q) G^{\mathrm{up}}(b), $$

*we have $\varepsilon_i(b) \le \varepsilon_i(b_1) + \varepsilon_i(b_2)$ for all $i \in I$ if $d^b_{b_1,b_2}(q) \ne 0$. An equality holds at least one $b$.*

If fact, we can prove $d^b_{b_1,b_2}(q) = 0$ if $\varepsilon_i(b) > \varepsilon_i(b_1) + \varepsilon_i(b_2)$ by the descending induction on $\varepsilon_i(b)$. In particular, the positivity of $d^b_{b_1,b_2}$, assumed in [51], is not used in the proof. The second assertion follows from

$$ \begin{aligned} &{}_ir^{(\varepsilon_i(b_1)+\varepsilon_i(b_2))} \big(G^{\mathrm{up}}(b_1) G^{\mathrm{up}}(b_2)\big) \\ &= q^N G^{\mathrm{up}}(\tilde{e}_i^{\max} b_1) G^{\mathrm{up}}(\tilde{e}_i^{\max} b_2) \\ &= \sum_{\varepsilon_i(b_1)+\varepsilon_i(b_2) = \varepsilon_i(b)} q^N d^b_{b_1,b_2}(q) G^{\mathrm{up}}(\tilde{e}_i^{\max} b) \end{aligned} \tag{3.18} $$

for some $N \in \mathbb{Z}$.

As a corollary of Corollaries 3.16 and 3.17, we obtain the following properties.

COROLLARY 3.19

(1) *If* $b_1 \perp b_2$, *then* $\widetilde{e}_i^{\max} b_1 \perp \widetilde{e}_i^{\max} b_2$ *for all* $i \in I$. *In fact, we have* $\varepsilon_i(b_1 \circledast b_2) = \varepsilon_i(b_1) + \varepsilon_i(b_2)$ *and* $\widetilde{e}_i^{\max}(b_1) \circledast \widetilde{e}_i^{\max}(b_2) = \widetilde{e}_i^{\max}(b_1 \circledast b_2)$. *A similar statement holds for* $\widetilde{e}_i^{*\max}$.

(2) *If* $b$ *is (resp., strongly) real,* $\widetilde{e}_i^{\max}(b)$ *is (resp., strongly) real for all* $i \in I$. *In fact, we have* $\varepsilon_i(b^{[m]}) = m\varepsilon_i(b)$ *and* $(\widetilde{e}_i^{\max} b)^{[m]} = \widetilde{e}_i^{\max}(b^{[m]})$ *for* $m = 2$ *(resp., any* $m$*). Similar statements hold for* $\widetilde{e}_i^{*\max}$.

LEMMA 3.20

*If* $b$ *is (resp., strongly) real, we have* $b^{[2]} = S_2(b)$ *(resp.,* $b^{[m]} = S_m(b)$*).*

*Proof*

For any $b$ with $|\operatorname{tr}(\mathrm{wt}(b))| > 0$, there exists $i \in I$ such that $\varepsilon_i(b) > 0$. Therefore we can connect $b$ to $u_\infty$ by a path consisting of (strongly) real elements by successive applications of $\widetilde{e}_i^{\max}$'s. From the formula in Corollary 3.19(2), we get the assertion. □

### 3.3. Compatibilities of the dual canonical basis

In this subsection, we study the dual canonical basis of integrable highest weight modules and its compatibilities with tensor products.

*3.3.1*

We recall the definition of the dual canonical basis of the integrable highest weight module $V(\lambda)$ following [28, Section 4.2]. Kashiwara calls it the *upper global basis*. Let $M$ be an integrable $\mathbf{U}_q(\mathfrak{g})$-module with a weight decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$. For $u \in \operatorname{Ker}(e_i) \cap M_\lambda$ and $0 \le n \le \langle h_i, \lambda\rangle$, we define other modified root operators called the *upper crystal operators*:

$$\widetilde{e}_i^{\mathrm{up}}(f_i^{(n)} u) = \frac{[\langle h_i, \lambda\rangle - n + 1]_i}{[n]_i} f_i^{(n-1)} u,$$

$$\widetilde{f}_i^{\mathrm{up}}(f_i^{(n)} u) = \frac{[n+1]_i}{[\langle h_i, \lambda\rangle - n]_i} f_i^{(n+1)} u.$$

We have a $\mathbb{Q}(q)$-linear antiautomorphism $\varphi$ on $\mathbf{U}_q(\mathfrak{g})$ defined by

$$\varphi(e_i) = f_i, \qquad \varphi(f_i) = e_i, \qquad \varphi(q^h) = q^h. \tag{3.21}$$

For $\lambda \in P_+$, we have a unique symmetric nondegenerate bilinear form $(\ ,\ )_\lambda : V(\lambda) \otimes V(\lambda) \to \mathbb{Q}(q)$ which satisfies

$$\big(\varphi(x)u, v\big)_\lambda = (u, xv)_\lambda \quad \text{for } u, v \in V(\lambda) \text{ and } x \in \mathbf{U}_q(\mathfrak{g}), \tag{3.22a}$$

$$(u_\lambda, u_\lambda)_\lambda = 1. \tag{3.22b}$$

Then we have

$$(\widetilde{e}_i^{\mathrm{up}} u, v)_\lambda = (u, \widetilde{f}_i^{\mathrm{low}} v)_\lambda, \tag{3.23a}$$

$$(\widetilde{f}_i^{\mathrm{up}} u, v)_\lambda = (u, \widetilde{e}_i^{\mathrm{low}} v)_\lambda. \tag{3.23b}$$

Using $(\cdot,\cdot)_\lambda$, we define the dual bar involution $\sigma_\lambda$ by

$$(\sigma_\lambda u, v)_\lambda := \overline{(u, \overline{v})_\lambda}.$$

This is well defined since $(\cdot,\cdot)_\lambda$ is a nondegenerate bilinear form. We set

$$V(\lambda)^{\mathrm{up}}_{\mathcal{A}} := \{u \in V(\lambda); (u, V(\lambda)_{\mathcal{A}})_\lambda \subset \mathcal{A}\}, \tag{3.24a}$$

$$\mathscr{L}^{\mathrm{up}}(\lambda) := \{u \in V(\lambda); (u, \mathscr{L}^{\mathrm{low}}(\lambda))_\lambda \subset \mathcal{A}_0\}. \tag{3.24b}$$

Then we have $\sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda)) = \{u \in V(\lambda); (u, \overline{\mathscr{L}^{\mathrm{low}}(\lambda)})_\lambda \subset \mathcal{A}_\infty\}$. Kashiwara denotes $\sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda))$ by $\overline{\mathscr{L}}^{\mathrm{up}}(\lambda)$. The triple $(\mathscr{L}^{\mathrm{up}}(\lambda), \sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda)), V(\lambda)^{\mathrm{up}}_{\mathcal{A}})$ is balanced by [28, Lemma 2.2.3].

PROPOSITION 3.25

*Let $\mathscr{B}^{\mathrm{up}}(\lambda)$ be the dual basis of $\mathscr{B}^{\mathrm{low}}(\lambda)$ with respect to the induced pairing $(\cdot,\cdot)_\lambda : \mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda) \times \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda) \to \mathbb{Q}$; then the pair $(\mathscr{L}^{\mathrm{up}}(\lambda), \mathscr{B}^{up}(\lambda))$ is an* upper crystal basis*; that is,*

(1) *$\mathscr{L}^{\mathrm{up}}(\lambda)$ is a free $\mathcal{A}_0$-module with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}^{\mathrm{up}}(\lambda) \simeq V(\lambda)$;*
(2) *$\widetilde{f}_i^{\mathrm{up}} \mathscr{L}^{\mathrm{up}}(\lambda) \subset \mathscr{L}^{\mathrm{up}}(\lambda)$ and $\widetilde{e}_i^{\mathrm{up}} \mathscr{L}^{\mathrm{up}}(\lambda) \subset \mathscr{L}^{\mathrm{up}}(\lambda)$;*
(3) *$\mathscr{B}^{\mathrm{up}}(\lambda) \subset \mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda)$ is a $\mathbb{Q}$-basis;*
(4) *$\widetilde{e}_i^{\mathrm{up}} \mathscr{B}^{\mathrm{up}}(\lambda) \subset \mathscr{B}^{\mathrm{up}}(\lambda) \sqcup \{0\}$ and $\widetilde{f}_i^{\mathrm{up}} \mathscr{B}^{\mathrm{up}}(\lambda) \subset \mathscr{B}^{\mathrm{up}}(\lambda) \sqcup \{0\}$;*
(5) *for $b, b' \in \mathscr{B}^{\mathrm{up}}(\lambda)$, $b = \widetilde{f}_i^{\mathrm{up}} b'$ is equivalent to $\widetilde{e}_i^{\mathrm{up}} b = b'$.*

Let $G^{\mathrm{up}}_\lambda$ be the inverse of $V(\lambda)^{\mathrm{up}}_{\mathcal{A}} \cap \mathscr{L}^{\mathrm{up}}(\lambda) \cap \sigma_\lambda(\mathscr{L}^{\mathrm{up}}(\lambda)) \xrightarrow{\sim} \mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda)$. The set $G^{\mathrm{up}}_\lambda(\mathscr{B}^{\mathrm{up}}(\lambda))$ is called the *dual canonical basis* of $V(\lambda)$. By its construction, the dual canonical basis is the dual basis of the canonical basis with respect to $(\ ,\ )_\lambda$. We also have

$$\mathscr{L}^{\mathrm{up}}(\lambda)_\mu = q^{(\lambda,\lambda)/2-(\mu,\mu)/2} \mathscr{L}^{\mathrm{low}}(\lambda)_\mu \quad \text{for } \mu \in P \tag{3.26}$$

(see [28, (4.2.9)]). By (3.26), we obtain an isomorphism $\mathscr{L}^{\mathrm{up}}(\lambda)/q\mathscr{L}^{\mathrm{up}}(\lambda) \simeq \mathscr{L}^{\mathrm{low}}(\lambda)/q\mathscr{L}^{\mathrm{low}}(\lambda)$. Through this identification, we have a bijection $\mathscr{B}^{\mathrm{up}}(\lambda) \simeq \mathscr{B}^{\mathrm{low}}(\lambda)$, and this bijection is an isomorphism of abstract crystals associated with the upper and lower crystal bases. Hence we can identify $\mathscr{B}^{\mathrm{up}}(\lambda)$ with $\mathscr{B}^{\mathrm{low}}(\lambda)$ and denote both by $\mathscr{B}(\lambda)$ hereafter. If $\mu \in W\lambda$, this identification is given by the identity as $(\lambda,\lambda) = (\mu,\mu)$. We can also prove that the canonical basis elements and the dual canonical basis elements coincide in this case.

REMARK 3.27

For $\mathbf{U}_q^-(\mathfrak{g})$, we consider the $\mathbb{Q}(q)$-linear antiautomorphism $a$ of the reduced $q$-analogue $\mathscr{B}_q(\mathfrak{g})$ defined by

$$a({}_ir) = f_i, \qquad a(f_i) = {}_ir. \tag{3.28}$$

Since the (lower) crystal lattice is self-dual with respect to Kashiwara's bilinear form $(\cdot,\cdot)_K$, we do not need to consider the dual lattice of $\mathscr{L}(\infty)$.

*3.3.2*

Using the pairing $(\ ,\ )_\lambda$, we consider a $\mathbb{Q}(q)$-linear embedding $j_\lambda : V(\lambda) \to \mathbf{U}_q^-(\mathfrak{g})$ which is defined in the following commutative diagram:

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\ \sim\ } & V(\lambda)^* \\ \Big\downarrow{\scriptstyle j_\lambda} & & \Big\downarrow{\scriptstyle \pi_\lambda^*} \\ \mathbf{U}_q^-(\mathfrak{g}) & \xrightarrow{\ \sim\ } & \mathbf{U}_q^-(\mathfrak{g})^* \end{array}$$

where the horizontal isomorphisms are induced by the nondegenerate inner products on $V(\lambda)$ and $\mathbf{U}_q^-(\mathfrak{g})$, and the right vertical homomorphism is the transpose of the $\mathbf{U}_q^-(\mathfrak{g})$-module homomorphism $\pi_\lambda : \mathbf{U}_q^-(\mathfrak{g}) \to V(\lambda)$ given by $x \mapsto xu_\lambda$ for $x \in \mathbf{U}_q^-(\mathfrak{g})$. Then for $b \in \mathscr{B}(\lambda)$, we have $j_\lambda G_\lambda^{\mathrm{up}}(b) = G^{\mathrm{up}}(j_\lambda(b))$, where $j_\lambda$ in the right-hand side was defined just after Theorem 2.25. Thanks to this equality, there is no fear of confusion even though we use the same symbol $j_\lambda$ for different maps.

*3.3.3*

We use the following result in [37, Section 7.3.2]. For $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_r \in P_+$ with $\lambda = \sum_j \lambda_j$, let $\Phi(\lambda_1, \ldots, \lambda_r) : V(\lambda) \to V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ be the unique $\mathbf{U}_q(\mathfrak{g})$-module homomorphism with $\Phi(\lambda_1, \ldots, \lambda_r)(u_\lambda) = u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_r}$. Then we have the corresponding embeddings

$$\Phi(\lambda_1, \ldots, \lambda_r) : \mathscr{B}(\lambda) \hookrightarrow \mathscr{B}(\lambda_1) \otimes \cdots \otimes \mathscr{B}(\lambda_r),$$

$$\Phi(\lambda_1, \ldots, \lambda_r)\big(\mathscr{L}^{\mathrm{low}}(\lambda)\big) \subset \mathscr{L}^{\mathrm{low}}(\lambda_1) \otimes_{\mathcal{A}_0} \cdots \otimes_{\mathcal{A}_0} \mathscr{L}^{\mathrm{low}}(\lambda_r)$$

(see [27, Section 4.2]). Hence we obtain

$$\Phi(\lambda_1, \ldots, \lambda_r)\big(G_\lambda^{\mathrm{low}}(b)\big) \equiv G_{\lambda_1}^{\mathrm{low}}(b_1) \otimes \cdots \otimes G_{\lambda_r}^{\mathrm{low}}(b_r) \\ \times \operatorname{mod} q\big(\mathscr{L}^{\mathrm{low}}(\lambda_1) \otimes \cdots \otimes \mathscr{L}^{\mathrm{low}}(\lambda_r)\big)$$

for $\Phi(\lambda_1, \ldots, \lambda_r)(b) = b_1 \otimes \cdots \otimes b_r$ for some $b_j \in \mathscr{B}^{\mathrm{low}}(\lambda_j)$.

Let $q_{\lambda_1, \ldots, \lambda_r} : V(\lambda_1) \otimes \cdots \otimes V(\lambda_r) \to V(\lambda)$ be the homomorphism defined by the commutative diagram

$$\begin{array}{ccc} V(\lambda_1) \otimes \cdots \otimes V(\lambda_r) & \xrightarrow{\ \sim\ } & V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_r)^* \\ \Big\downarrow{\scriptstyle q_{\lambda_1, \ldots, \lambda_r}} & & \Big\downarrow{\scriptstyle \Phi(\lambda_1, \ldots, \lambda_r)^*} \\ V(\lambda) & \xrightarrow{\ \sim\ } & V(\lambda)^* \end{array}$$

where the upper horizontal isomorphism is induced by the nondegenerate inner product $(\cdot, \cdot)_{\lambda_1, \ldots, \lambda_r} := (\cdot, \cdot)_{\lambda_1} \cdots (\cdot, \cdot)_{\lambda_r}$ on $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$, the lower horizontal isomorphism is induced by the nondegenerate inner product $(\cdot, \cdot)_\lambda$ on $V(\lambda)$, and the right vertical homomorphism is the transpose of $\Phi(\lambda_1, \ldots, \lambda_r)$.

PROPOSITION 3.29

*Let $\lambda, \lambda_1, \dots, \lambda_r \in P_+$ with $\lambda = \sum_{1 \le j \le r} \lambda_j$, and let $b_j \in \mathscr{B}(\lambda_j)$ $(1 \le j \le r)$. Assume that there exists $b_1 \diamond \cdots \diamond b_r \in \mathscr{B}(\lambda)$ with $\Phi(\lambda_1, \lambda_2, \dots, \lambda_r)(b_1 \diamond \cdots \diamond b_r) = b_1 \otimes \cdots \otimes b_r \in \mathscr{B}(\lambda_1) \otimes \cdots \otimes \mathscr{B}(\lambda_r)$. Then we have the following equality:*

$$q_{\lambda_1,\dots,\lambda_r}\big(G^{\mathrm{up}}_{\lambda_1}(b_1) \otimes \cdots \otimes G^{\mathrm{up}}_{\lambda_r}(b_r)\big) \equiv G^{\mathrm{up}}_{\lambda}(b_1 \diamond b_2 \diamond \cdots \diamond b_r) \bmod q\mathscr{L}^{\mathrm{up}}(\lambda).$$

We give the proof for completeness.

*Proof*

We have $q_{\lambda_1,\dots,\lambda_r}(\mathscr{L}^{\mathrm{up}}(\lambda_1) \otimes_{\mathcal{A}_0} \cdots \otimes_{\mathcal{A}_0} \mathscr{L}^{\mathrm{up}}(\lambda_r)) \subset \mathscr{L}^{\mathrm{up}}(\lambda)$; in particular, we have $q_{\lambda_1,\dots,\lambda_r}(G^{\mathrm{up}}_{\lambda_1}(b_1) \otimes \cdots \otimes G^{\mathrm{up}}_{\lambda_r}(b_r)) \in \mathscr{L}^{\mathrm{up}}(\lambda)$.

Hence to show the statement, it suffices to compute the following inner product:

$$\big(q_{\lambda_1,\dots,\lambda_r}(G^{\mathrm{up}}_{\lambda_1}(b_1) \otimes \cdots \otimes G^{\mathrm{up}}_{\lambda_r}(b_r)), G^{\mathrm{low}}_{\lambda}(b)\big)_{\lambda}|_{q=0}$$

for $b \in \mathscr{B}(\lambda)$. By its definition of $q_{\lambda_1,\dots,\lambda_r}$, this is equal to $(b_1 \otimes \cdots \otimes b_r, \Phi(\lambda_1, \dots, \lambda_r)(b))_{\lambda_1,\dots,\lambda_r}|_{q=0}$. Since the tensor product of the dual canonical basis is the dual of the tensor product of the canonical basis, this is equal to $\delta_{b_1 \otimes \cdots \otimes b_r, \Phi(\lambda_1,\dots,\lambda_r)(b)} = \delta_{b_1 \diamond b_2 \diamond \cdots \diamond b_r, b}$. Hence we have obtained the assertion. ☐

*3.3.4*

To compute a product of dual canonical basis elements of integrable highest weight modules, we need the following modification of the coproduct as in [37, Lemma 32].

LEMMA 3.30

*For $\lambda, \mu \in P_+$, let $r_{\lambda,\mu} : \mathbf{U}_q^-(\mathfrak{g}) \to \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g})$ be the $\mathbb{Q}(q)$-linear map defined by*

$$r_{\lambda,\mu} G^{\mathrm{low}}(b) = \sum_{b_1, b_2} d^{b}_{b_1,b_2}(q) q^{-(\mathrm{wt}(b_2),\lambda)} G^{\mathrm{low}}(b_1) \otimes G^{\mathrm{low}}(b_2)$$

*for $G^{\mathrm{low}}(b) \in \mathbf{U}_q^-(\mathfrak{g})$ with $r(G^{\mathrm{low}}(b)) = \sum_{b_1,b_2} d^{b}_{b_1,b_2}(q) G^{\mathrm{low}}(b_1) \otimes G^{\mathrm{low}}(b_2)$. Then we have the commutative diagram of $\mathbb{Q}(q)$-vector spaces*

$$\begin{array}{ccc} \mathbf{U}_q^-(\mathfrak{g}) & \xrightarrow{\pi_{\lambda+\mu}} & V(\lambda+\mu) \\ \Big\downarrow{\scriptstyle r_{\lambda,\mu}} & & \Big\downarrow{\scriptstyle \Phi(\lambda,\mu)} \\ \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(\mathfrak{g}) & \xrightarrow[\pi_\lambda \otimes \pi_\mu]{} & V(\lambda) \otimes V(\mu) \end{array}$$

Using the above modification, we obtain the following formula.

PROPOSITION 3.31

*For $b_1 \in \mathscr{B}(\lambda)$ and $b_2 \in \mathscr{B}(\mu)$, we have*

$$q^{(\mathrm{wt}\, b_2 - \mu, \lambda)} G^{\mathrm{up}}\big(j_\lambda(b_1)\big) G^{\mathrm{up}}\big(j_\mu(b_2)\big) = j_{\lambda+\mu} q_{\lambda,\mu}\big(G^{\mathrm{up}}_\lambda(b_1) \otimes G^{\mathrm{up}}_\mu(b_2)\big).$$

*3.3.5*

Combining Proposition 3.31 with Proposition 3.29, we obtain the following proposition.

PROPOSITION 3.32

*Let $\lambda, \lambda_1, \ldots, \lambda_r \in P_+$ with $\lambda = \sum_{1 \le j \le r} \lambda_j$ and $b_j \in \mathscr{B}(\lambda_j)$ $(1 \le j \le r)$. Assume that there exists $b_1 \diamond \cdots \diamond b_r \in \mathscr{B}(\lambda)$ with $\Phi(\lambda_1, \lambda_2, \ldots, \lambda_r)(b_1 \diamond \cdots \diamond b_r) = b_1 \otimes \cdots \otimes b_r \in \mathscr{B}(\lambda_1) \otimes \cdots \otimes \mathscr{B}(\lambda_r)$. Then there exists a unique $m \in \mathbb{Z}$ such that*

$$q^m G^{\mathrm{up}}\big(j_{\lambda_1}(b_1)\big) \cdots G^{\mathrm{up}}\big(j_{\lambda_r}(b_r)\big) \equiv G^{\mathrm{up}}\big(j_\lambda(b_1 \diamond \cdots \diamond b_r)\big) \mod q\mathscr{L}(\infty).$$

## 4. Quantum unipotent subgroup and the dual canonical basis

### 4.1. The Lie algebra $\mathfrak{n}(w)$

*4.1.1*

Let $w \in W$ be an element of the Weyl group associated with $\mathfrak{g}$, and let $\ell : W \to \mathbb{Z}_{\ge 0}$ be the length function. Let $\Delta^+(w) := \Delta^+ \cap w\Delta^- = \{\alpha \in \Delta^+ \mid w^{-1}\alpha < 0\} \subset \Delta^+$. We have the following description of $\Delta^+(w)$ as follows (see [33, Lemma 1.3.14]).

For a Weyl group element $w$, let $\widetilde{w} = (i_1, i_2, \ldots, i_\ell) \in R(w)$ be a reduced expression of $w$, where $R(w)$ is the set of reduced expressions of $w$. For each $1 \le k \le \ell = \ell(w)$, we set

$$\beta_k := s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

Then $\Delta^+(w)$ has cardinality exactly equal to $\ell = \ell(w)$, and we have

$$\Delta^+(w) = \{\beta_k\}_{1 \le k \le \ell}.$$

Let

$$\mathfrak{n}(w) := \bigoplus_{\alpha \in \Delta^+(w)} \mathfrak{g}_{-\alpha}.$$

Let $N(w)$ be the corresponding (pro)unipotent (pro)group in [33, Chapter VI]. Then $N(w)$ is a unipotent algebraic group of dimension $\ell(w)$, and its Lie algebra is $\mathfrak{n}(w)$. We can identify the restricted dual $U(\mathfrak{n}(w))^*_{\mathrm{gr}}$ of $U(\mathfrak{n}(w))$ with the coordinate ring of $N(w)$; that is, $U(\mathfrak{n}(w))^*_{\mathrm{gr}} \simeq \mathbb{C}[N(w)]$ (see [22, Section 5.2] for more details).

### 4.2. Braid group symmetry on $\mathbf{U}_q(\mathfrak{g})$

We define (quantum) root vectors, using Lusztig's braid group symmetry $\{T_i\}$ on $\mathbf{U}_q(\mathfrak{g})$ (see [45, Chapter 32] for more details).

*4.2.1*
Following [45, Section 37.1.3], we define the $\mathbb{Q}(q)$-algebra automorphisms $T'_{i,\epsilon}$ : $\mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ for $i \in I$ and $\epsilon \in \{\pm 1\}$ by

$$T'_{i,\epsilon}(q^h) = q^{s_i(h)}, \tag{4.1a}$$

$$T'_{i,\epsilon}(e_i) = -t_i^{\epsilon} f_i, \tag{4.1b}$$

$$T'_{i,\epsilon}(f_i) = -e_i t_i^{-\epsilon}, \tag{4.1c}$$

$$T'_{i,\epsilon}(e_j) = \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{\epsilon r} e_i^{(r)} e_j e_i^{(s)} \quad \text{for } j \neq i, \tag{4.1d}$$

$$T'_{i,\epsilon}(f_j) = \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-\epsilon r} f_i^{(s)} f_j f_i^{(r)} \quad \text{for } j \neq i. \tag{4.1e}$$

For $i \in I$ and $\epsilon \in \{\pm 1\}$, we also define the $\mathbb{Q}(q)$-algebra automorphisms $T''_{i,\epsilon} : \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$ by

$$T''_{i,-\epsilon}(q^h) = q^{s_i(h)}, \tag{4.2a}$$

$$T''_{i,-\epsilon}(e_i) = -f_i t_i^{-\epsilon}, \tag{4.2b}$$

$$T''_{i,-\epsilon}(f_i) = -t_i^{\epsilon} e_i, \tag{4.2c}$$

$$T''_{i,-\epsilon}(e_j) = \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{\epsilon r} e_i^{(s)} e_j e_i^{(r)} \quad \text{for } j \neq i, \tag{4.2d}$$

$$T''_{i,-\epsilon}(f_j) = \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-\epsilon r} f_i^{(r)} f_j f_i^{(s)} \quad \text{for } j \neq i. \tag{4.2e}$$

We have

$$T'_{i,\epsilon} T''_{i,-\epsilon} = T''_{i,-\epsilon} T'_{i,\epsilon} = \mathrm{id}. \tag{4.3}$$

In the following, we write $T_i = T''_{i,1}$ and $T_i^{-1} = T'_{i,-1}$ as in [53, Proposition 1.3.1].

*4.2.2*
We define braid group action on integrable modules following [45, Chapter 5] and [53]. We use a $q$-analogue of exponential $\exp_q(x)$ defined by

$$\exp_q(x) := \sum_{n \geq 0} \frac{q^{n(n-1)/2}}{[n]_q!} x^n.$$

We have

$$\exp_q(x) \exp_{q^{-1}}(-x) = 1. \tag{4.4}$$

For $i \in I$, we define $S_i$ (see [53, (1.2.2), (1.2.13)]) by

$$S_i = \exp_{q_i^{-1}}(q_i^{-1} e_i t_i^{-1}) \exp_{q_i^{-1}}(-f_i) \exp_{q_i^{-1}}(q_i e_i t_i) q_i^{h_i(h_i+1)/2} \tag{4.5a}$$

$$= \exp_{q_i^{-1}}(-q_i^{-1} f_i t_i) \exp_{q_i^{-1}}(e_i) \exp_{q_i^{-1}}(-q_i f_i t_i^{-1}) q_i^{h_i(h_i+1)/2}. \tag{4.5b}$$

The operator $q_i^{h_i(h_i+1)/2}$ acts on the weight space of weight $\lambda$ by the multiplication of $q_i^{\langle h_i,\lambda\rangle(\langle h_i,\lambda\rangle+1)/2}$. Since the action of $e_i$ and $f_i$ are locally nilpotent, $S_i$ defines an endomorphism of integrable modules. It is known that the action of $\{S_i\}_{i\in I}$ satisfies the braid group relations for the Weyl group $W$.

The braid group symmetry $\{T_i\}_{i\in I}$ defined above is described as

$$T_i(x) = S_i x S_i^{-1}, \tag{4.6}$$

where the elements are considered in the endomorphism ring of integrable modules (see [53, 1.3] for more details).

*4.2.3*

We have the following relationship between $T_i, T_i^{-1}$ and the $*$-involution.

PROPOSITION 4.7 ([45, SECTION 37.2.4])

*We have*

$$* \circ T_i \circ * = T_i^{-1}. \tag{4.8}$$

### 4.3. Quantum nilpotent subalgebra $\mathbf{U}_q^-(w,\epsilon)$

*4.3.1*

We define root vectors associated with $\widetilde{w} = (i_1, \ldots, i_\ell) \in R(w)$ for $w \in W$ (see [45, Propositions 40.1.3, 41.1.4] for more detail). For $w \in W$ and $\widetilde{w} \in R(w)$, we define $\beta_k$ as above. We define the *root vectors* $F_\epsilon(\beta_k)$ associated with $\beta_k \in \Delta(w)$ and $\epsilon \in \{\pm 1\}$ by

$$F_\epsilon(\beta_k) := T_{i_1}^\epsilon \cdots T_{i_{k-1}}^\epsilon (f_{i_k}).$$

It is known that $F_\epsilon(\beta_k) \in \mathbf{U}_q^-(\mathfrak{g})$. We note that $F_\epsilon(\beta_k)$ *does* depend on the choice of $\widetilde{w} \in R(w)$. We define their divided powers by

$$F_\epsilon(c\beta_k) := T_{i_1}^\epsilon \cdots T_{i_{k-1}}^\epsilon (f_{i_k}^{(c)})$$

for $c \geq 1$. It is known that $F_\epsilon(c\beta_k) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$.

*4.3.2*

THEOREM 4.9 ([45, PROPOSITIONS 40.2.1, 41.1.3])

(1) *For $w \in W$, $\widetilde{w} \in R(w)$, $\epsilon \in \{\pm 1\}$, and $\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell$, we set*

$$F_\epsilon(\mathbf{c}, \widetilde{w}) := \begin{cases} F_\epsilon(c_1\beta_1) \cdots F_\epsilon(c_\ell\beta_\ell) & \text{if } \epsilon = +1, \\ F_\epsilon(c_\ell\beta_\ell) \cdots F_\epsilon(c_1\beta_1) & \text{if } \epsilon = -1. \end{cases}$$

*Then $\{F_\epsilon(\mathbf{c}, \widetilde{w})\}_{\mathbf{c}\in\mathbb{Z}_{\geq 0}^\ell}$ forms a basis of a subspace defined to be $\mathbf{U}_q^-(w,\epsilon)$ of $\mathbf{U}_q^-(\mathfrak{g})$ which does not depend on $\widetilde{w}$.*

(2) *We have $F_\epsilon(\mathbf{c}, \widetilde{w}) \in \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ for all $\mathbf{c} \in \mathbb{Z}_{\geq 0}^\ell$.*

*4.3.3*

We recall commutation relations for root vectors and their divided powers $\{F_\epsilon(c_k\beta_k)\}_{1\le k\le \ell, c_k\ge 1}$, known as the *Levendorskii–Soibelman formula* (see [39], [1], [49] for more details).

In this subsection, we give statements for the $\epsilon=+1$ case. We can obtain the corresponding results for the $\epsilon=-1$ case, applying the $*$-involution (4.8). So we denote $F_\epsilon(c\beta)$, $F_\epsilon(\mathbf{c},\widetilde{w})$ by $F(c\beta)$, $F(\mathbf{c},\widetilde{w})$ by omitting $\epsilon$.

Let $w\in W$, $\widetilde{w}=(i_1,i_2,\dots,i_\ell)\in R(w)$, and fix a total order on $\Delta^+(w)$ given by

$$\beta_1<\beta_2<\cdots<\beta_\ell.$$

THEOREM 4.10 ([49, PROPOSITION 3.6], [39, SECTION 5.5.2, PROPOSITION])

*For $j<k$, let us write*

$$F(c_k\beta_k)F(c_j\beta_j)-q^{-(c_j\beta_j,c_k\beta_k)}F(c_j\beta_j)F(c_k\beta_k)=\sum f_{\mathbf{c}'}F(\mathbf{c}',\widetilde{w})$$

*with $f_{\mathbf{c}'}\in\mathbb{Q}(q)$. If $f_{\mathbf{c}'}\neq 0$, then $c'_j<c_j$ and $c'_k<c_k$ with $\sum_{j\le m\le k}c'_m\beta_m=c_j\beta_j+c_k\beta_k$.*

*4.3.4*

The following proposition is a consequence of Theorem 4.10. (cf. [38, Section 2.4.2, Proposition, Theorem (b)] and [11, Proposition 2.2].)

PROPOSITION 4.11

*Let $\widetilde{w}=(i_1,i_2,\dots,i_\ell)$ be a reduced expression for $w\in W$ and $\epsilon\in\{\pm1\}$. Then the subspace $\mathbf{U}_q^-(w,\epsilon)$ is the $\mathbb{Q}(q)$-subalgebra generated by $\{F_\epsilon(\beta_k)\}_{1\le k\le\ell}$.*

We call it the *quantum nilpotent subalgebra* associated with $w\in W$.

*4.3.5*

We define a lexicographic order $\le$ on $\mathbb{Z}_{\ge0}^\ell$ associated with $\widetilde{w}\in R(w)$ by

$$\begin{aligned}&\mathbf{c}=(c_1,c_2,\dots,c_\ell)<\mathbf{c}'=(c'_1,c'_2,\dots,c'_\ell)\\&\quad\iff \text{there exists } 1\le p\le\ell \text{ such that } c_1=c'_1,\dots,c_{p-1}=c'_{p-1},c_p<c'_p.\end{aligned}$$

The following theorem is obtained as a consequence of the Levendorskii–Soibelmann formula.

THEOREM 4.12

*Let $w\in W$, and let $\widetilde{w}\in R(w)$ be a reduced expression. For $\mathbf{c}\in\mathbb{Z}_{\ge0}^\ell$, we consider the following $\mathbb{Q}(q)$-subspace $\mathcal{F}^{\widetilde{w}}_{\le\mathbf{c}}\mathbf{U}_q^-(w)$:*

$$\mathcal{F}^{\widetilde{w}}_{\le\mathbf{c}}\mathbf{U}_q^-(w):=\bigoplus_{\mathbf{c}'\le\mathbf{c}}\mathbb{Q}(q)F(\mathbf{c}',\widetilde{w}).$$

*Then*

(1) $\{\mathcal{F}^{\widetilde{w}}_{\leq \mathbf{c}}\mathbf{U}_q^-(w)\}_{\mathbf{c}\in\mathbb{Z}^{\ell}_{\geq 0}}$ *forms an increasing filtration on* $\mathbf{U}_q^-(w)$*;*

(2) *the associated graded algebra* $\mathrm{gr}^{\widetilde{w}}\mathbf{U}_q^-(w)$ *is generated by* $\{\mathrm{gr}^{\widetilde{w}}(F(\beta_k))\mid 1\leq k\leq \ell\}$*, with relations*

$$\mathrm{gr}^{\widetilde{w}}\big(F(\beta_k)\big)\mathrm{gr}^{\widetilde{w}}\big(F(\beta_j)\big) = q^{-(\beta_j,\beta_k)}\mathrm{gr}^{\widetilde{w}}\big(F(\beta_j)\big)\mathrm{gr}^{\widetilde{w}}\big(F(\beta_k)\big)(j<k).$$

*Here the associated graded algebra* $\mathrm{gr}^{\widetilde{w}}\mathbf{U}_q^-(w) := \bigoplus_{\mathbf{c}\in\mathbb{Z}^{\ell}_{\geq 0}}\mathrm{gr}^{\widetilde{w}}_{\mathbf{c}}\mathbf{U}_q^-(w)$ *is defined by*

$$\mathrm{gr}^{\widetilde{w}}_{\mathbf{c}}\mathbf{U}_q^-(w) := \mathcal{F}^{\widetilde{w}}_{\leq\mathbf{c}}\mathbf{U}_q^-(w)/\sum_{\mathbf{c}'<\mathbf{c}}\mathbb{Q}(q)F(\mathbf{c}',\widetilde{w}),$$

*and we set* $\mathrm{gr}^{\widetilde{w}}F(\mathbf{c},\widetilde{w}) := F(\mathbf{c},\widetilde{w}) \bmod \sum_{\mathbf{c}'<\mathbf{c}}\mathbb{Q}(q)F(\mathbf{c}',\widetilde{w})$ *for* $\mathbf{c}\in\mathbb{Z}^{\ell}_{\geq 0}$.

We call this the *De Concini–Kac filtration.*

## 4.4. PBW basis and the canonical basis

In this subsection, we recall compatibilities between Lusztig's braid symmetry $\{T_i\}_{i\in I}$ and the canonical basis (for more details, see [45, Chapter 38], [43], [53]).

*4.4.1*

We have the following orthogonal decomposition with respect to Kashiwara's form $(\ ,\ )_K$ and its compatibility with the canonical basis.

LEMMA 4.13 ([45, PROPOSITION 38.1.6, LEMMA 38.1.5])

(1) *For* $i\in I$*, we have*

$$\mathbf{U}_q^-[i] := \{x\in\mathbf{U}_q^-; {}_ir(x)=0\} = \{x\in\mathbf{U}_q^-; T_i^{-1}(x)\in\mathbf{U}_q^-\},$$
$${}^*\mathbf{U}_q^-[i] := \{x\in\mathbf{U}_q^-; r_i(x)=0\} = \{x\in\mathbf{U}_q^-; T_i(x)\in\mathbf{U}_q^-\}.$$

(2) *For* $i\in I$*, we have the following orthogonal decompositions with respect to* $(\ ,\ )_K$*:*

$$\mathbf{U}_q^- = \mathbf{U}_q^-[i]\oplus f_i\mathbf{U}_q^- = {}^*\mathbf{U}_q^-[i]\oplus\mathbf{U}_q^- f_i.$$

From Lemma 4.13 and Theorem 2.29, we obtain the following result.

PROPOSITION 4.14

*For* $n\geq 0$ *and* $i\in I$*, the subspaces* $\bigoplus_{k=0}^n f_i^k\mathbf{U}_q^-[i]$ *and* $\bigoplus_{k=0}^n {}^*\mathbf{U}_q^-[i]f_i^k$*, which are orthogonal decompositions with respect to* $(\ ,\ )_K$*, are compatible with the dual canonical basis; that is, we have*

$$\bigoplus_{k=0}^n f_i^k\mathbf{U}_q^-[i] = \bigoplus_{b\in\mathscr{B}(\infty),\varepsilon_i(b)\leq n}\mathbb{Q}(q)G^{\mathrm{up}}(b),$$

$$\bigoplus_{k=0}^n {}^*\mathbf{U}_q^-[i]f_i^k = \bigoplus_{b\in\mathscr{B}(\infty),\varepsilon_i^*(b)\leq n}\mathbb{Q}(q)G^{\mathrm{up}}(b).$$

*4.4.2*
The following result is due to Saito.

PROPOSITION 4.15 ([53, PROPOSITION 3.4.7, COROLLARY 3.4.8])

(1) *Let* $x \in \mathscr{L}(\infty) \cap T_i^{-1}\mathbf{U}_q^-(\mathfrak{g})$ *with* $b = x \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty)$. *We note that* $r_i(x) = 0$ *and* $\varepsilon_i^*(b) = 0$. *We have*

$$T_i(x) \in \mathscr{L}(\infty), \tag{4.16}$$

$$T_i(x) \equiv \widetilde{f}_i^{*\varphi_i(b)} \widetilde{e}_i^{\varepsilon_i(b)} x \quad \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty). \tag{4.17}$$

(2) *Let* $\Lambda_i : \{b \in \mathscr{B}(\infty); \varepsilon_i^*(b) = 0\} \to \{b \in \mathscr{B}(\infty); \varepsilon_i(b) = 0\}$ *be the map defined by* $\Lambda_i(b) = \widetilde{f}_i^{*\varphi_i(b)} \widetilde{e}_i^{\varepsilon_i(b)} b$. *Then* $\Lambda_i$ *is bijective and its inverse is given by* $\Lambda_i^{-1}(b) = \widetilde{f}_i^{\varphi_i^*(b)} \widetilde{e}_i^{*\varepsilon_i^*(b)} b$.

By Proposition 4.15, we can show the following result by using the induction on the length of $w$.

THEOREM 4.18 ([53, THEOREM 4.1.2], [43, PROPOSITION 8.2])
*For* $w \in W$, $\widetilde{w} = (i_1, i_2, \dots, i_\ell) \in R(w)$, *and* $\epsilon \in \{\pm 1\}$,

(1) *we have* $F_\epsilon(\mathbf{c}, \widetilde{w}) \in \mathscr{L}(\infty)$ *and*

$$b_\epsilon(\mathbf{c}, \widetilde{w}) := F_\epsilon(\mathbf{c}, \widetilde{w}) \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty);$$

(2) *the map* $\mathbb{Z}_{\geq 0}^\ell \to \mathscr{B}(\infty)$ *which is defined by* $\mathbf{c} \mapsto b_\epsilon(\mathbf{c}, \widetilde{w})$ *is injective. We denote the image by* $\mathscr{B}(w, \epsilon)$, *and this does not depend on the choice of* $\widetilde{w} \in R(w)$.

For fixed $\widetilde{w} \in R(w)$, we denote the inverse of $\mathbf{c} \mapsto b_\epsilon(\mathbf{c}, \widetilde{w})$ by $L_{\epsilon,\widetilde{w}} : \mathscr{B}(w, \epsilon) \to \mathbb{Z}_{\geq 0}^\ell$. This map is called the *Lusztig data* of $b$ associated with $\widetilde{w}$.

*4.4.3*
Let ${}^i\pi : \mathbf{U}_q^- \to \mathbf{U}_q^-[i]$ (resp., $\pi^i : \mathbf{U}_q^- \to {}^*\mathbf{U}_q^-[i]$) be the orthogonal projection whose kernel is $f_i\mathbf{U}_q^-(\mathfrak{g})$ (resp., $\mathbf{U}_q^-(\mathfrak{g})f_i$) in Lemma 4.13. The following result is due to Lusztig.

THEOREM 4.19 ([43, THEOREM 1.2])
*For* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i^*(b) = 0$, *we have*

$$T_i\big(\pi^i G^{\mathrm{low}}(b)\big) = {}^i\pi\big(G^{\mathrm{low}}(\Lambda_i(b))\big).$$

*4.4.4*
As a corollary of the above description, we have the following properties of the inflation $S_m$ and the $*$-involution.

COROLLARY 4.20

(1) *We have* $\Lambda_i \mathcal{S}_m(b) = \mathcal{S}_m \Lambda_i(b)$ *for* $b \in \{b \in \mathscr{B}(\infty); \varepsilon_i^*(b) = 0\}$.

(2) *We have* $\mathcal{S}_m : \mathscr{B}(w,\epsilon) \to \mathscr{B}(w,\epsilon)$ *for all* $m \geq 1$ *and* $\mathcal{S}_m(b_\epsilon(\mathbf{c},\widetilde{w})) = b_\epsilon(m\mathbf{c},\widetilde{w})$.

(3) *We have* $*(\mathscr{B}(w,\epsilon)) = \mathscr{B}(w,-\epsilon)$ *and* $*b_\epsilon(\mathbf{c},\widetilde{w}) = b_{-\epsilon}(\mathbf{c},\widetilde{w})$.

### 4.5. Inner products of the PBW basis

By Lemma 2.12, we have the following modification of [45, Proposition 38.2.1].

PROPOSITION 4.21

*For* $x,y \in \mathbf{U}_q^-(\mathfrak{g})_\xi$ *with* $x,y \in \mathbf{U}_q^-[i]$ *(resp., with* $x,y \in {}^*\mathbf{U}_q^-[i]$*), we have*

$$(x,y)_K = (1-q_i^2)^{-\langle h_i,\xi\rangle}(T_i^{-1}x, T_i^{-1}y)_K \quad (\textit{resp.}, \ (1-q_i^2)^{-\langle h_i,\xi\rangle}(T_i x, T_i y)_K).$$

*4.5.1*

We have the following formula for the inner product of the PBW basis with respect to Lusztig's bilinear form $(\ ,\ )_\ell$ (for more details, see [45, Proposition 38.2.3]).

PROPOSITION 4.22

*Let* $w \in W$ *and* $\widetilde{w} \in R(w)$ *with* $\ell = \ell(w)$. *We have*

$$\big(F(\mathbf{c},\widetilde{w}), F(\mathbf{c}',\widetilde{w})\big)_L = \prod_{k=1}^{\ell} \delta_{c_k,c'_k} \prod_{s=1}^{c_k} \frac{1}{1-q_i^{2s}} = \prod_{k=1}^{\ell} \delta_{c_k,c'_k} (-1)^{c_k} \frac{q_{i_k}^{-(c_k(c_k+1))/2}}{(q_{i_k}-q_{i_k}^{-1})^{c_k}[c_k]_{i_k}^!}.$$

### 4.6. Compatibility with $T_i$ and the dual canonical basis

*4.6.1*

By using the above results, we obtain the following compatibility between the dual canonical basis and Lusztig's braid group symmetry $T_i$.

THEOREM 4.23

*For* $b \in \mathscr{B}(\infty)_\xi$ *with* $\varepsilon_i^*(b) = 0$, *we have*

$$(1-q_i^2)^{\langle h_i,\xi\rangle} T_i G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\Lambda_i b).$$

*Proof*

We shall prove that $((1-q_i^2)^{\langle h_i,\xi\rangle} T_i G^{\mathrm{up}}(b), G^{\mathrm{low}}(b'))_K = \delta_{b',\Lambda_i(b)}$. By Lemma 4.13, $(1-q_i^2)^{\langle h_i,\xi\rangle}(T_i G^{\mathrm{up}}(b), G^{\mathrm{low}}(b'))_K$ is equal to $(1-q_i^2)^{\langle h_i,\xi\rangle}(T_i G^{\mathrm{up}}(b), {}^i\pi \times G^{\mathrm{low}}(b'))_K$. By Proposition 4.21, this is equal to $(G^{\mathrm{up}}(b), T_i^{-1}\,{}^i\pi G^{\mathrm{low}}(b'))_K$. Using Theorem 4.19, we have

$$\begin{aligned}\big(G^{\mathrm{up}}(b), T_i^{-1}{}^i\pi G^{\mathrm{low}}(b')\big)_K &= \big(G^{\mathrm{up}}(b), \pi^i G^{\mathrm{low}}(\Lambda_i^{-1}b')\big)_K \\ &= \big(G^{\mathrm{up}}(b), G^{\mathrm{low}}(\Lambda_i^{-1}b')\big)_K = \delta_{b,\Lambda_i^{-1}(b')}.\end{aligned}$$

Then we obtain the assertion. □

As a corollary, we obtain the following multiplicative properties.

COROLLARY 4.24

(1) *For* $b_1, b_2 \in \mathscr{B}(\infty)$ *with* $\varepsilon_i^*(b_1) = \varepsilon_i^*(b_2) = 0$ *(resp.,* $\varepsilon_i(b_1) = \varepsilon_i(b_2) = 0$*) and* $b_1 \perp b_2$*, we have* $\Lambda_i(b_1) \perp \Lambda_i(b_2)$ *(resp.,* $\Lambda_i^{-1}(b_1) \perp \Lambda_i^{-1}(b_2)$*).*

(2) *If* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i^*(b) = 0$ *(resp.,* $\varepsilon_i(b) = 0$*) is (strongly) real, then* $\Lambda_i(b)$ *(resp.,* $\Lambda_i^{-1}(b)$*) is also (strongly) real.*

## 4.7. Compatibility with the dual canonical basis

In this subsection, we prove the compatibility of the dual canonical basis with the $\mathbb{Q}(q)$-subalgebra $\mathbf{U}_q^-(w,\epsilon)$. This is a straightforward generalization of [8, Section 2.2, Proposition] and [34, Theorem 4.1]. Here we fix $w \in W$ and $\widetilde{w} \in R(w)$.

THEOREM 4.25

*For* $w \in W$ *and* $\epsilon \in \{\pm 1\}$*, the triple* $(\mathbf{U}_q^-(w,\epsilon) \cap \mathscr{L}(\infty), \mathbf{U}_q^-(w,\epsilon) \cap \sigma(\mathscr{L}(\infty)), \mathbf{U}_q^-(w,\epsilon) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}})$ *is balanced. In particular we have*

$$\mathbf{U}_q^-(w,\epsilon) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} = \bigoplus_{b \in \mathscr{B}(w,\epsilon)} \mathcal{A} G^{\mathrm{up}}(b).$$

The proof of this theorem occupies the rest of this subsection. As in [37, Section 3.4], [36, Proposition 31, Corollary 41], and [8], we first prove that the dual root vectors are contained in the dual canonical basis and then prove the unitriangular property of upper global basis with respect to the dual PBW basis. The compatibility with the dual canonical basis is its direct consequence.

Our proof needs an extra step from ones in [37], [36], and [8], as it is not known that the PBW basis is an $\mathcal{A}$-basis of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}} \cap \mathbf{U}_q^-(w)$ unless $\mathfrak{g}$ is of finite or affine type.

*4.7.1*

PROPOSITION 4.26

(1) *For* $i \in I$ *and* $n \geq 1$*, let*

$$F^{\mathrm{up}}(n\alpha_i) := \frac{f_i^{(n)}}{(f_i^{(n)}, f_i^{(n)})_K}.$$

*Then we have* $F^{\mathrm{up}}(n\alpha_i) \in \mathbf{B}^{\mathrm{up}}$*,* $(F^{\mathrm{up}}(\alpha_i))^n \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$*, and* $F^{\mathrm{up}}(n\alpha_i)F^{\mathrm{up}}(m\alpha_i) = q_i^{-mn}F^{\mathrm{up}}((m+n)\alpha_i)$

(2) *For* $n \geq 1$ *and* $1 \leq k \leq \ell$*, let*

$$F^{\mathrm{up}}(n\beta_k) := \frac{F(n\beta_k)}{(F(n\beta_k), F(n\beta_k))_K}.$$

*Then we have* $F^{\mathrm{up}}(n\beta_k) \in \mathbf{B}^{\mathrm{up}}$*,* $F^{\mathrm{up}}(\beta_k)^n \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$*, and* $F^{\mathrm{up}}(n\beta_k)F^{\mathrm{up}}(m\beta_k) = q_{i_k}^{-mn}F^{\mathrm{up}}((m+n)\beta_k)$*.*

*Proof*
Since $f_i^{(n)}$ are the canonical basis elements and $\dim \mathbf{U}_q^-(\mathfrak{g})_{-n\alpha_i} = 1$ for all $n \geq 1$, $F^{\mathrm{up}}(n\alpha_i)$ are the dual canonical basis elements by its definition. By Proposition 4.22 and Lemma 2.12, we have

$$(f_i^{(n)}, f_i^{(n)})_K = (1-q_i^2)^n / \prod_{j=1}^{n}(1-q_i^{2j}) = \frac{1}{[n]_i!} q_i^{-(n(n-1))/2}.$$

Therefore we have $(F^{\mathrm{up}}(\alpha_i))^n = q_i^{-(n(n-1))/2} F^{\mathrm{up}}(n\alpha_i) \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$. Applying Theorem 4.23, we obtain the result for $F^{\mathrm{up}}(n\beta_k)$ for $1 \leq k \leq \ell$. □

*4.7.2*
For the computation of the action of the dual bar involution $\sigma$, we need an integrality property of the Levendorskii–Soibelman formula for the dual root vectors and their multiples. For $w \in W$, $\widetilde{w} \in R(w)$, and $\epsilon \in \{\pm 1\}$, we set

$$F_\epsilon^{\mathrm{up}}(\mathbf{c}, \widetilde{w}) := \frac{1}{(F_\epsilon(\mathbf{c}, \widetilde{w}), F_\epsilon(\mathbf{c}, \widetilde{w}))_K} F_\epsilon(\mathbf{c}, \widetilde{w}).$$

This is the dual basis of $\{F_\epsilon(\mathbf{c}, \widetilde{w})\}$ with respect to Kashiwara's bilinear form $(\cdot,\cdot)_K$. As before, when we consider only the $\epsilon = 1$ case, we omit the subscript $\epsilon$.

THEOREM 4.27 (DUAL LEVENDORSKII–SOIBELMAN FORMULA)
*For $j < k$, we write*

$$F^{\mathrm{up}}(c_k\beta_k)F^{\mathrm{up}}(c_j\beta_j) - q^{-(c_j\beta_j, c_k\beta_k)} F^{\mathrm{up}}(c_j\beta_j)F^{\mathrm{up}}(c_k\beta_k) = \sum f_{\mathbf{c}'}^* F^{\mathrm{up}}(\mathbf{c}', \widetilde{w}).$$

*Then $f_{\mathbf{c}'}^* \in \mathcal{A}$, and if $f_{\mathbf{c}'}^* \neq 0$, then $c_j' < c_j$ and $c_k' < c_k$ with $\sum_{j \leq m \leq k} c_m'\beta_m = c_j\beta_j + c_k\beta_k$.*

*Proof*
Firstly, a weaker statement that $f_{\mathbf{c}'}^* \in \mathbb{Q}(q)$ with the above conditions follows from Theorem 4.10 and Proposition 4.22. Let us prove that $f_{\mathbf{c}'}^* \in \mathcal{A}$. Since the twisted coproduct $r$ preserves the $\mathcal{A}$-form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$, the dual integral form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ is an $\mathcal{A}$-subalgebra of $\mathbf{U}_q^-(\mathfrak{g})$. Therefore the left-hand side belongs to $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ by Proposition 4.26. Taking the inner product with $F(\mathbf{c}', \widetilde{w})$, we find $f_{\mathbf{c}'}^* \in \mathcal{A}$ thanks to Theorem 4.9. □

We define an $\mathcal{A}$-form $\mathbf{U}_q^-(w,\epsilon)_{\mathcal{A}}^{\mathrm{up}}$ of $\mathbf{U}_q^-(w,\epsilon)$ by the $\mathcal{A}$-span of $\{F_\epsilon^{\mathrm{up}}(\mathbf{c}, \widetilde{w}); \mathbf{c} \in \mathbb{Z}_{\geq 0}^{\ell}\}$. By Theorem 4.27, it is an $\mathcal{A}$-subalgebra and generated by $\{F_\epsilon^{\mathrm{up}}(\beta_k)\}_{1 \leq k \leq \ell}$.

*4.7.3*
We compute the action of the dual bar involution $\sigma$ on the dual PBW basis. The following is a straightforward generalization of [8, Section 2.1, Corollary(i)] and follows from (3.7) and Theorem 4.27.

PROPOSITION 4.28

*We have*

$$\sigma\big(F^{\mathrm{up}}(\mathbf{c},\widetilde{w})\big) = F^{\mathrm{up}}(\mathbf{c},\widetilde{w}) + \sum_{\mathbf{c}'<\mathbf{c}} f^*_{\mathbf{c},\mathbf{c}'}(q) F^{\mathrm{up}}(\mathbf{c}',\widetilde{w}),$$

*where* $f^*_{\mathbf{c},\mathbf{c}'}(q) \in \mathcal{A}$.

*4.7.4*

THEOREM 4.29

(1) *Let* $w \in W$ *and* $\widetilde{w} \in R(w)$. *Then there exists a unique* $\mathcal{A}$*-basis* $\{B^{\mathrm{up}}(\mathbf{c},\widetilde{w});\ \mathbf{c} \in \mathbb{Z}^{\ell}_{\geq 0}\}$ *of* $\mathbf{U}^-_q(w,\epsilon)^{\mathrm{up}}_{\mathcal{A}}$ *with the following properties:*

$$\sigma\big(B^{\mathrm{up}}(\mathbf{c},\widetilde{w})\big) = B^{\mathrm{up}}(\mathbf{c},\widetilde{w}), \tag{4.30a}$$

$$F^{\mathrm{up}}(\mathbf{c},\widetilde{w}) = B^{\mathrm{up}}(\mathbf{c},\widetilde{w}) + \sum_{\mathbf{c}'<\mathbf{c}} \varphi_{\mathbf{c},\mathbf{c}'} B^{\mathrm{up}}(\mathbf{c}',\widetilde{w}), \quad \varphi_{\mathbf{c},\mathbf{c}'} \in q\mathbb{Z}[q]. \tag{4.30b}$$

(2) *We have* $B^{\mathrm{up}}(\mathbf{c},\widetilde{w}) = G^{\mathrm{up}}(b(\mathbf{c},\widetilde{w}))$.

*Proof*

The proof of (1) is the same as one for the existence of Kazhdan–Lusztig polynomials. The only claim we need is Proposition 4.28.

For (2), since we have $g_{\mathbf{c}}(q) := (F(\mathbf{c},\widetilde{w}), F(\mathbf{c},\widetilde{w}))_K \in \mathcal{A}_0$ and $g_{\mathbf{c}}(0) = 1$, we obtain

$$B^{\mathrm{up}}(\mathbf{c},\widetilde{w}) \equiv F^{\mathrm{up}}(\mathbf{c},\widetilde{w}) \equiv b(\mathbf{c},\widetilde{w}) \mod q\mathscr{L}(\infty).$$

Therefore (2) follows from (1) and Corollary 3.4. □

As a corollary, we have $\mathbf{U}^-_q(w,\epsilon)^{\mathrm{up}}_{\mathcal{A}} = \mathbf{U}^-_q(w,\epsilon) \cap \mathbf{U}^-_q(\mathfrak{g})^{\mathrm{up}}_{\mathcal{A}}$ since $\{G^{\mathrm{up}}(b)\}_{b\in\mathscr{B}(\infty)}$ is an $\mathcal{A}$-basis of $\mathbf{U}^-_q(\mathfrak{g})^{\mathrm{up}}_{\mathcal{A}}$. Together with this result, Theorem 4.29 implies Theorem 4.25.

## 4.8

In this subsection, we study basic commutation relations among the dual canonical basis elements of $\mathbf{U}^-_q(w,\epsilon=1)$. The following is a generalization of [51, Proposition 4.2], and follows from the characterization of the dual canonical basis in terms of the dual PBW basis. For $\mathbf{c},\mathbf{c}' \in \mathbb{Z}^{\ell}_{\geq 0}$, we set

$$c_{\widetilde{w}}(\mathbf{c},\mathbf{c}') := \sum_{k'<k} c_k c'_{k'}(\beta_k,\beta_{k'}) - \frac{1}{2}\sum_k c_k c'_k(\beta_k,\beta_k).$$

PROPOSITION 4.31

*We have*

$$G^{\mathrm{up}}\big(b(\mathbf{c},\widetilde{w})\big)G^{\mathrm{up}}\big(b(\mathbf{c}',\widetilde{w})\big) = q^{-c_{\widetilde{w}}(\mathbf{c},\mathbf{c}')}G^{\mathrm{up}}\big(b(\mathbf{c}+\mathbf{c}',\widetilde{w})\big) + \sum d^{\mathbf{d}}_{\mathbf{c},\mathbf{c}'}(q) G^{\mathrm{up}}\big(b(\mathbf{d},\widetilde{w})\big),$$

*where* $\mathbf{d} < \mathbf{c}+\mathbf{c}'$ *and* $d^{\mathbf{d}}_{\mathbf{c},\mathbf{c}'}(q) \in \mathcal{A}$.

COROLLARY 4.32

*If* $b(\mathbf{c},\widetilde{w}) \perp b(\mathbf{c}',\widetilde{w})$, *we have*

$$G^{\mathrm{up}}\big(b(\mathbf{c},\widetilde{w})\big)G^{\mathrm{up}}\big(b(\mathbf{c},\widetilde{w})\big) \simeq G^{\mathrm{up}}\big(b(\mathbf{c}+\mathbf{c}',\widetilde{w})\big),$$

*that is,* $b(\mathbf{c},\widetilde{w}) \circledast b(\mathbf{c}',\widetilde{w}) = b(\mathbf{c}+\mathbf{c}',\widetilde{w})$.

*4.8.1*

Using Proposition 4.31, we have the following expression of the exponent in $q$ of the $q$-commuting dual canonical basis elements in $\mathscr{B}(w,\epsilon=1)$ as in [37, Proposition 18].

PROPOSITION 4.33

*Let*

$$N_{\widetilde{w}}(\mathbf{c},\mathbf{c}') := c_{\widetilde{w}}(\mathbf{c},\mathbf{c}') - c_{\widetilde{w}}(\mathbf{c}',\mathbf{c}). \tag{4.34}$$

*If* $G^{\mathrm{up}}(b(\mathbf{c},\widetilde{w}))G^{\mathrm{up}}(b(\mathbf{c}',\widetilde{w})) = q^N G^{\mathrm{up}}(b(\mathbf{c}',\widetilde{w}))G^{\mathrm{up}}(b(\mathbf{c},\widetilde{w}))$, *then we have* $N = -N_{\widetilde{w}}(\mathbf{c},\mathbf{c}')$.

**4.9**

In this subsection, we study the specialization of $\mathbf{U}_q^-(w,\epsilon)$ at $q=1$. Throughout this subsection we consider $\mathbb{C}$ as an $\mathcal{A}$-algebra by the homomorphism $\mathcal{A}\to\mathbb{C}$ defined by $q\mapsto 1$.

*4.9.1*

We have the following property of the specialization of $\mathbf{U}_q^-$ at $q=1$.

THEOREM 4.35 ([45, SECTION 33.1])

*There is an isomorphism of algebras*

$$\Phi : U(\mathfrak{n}) \xrightarrow{\sim} \mathbb{C}\otimes_{\mathcal{A}} \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$$

*which sends* $f_i$ *to* $f_i$.

Let $r: U(\mathfrak{n}) \to U(\mathfrak{n})\otimes U(\mathfrak{n})$ be the coproduct defined by $r(f) = f\otimes 1 + 1\otimes f$ for $f\in\mathfrak{n}$. Here we note that $U(\mathfrak{n})$ is generated by $\{f_i\}_{i\in I}$ as an algebra. Since the specializatioin of the twisted coproduct satisfies this relation on the generators, the above is an isomorphism of bialgebras.

*4.9.2*

Let $\mathbb{C}[N]$ be the restricted dual of the universal enveloping algebra $U(\mathfrak{n})$ of the Lie algebra $\mathfrak{n}$; that is,

$$\mathbb{C}[N] := \bigoplus_{\xi\in Q} U(\mathfrak{n})^*_{\xi}.$$

We take the dual $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ of $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ as before. Since the multiplication of $\mathbf{U}_q^-(\mathfrak{g})$ preserves $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}$, the twisted coproduct $r$ preserves the dual integral form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$; that is, $r(\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}) \subset \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \otimes \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$.

Let $r^* : \mathbb{C}[N] \otimes \mathbb{C}[N] \to \mathbb{C}[N]$ be a product so that $\langle r^*(\varphi \otimes \varphi'), x\rangle = \langle \varphi \otimes \varphi', r(x)\rangle$ holds for any $x \in U(\mathfrak{n})$, and let $\mu^* : \mathbb{C}[N] \to \mathbb{C}[N] \otimes \mathbb{C}[N]$ be a coproduct so that $\langle \mu^*(\varphi), x \otimes x'\rangle = \langle \varphi, \mu(x \otimes x')\rangle$ holds for any $x, x' \in U(\mathfrak{n})$, where $\mu : U(\mathfrak{n}) \otimes U(\mathfrak{n}) \to U(\mathfrak{n})$ is the product on $U(\mathfrak{n})$. The above isomorphism $\Phi$ induces the following.

PROPOSITION 4.36

*There is an isomorphism of bialgebras*

$$\Phi^{\mathrm{up}} : \mathbb{C} \otimes_{\mathcal{A}} \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \mathbb{C}[N];$$

*that is, we have*

$$(\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}}) \circ r = \mu^* \circ \Phi^{\mathrm{up}},$$
$$\Phi^{\mathrm{up}} \circ \mu = r^* \circ (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}}),$$

*where $r$ and $\mu$ in the left-hand sides are specializations of the twisted comultiplication and multiplication in $\mathbf{U}_q^-(\mathfrak{g})$, respectively.*

*4.9.3*

Let

$$\begin{aligned}\sigma_i &:= \exp(-f_i)\exp(e_i)\exp(-f_i)\\ &= \exp(e_i)\exp(-f_i)\exp(e_i),\end{aligned}$$

for $i \in I$. Then we have

$$\begin{aligned}(\sigma_i)^{-1} &= \exp(f_i)\exp(-e_i)\exp(f_i)\\ &= \exp(-e_i)\exp(f_i)\exp(-e_i).\end{aligned}$$

(This $(\sigma_i)^{-1}$ is equal to $\overline{s}_i$ used in [22, Section 7.1].) The action of $\sigma_i$ is well defined on integrable $\mathfrak{g}$-modules, especially on the adjoint representation of $\mathfrak{g}$. Under the specialization at $q = 1$, we have $\sigma_i = S_i|_{q=1}$.

*4.9.4*

For $\widetilde{w} \in R(w)$ and $\epsilon \in \{\pm 1\}$, let

$$f_\epsilon(\beta_k) := \sigma_{i_1}^{\epsilon} \cdots \sigma_{i_{k-1}}^{\epsilon}(f_{i_k}).$$

Then we have $f_\epsilon(\beta_k) \in \mathfrak{g}_{-\beta_k}$ and

$$\mathfrak{n}(w) = \bigoplus_{1 \le k \le \ell} \mathbb{C} f_\epsilon(\beta_k).$$

By the definition, $f_\epsilon(\beta_k)$ is the specialization of $F_\epsilon(\beta_k)$.

*4.9.5*
Let $\mathbb{C}[N(w)]$ be the restricted dual of the universal enveloping algebra $U(\mathfrak{n}(w))$ associated with $\mathfrak{n}(w)$. We consider a basis of $\mathfrak{n}(w)$ given by $\{f_\epsilon(\beta_k)\}_{1\le k\le \ell}$ and also a basis $\{f_\epsilon(\beta_k)\}_{1\le k\le \ell}\cup\{f'_k\}$ of $\mathfrak{n}$ which includes $\{f_\epsilon(\beta_k)\}_{1\le k\le \ell}$ as in [22, 4.3]. Here we fix a total order on the basis of $\mathfrak{n}$ by

$$f_\epsilon(\beta_1)<\cdots<f_\epsilon(\beta_\ell)<f'_1<f'_2<\cdots.$$

By the Poincaré–Birkhoff–Witt basis theorem, we have a basis of $U(\mathfrak{n})$ given by

$$f_\epsilon\big((\mathbf{c},\mathbf{d}),\widetilde{w}\big):=\begin{cases} f_\epsilon(\beta_1)^{(c_1)}\cdots f_\epsilon(\beta_\ell)^{(c_\ell)}f_1'^{(d_1)}\cdots & \text{when } \epsilon=1,\\ \cdots f_1'^{(d_1)}f_\epsilon(\beta_\ell)^{(c_\ell)}\cdots f_\epsilon(\beta_1)^{(c_1)} & \text{when } \epsilon=-1,\end{cases}$$

and also a basis of $U(\mathfrak{n}(w))$ given by

$$f_\epsilon(\mathbf{c},\widetilde{w}):=\begin{cases} f_\epsilon(\beta_1)^{(c_1)}\cdots f_\epsilon(\beta_\ell)^{(c_\ell)} & \text{when } \epsilon=1,\\ f_\epsilon(\beta_\ell)^{(c_\ell)}\cdots f_\epsilon(\beta_1)^{(c_1)} & \text{when } \epsilon=-1,\end{cases}$$

where $x^{(c)}=x^c/c!$ for $x\in\mathfrak{g}$ and $c\in\mathbb{Z}_{\ge 0}$. We have $\Phi(f_\epsilon(\mathbf{c},\widetilde{w}))=F_\epsilon(\mathbf{c},\widetilde{w})|_{q=1}$.

*4.9.6*
Let $\{f_\epsilon^*(\mathbf{c},\widetilde{w})\}$ (resp., $\{f_\epsilon^*((\mathbf{c},\mathbf{d}),\widetilde{w})\}$) be the dual basis of $\{f_\epsilon(\mathbf{c},\widetilde{w})\}$ (resp., $\{f_\epsilon((\mathbf{c},\mathbf{d}),\widetilde{w})\}$). Using these, we obtain a section of $\mathbb{C}[N]\to\mathbb{C}[N(w)]$ as algebras.

LEMMA 4.37
*Let $\widetilde{\pi}_w^*:\mathbb{C}[N(w)]\to\mathbb{C}[N]$ be a $\mathbb{C}$-linear homomorphism defined by*

$$\widetilde{\pi}_w^*\big(f_\epsilon^*(\mathbf{c},\widetilde{w})\big):=f_\epsilon^*\big((\mathbf{c},0),\widetilde{w}\big).$$

*Then it is an algebra embedding.*

*Proof*
First, $\langle\widetilde{\pi}_w^*(f_\epsilon^*(\mathbf{c}_1,\widetilde{w}))\cdot\widetilde{\pi}_w^*(f_\epsilon^*(\mathbf{c}_2,\widetilde{w})),f_\epsilon((\mathbf{c}',\mathbf{d}'),\widetilde{w})\rangle$ is equal to

$$\big\langle\widetilde{\pi}_w^*\big(f_\epsilon^*(\mathbf{c}_1,\widetilde{w})\big)\otimes\widetilde{\pi}_w^*\big(f_\epsilon^*(\mathbf{c}_2,\widetilde{w})\big),r\big(f_\epsilon((\mathbf{c}',\mathbf{d}'),\widetilde{w})\big)\big\rangle, \tag{4.38}$$

where we consider the pairing between $\mathbb{C}[N]$ and $U(\mathfrak{n})$. We note that

$$r\big(f_\epsilon((\mathbf{c}',\mathbf{d}'),\widetilde{w})\big)=\sum_{\mathbf{c}'_1+\mathbf{c}'_2=\mathbf{c}',\mathbf{d}'_1+\mathbf{d}'_2=\mathbf{d}'}f_\epsilon\big((\mathbf{c}'_1,\mathbf{d}'_1),\widetilde{w}\big)\otimes f_\epsilon\big((\mathbf{c}'_2,\mathbf{d}'_2),\widetilde{w}\big). \tag{4.39}$$

Hence (4.38) is equal to $\delta_{\mathbf{c}_1+\mathbf{c}_2,\mathbf{c}'}\delta_{0,\mathbf{d}'}$.

On the other hand, we consider

$$\big\langle\widetilde{\pi}_w^*\big(f_\epsilon^*(\mathbf{c}_1,\widetilde{w})\cdot f_\epsilon^*(\mathbf{c}_2,\widetilde{w})\big),f_\epsilon\big((\mathbf{c}',\mathbf{d}'),\widetilde{w}\big)\big\rangle, \tag{4.40}$$

which is the pairing between $\mathbb{C}[N(w)]$ and $U(\mathfrak{n}(w))$. By (4.39), we have $f_\epsilon^*(\mathbf{c}_1,\widetilde{w})\cdot f_\epsilon^*(\mathbf{c}_2,\widetilde{w})=r^*(f_\epsilon^*(\mathbf{c}_1,\widetilde{w})\otimes f_\epsilon^*(\mathbf{c}_2,\widetilde{w}))=f_\epsilon^*(\mathbf{c}_1+\mathbf{c}_2,\widetilde{w})$. Then (4.40) is equal to $\delta_{\mathbf{c}_1+\mathbf{c}_2,\mathbf{c}'}\delta_{0,\mathbf{d}'}$. Then the assertion holds. □

By [22, Proposition 8.2], this embedding does not depend on the choice of $\widetilde{w} \in R(w)$ or of the basis of $\mathfrak{n}$ which completes $\{f_\epsilon(\beta_k)\}_{1\le k\le \ell}$.

*4.9.7*

We study the image of $\mathbb{C} \otimes_{\mathcal{A}} \mathbf{U}_q^-(w,\epsilon)_{\mathcal{A}}^{\mathrm{up}}$ under the isomorphism $\Phi^{\mathrm{up}}$.

LEMMA 4.41

*Let $f \in \mathfrak{g}_{-\alpha}$ with $\alpha \in \Delta_+ \setminus \Delta_+(w)$;, we have*

$$\langle f, \Phi^{\mathrm{up}}\big(G^{\mathrm{up}}(b)|_{q=1}\big)\rangle = 0$$

*for $b \in \mathscr{B}(w,\epsilon)$.*

*Proof*

Suppose that $b \in \mathscr{B}(w,\epsilon)$ and $f \in \mathfrak{g}_{-\alpha}$ with $\langle f, \Phi^{\mathrm{up}}(G^{\mathrm{up}}(b)|_{q=1})\rangle \neq 0$. Then we have

$$\alpha = \sum_{1\le k\le \ell} a_k\beta_k$$

for some $a_k \in \mathbb{Z}_{\ge 0}$. By the definition of $\Delta_+(w)$, we have $w^{-1}\alpha \in \Delta_+$ and $w^{-1} \times (\sum_{1\le k\le\ell} a_k\beta_k) \in Q_-$. This is a contradiction. Hence we get the assertion. □

*4.9.8*

We have the following formula of the (twisted) coproduct of the root vectors $F(\beta_k)$ (see [10, 3.5 Corollary 3]).

PROPOSITION 4.42

*We have the following expansion:*

$$r\big(F(\beta_k)\big) - \big(1 \otimes F(\beta_k) + F(\beta_k) \otimes 1\big) = \sum_{\mathbf{c}} x_{\mathbf{c}} \otimes F(\mathbf{c}, \widetilde{w}),$$

*where $x_{\mathbf{c}} \in \mathbf{U}_q^-(\mathfrak{g})$, and if $x_{\mathbf{c}} \neq 0$, then $c_{k'} = 0$ for $k' \ge k$.*

We have the compatibility of the twisted coproduct $r$ with $\mathbf{U}_q^-(w,\epsilon)$ (cf. [38, Section 2.4.2, Theorem(c))]).

PROPOSITION 4.43

*We have*

$$r\big(\mathbf{U}_q^-(w,+1)_{\mathcal{A}}^{\mathrm{up}}\big) \subset \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \otimes \mathbf{U}_q^-(w,+1)_{\mathcal{A}}^{\mathrm{up}},$$
$$r\big(\mathbf{U}_q^-(w,-1)_{\mathcal{A}}^{\mathrm{up}}\big) \subset \mathbf{U}_q^-(w,-1)_{\mathcal{A}}^{\mathrm{up}} \otimes \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}},$$

*that is, $\mathbf{U}_q^-(w,+1)_{\mathcal{A}}^{\mathrm{up}}$ (resp., $\mathbf{U}_q^-(w,-1)_{\mathcal{A}}^{\mathrm{up}}$) is a left (resp., right) $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$-comodule.*

*Proof*
Recall that we proved $\mathbf{U}_q^-(w,\epsilon)_{\mathcal{A}}^{\mathrm{up}} = \mathbf{U}_q^-(w,\epsilon) \cap \mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ during the proof of Theorem 4.25. Since $r$ preserves the dual $\mathcal{A}$-form $\mathbf{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$, it suffices to prove a weaker statement; that is,

$$r\big(\mathbf{U}_q^-(w,+1)\big) \subset \mathbf{U}_q^-(\mathfrak{g}) \otimes \mathbf{U}_q^-(w,+1),$$
$$r\big(\mathbf{U}_q^-(w,-1)\big) \subset \mathbf{U}_q^-(w,-1) \otimes \mathbf{U}_q^-(\mathfrak{g}).$$

Moreover, if we apply the $*$-involution, we obtain the claim for the $\epsilon = -1$ case from the claim for the $\epsilon = 1$ case. So it is enough to prove the $\epsilon = 1$ case. This assertion is a consequence of Propositions 4.11 and 4.42. □

*4.9.9*
THEOREM 4.44
*Under the algebra homomorphism $\Phi^{\mathrm{up}}$, we have*

$$\mathbb{C} \otimes_{\mathcal{A}} \mathbf{U}_q^-(w,\epsilon)_{\mathcal{A}}^{\mathrm{up}} \simeq \mathbb{C}[N(w)].$$

In view of this theorem, the quantum nilpotent subalgebra $\mathbf{U}_q^-(w,\epsilon)$ can be considered as the "quantum coordinate ring" of the corresponding unipotent subgroup $N(w)$, so we call it the *quantum unipotent subgroup* and denote it by $\mathcal{O}_q[N(w)]$.

*Proof*
We compute the following inner product:

$$\big\langle \Phi^{\mathrm{up}}\big(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}\big), f_\epsilon\big((\mathbf{c}',\mathbf{d}'),\widetilde{w}\big)\big\rangle.$$

First we have

$$\begin{aligned}
&\big\langle \Phi^{\mathrm{up}}\big(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}\big), f_\epsilon\big((\mathbf{c}',\mathbf{d}'),\widetilde{w}\big)\big\rangle \\
&\quad= \begin{cases} \big\langle \mu^*\big(\Phi^{\mathrm{up}}(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_\epsilon((\mathbf{c}',0),\widetilde{w}) \otimes f_\epsilon((0,\mathbf{d}'),\widetilde{w})\big\rangle & \text{when } \epsilon = 1, \\ \big\langle \mu^*\big(\Phi^{\mathrm{up}}(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}), f_\epsilon((0,\mathbf{d}'),\widetilde{w}) \otimes f_\epsilon((\mathbf{c}',0),\widetilde{w})\big\rangle & \text{when } \epsilon = -1 \end{cases} \\
&\quad= \begin{cases} \big\langle (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}})\big(r(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1})\big), f_\epsilon((\mathbf{c}',0),\widetilde{w}) \otimes f_\epsilon((0,\mathbf{d}'),\widetilde{w})\big\rangle \\ \quad \text{when } \epsilon = 1, \\ \big\langle (\Phi^{\mathrm{up}} \otimes \Phi^{\mathrm{up}})\big(r(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1})\big), f_\epsilon\big((0,\mathbf{d}'),\widetilde{w}\big) \otimes f_\epsilon\big((\mathbf{c}',0),\widetilde{w}\big)\big\rangle \\ \quad \text{when } \epsilon = -1 \end{cases} \\
&\quad= 0
\end{aligned}$$

if $\mathbf{d}' \neq 0$. This follows from Lemma 4.41 and Proposition 4.43. Hence it suffices to compute the form

$$\big\langle \Phi^{\mathrm{up}}\big(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}\big), f_\epsilon(\mathbf{c}',\widetilde{w})\big\rangle.$$

This is equal to $\langle F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}, \Phi(f_\epsilon(\mathbf{c}',\widetilde{w}))\rangle = \langle F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}, F_\epsilon(\mathbf{c}',\widetilde{w})|_{q=1}\rangle = \delta_{\mathbf{c},\mathbf{c}'}$. Hence we have $\Phi^{\mathrm{up}}(F_\epsilon^{\mathrm{up}}(\mathbf{c},\widetilde{w})|_{q=1}) = f_\epsilon^*((\mathbf{c},0),\widetilde{w})$ and the assertion. □

## 5. Quantum closed unipotent cell and the dual canonical basis

### 5.1. Demazure–Schubert filtration $\mathbf{U}_w^-$

We recall the definition of the Demazure–Schubert filtration $\mathbf{U}_w^-$ associated with a Weyl group element $w \in W$.

*5.1.1*

Let $\mathbf{i} = (i_1, \dots, i_\ell)$ be a sequence in $I$, and let $\mathbf{U}_\mathbf{i}^-$ be the $\mathbb{Q}(q)$-linear subspace spanned by the monomials $f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)}$ for all $(a_1, a_2, \dots, a_l) \in \mathbb{Z}_{\geq 0}^\ell$; that is,

$$\mathbf{U}_\mathbf{i}^- := \sum_{a_1, a_2, \dots, a_\ell \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)}.$$

By its definition, this is a $\mathbb{Q}(q)$-subcoalgebra of $\mathbf{U}_q^-$. We have the following compatibility with the canonical basis.

PROPOSITION 5.1 ([42, SECTION 4.2])

*The subcoalgebra $\mathbf{U}_\mathbf{i}^-$ is compatible with the canonical basis $\mathbf{B}$; that is, there exists a subset $\mathscr{B}_\mathbf{i}(\infty)$ of $\mathscr{B}(\infty)$ such that*

$$\mathbf{U}_\mathbf{i}^- = \bigoplus_{b \in \mathscr{B}_\mathbf{i}(\infty)} \mathbb{Q}(q) G^{\mathrm{low}}(b).$$

REMARK 5.2

If we consider the $\mathcal{A}$-subspace $(\mathbf{U}_\mathbf{i}^-)_\mathcal{A}$ spanned by the monomials $f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)}$, then $(\mathbf{U}_\mathbf{i}^-)_\mathcal{A}$ is a $\mathcal{A}$-subcoalgebra of $\mathbf{U}_q^-$ and we have

$$(\mathbf{U}_\mathbf{i}^-)_\mathcal{A} = \bigoplus_{b \in \mathscr{B}_\mathbf{i}(\infty)} \mathcal{A} G^{\mathrm{low}}(b).$$

REMARK 5.3

By the construction of $\mathbf{U}_\mathbf{i}^-$, it is clear that

$$*(\mathbf{U}_\mathbf{i}^-) = \mathbf{U}_{\mathbf{i}^{\mathrm{opp}}}^-,$$

$$*\big(\mathscr{B}_\mathbf{i}(\infty)\big) = \mathscr{B}_{\mathbf{i}^{\mathrm{opp}}}(\infty),$$

where $\mathbf{i}^{\mathrm{opp}} = (i_\ell, i_{\ell-1}, \dots, i_1)$ for $\mathbf{i} = (i_1, i_2, \dots, i_\ell)$.

*5.1.2*

For $w \in W$, we consider $\mathbf{U}_{\widetilde{w}}^-$ associated with $\widetilde{w} = (i_1, \dots, i_\ell) \in R(w)$. Then it is known that $\mathbf{U}_{\widetilde{w}}^-$ does not depend on the choice of the reduced expression $\widetilde{w}$ (see [42, Section 5.3]). Therefore we denote $\mathbf{U}_{\widetilde{w}}^-$ by $\mathbf{U}_w^-$ and also $\mathscr{B}_{\widetilde{w}}(\infty)$ by $\mathscr{B}_w(\infty)$ by abuse of notation. By construction, we have

$$*(\mathbf{U}_w^-) = \mathbf{U}_{w^{-1}}^-, \tag{5.4a}$$

$$*\big(\mathscr{B}_w(\infty)\big) = \mathscr{B}_{w^{-1}}(\infty). \tag{5.4b}$$

*5.1.3*

Following [4, Section 9.3], we define the *quantum closed unipotent cell* $\mathcal{O}_q[\overline{N_w}]$ associated with $w$ by

$$\mathcal{O}_q[\overline{N_w}] := \mathbf{U}_q^-(\mathfrak{g})/(\mathbf{U}_w^-)^\perp,$$

where $(\mathbf{U}_w^-)^\perp := \{x \in \mathbf{U}_q^-(\mathfrak{g}); (x, \mathbf{U}_w^-)_K = 0\}$. Let $\iota_w^* : \mathbf{U}_q^-(\mathfrak{g}) \to \mathcal{O}_q[\overline{N_w}]$ be the natural projection. Since $(\mathbf{U}_w^-)^\perp$ is compatible with $\mathbf{B}^{\mathrm{up}}$, that is, $(\mathbf{U}_w^-)^\perp = \bigoplus_{b \notin \mathscr{B}_w(\infty)} \mathbb{Q}(q) G^{\mathrm{up}}(b)$, the natural projection $\iota_w^*$ induces an bijection $\{G^{\mathrm{up}}(b); b \in \mathscr{B}_w(\infty)\} \simeq \{\iota_w^*(G^{\mathrm{up}}(b)); b \in \mathscr{B}_w(\infty)\}$ and $\{\iota_w^*(G^{\mathrm{up}}(b)); b \in \mathscr{B}_w(\infty)\}$ is a basis of $\mathcal{O}_q[\overline{N_w}]$. Moreover, $(\mathbf{U}_w^-)^\perp$ is a two-sided ideal since $\mathbf{U}_w^-$ is a subcoalgebra. Thus $\mathcal{O}_q[\overline{N_w}]$ has an induced algebra structure.

## 5.2. Demazure module and its crystal

In this subsection, we recall the definition of the extremal vector $u_{w\lambda}$ and the associated Demazure module $V_w(\lambda)$. In particular, we remind the reader that $\mathscr{B}_w(\infty)$ can be considered as a certain limit of the Demazure crystal $\mathscr{B}_w(\lambda)$.

*5.2.1*

For $i \in I$, we consider the subalgebra $\mathbf{U}_q(\mathfrak{g})_i$ generated by $e_i, f_i, t_i$. Consider the $(l+1)$-dimensional irreducible representation of $\mathbf{U}_q(\mathfrak{g})_i$ with a highest weight vector $u_0^{(l)}$, and let $u_k^{(l)} := f_i^{(k)} u_0^{(l)}$ $(1 \le k \le l)$. We have

$$S_i(u_k^{(l)}) = (-1)^{l-k} q_i^{(l-k)(k+1)} u_{l-k}^{(l)}. \tag{5.5}$$

In particular, we have

$$S_i(u_l^{(l)}) = u_0^{(l)}, \tag{5.6a}$$

$$S_i(u_0^{(l)}) = (-q_i)^l u_l^{(l)}. \tag{5.6b}$$

*5.2.2*

We recall basic properties of the Demazure module (see [29, Section 3], [30, Chapitre 9] for more details). For $\lambda \in P_+$ and $w \in W$, let us denote by $u_{w\lambda}$ the canonical basis element of weight $w\lambda$. We have the following description (see [29, Section 3.2], [45, Lemma 39.1.2]):

$$\begin{aligned} u_{w\lambda} &= u_\lambda && \text{if } w = 1, \\ u_{s_i w\lambda} &= f_i^{(m)} u_{w\lambda} = S_i^{-1} u_{w\lambda} && \text{if } m = \langle h_i, w\lambda \rangle \ge 0. \end{aligned}$$

Recall that $u_{w\lambda}$ is also a dual canonical basis element. For $\widetilde{w} = (i_1, \ldots, i_\ell) \in R(w)$, we have

$$u_{w\lambda} = S_{i_1}^{-1} \cdots S_{i_\ell}^{-1} u_\lambda. \tag{5.7}$$

*5.2.3*

Let $\lambda \in P_+$, and let $V(\lambda)$ be the integrable highest weight $\mathbf{U}_q(\mathfrak{g})$-module with a highest weight vector $u_\lambda$ of weight $\lambda$. Let $V_w(\lambda) := \mathbf{U}_q^+(\mathfrak{g}) u_{w\lambda}$. This $\mathbf{U}_q^+(\mathfrak{g})$-

module is called the *Demazure module* associated with $w$ and $\lambda$. We have the following properties of the Demazure module $V_w(\lambda)$.

PROPOSITION 5.8

*Let $w \in W$, and let $\widetilde{w} = (i_1, \dots, i_l) \in R(w)$ be a reduced expression of $w$.*

(1) *We have*

$$V_w(\lambda) = \sum_{a_1,\dots,a_\ell \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) f_{i_1}^{(a_1)} \cdots f_{i_\ell}^{(a_\ell)} u_\lambda.$$

(2) *We define $\mathscr{B}_w(\lambda) \subset \mathscr{B}(\lambda)$ by*

$$\mathscr{B}_w(\lambda) := \{ \widetilde{f}_{i_1}^{a_1} \cdots \widetilde{f}_{i_\ell}^{a_\ell} u_\lambda \in \mathscr{B}(\lambda); (a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell \setminus \{0\} \} \tag{5.9a}$$

$$= \{ b \in \mathscr{B}(\lambda); \widetilde{e}_{i_\ell}^{\max} \cdots \widetilde{e}_{i_1}^{\max} b = u_\lambda \}. \tag{5.9b}$$

*Then we have*

$$V_w(\lambda) = \bigoplus_{b \in \mathscr{B}_w(\lambda)} \mathbb{Q}(q) G_\lambda^{\mathrm{low}}(b).$$

(3) *For $i \in I$, we have*

$$\widetilde{e}_i \mathscr{B}_w(\lambda) \subset \mathscr{B}_w(\lambda) \sqcup \{0\}.$$

We call $\mathscr{B}_w(\lambda)$ the *Demazure crystal.*

*5.2.4*

We have a similar description of $\mathscr{B}_w(\infty)$ as $\mathscr{B}_w(\lambda)$. Thus $\mathscr{B}_w(\infty)$ can be interpreted as certain limit of the Demazure crystals $\mathscr{B}_w(\lambda)$.

PROPOSITION 5.10 ([29, COROLLARY 3.2.2])

*Let $w \in W$, and let $(i_1, \dots, i_\ell) \in R(w)$ be its reduced expression.*

(1) *We have*

$$\mathscr{B}_w(\infty) = \{ \widetilde{f}_{i_1}^{a_1} \cdots \widetilde{f}_{i_\ell}^{a_\ell} u_\infty \in \mathscr{B}(\infty); (a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell \setminus \{0\} \} \tag{5.11a}$$

$$= \{ b \in \mathscr{B}(\infty); \widetilde{e}_{i_\ell}^{\max} \cdots \widetilde{e}_{i_1}^{\max} b = u_\infty \}. \tag{5.11b}$$

(2) *For $i \in I$, we have*

$$\widetilde{e}_i \mathscr{B}_w(\infty) \subset \mathscr{B}_w(\infty) \sqcup \{0\}. \tag{5.12}$$

## 5.3

To study multiplicative properties of $\mathbf{U}_q^-(w, \epsilon)$, we relate it to the quantum closed unipotent cell $\mathcal{O}_q[\overline{N_w}]$. The following is a generalization of [8, Section 3.2, Lemma], but its proof is different from that in [8], which works only for finite type. This can be considered as a quantum analogue of [22, Corollary 15.7].

THEOREM 5.13
*For $w \in W$ and $\epsilon \in \{\pm 1\}$, we have the following embedding of algebras:*

$$\mathbf{U}_q^-(w,\epsilon) \hookrightarrow \mathcal{O}_q[\overline{N_{w^{-\epsilon}}}].$$

*Proof*
We consider the composite of the inclusion $\mathbf{U}_q^-(w,\epsilon) \hookrightarrow \mathbf{U}_q^-(\mathfrak{g})$ and the natural projection $\iota^*_{w^{-\epsilon}} : \mathbf{U}_q^-(\mathfrak{g}) \to \mathcal{O}_q[\overline{N_{w^{-\epsilon}}}]$. Since both homomorphisms are algebra homomorphisms, we obtain an algebra homomorphism

$$\mathbf{U}_q^-(w,\epsilon) \to \mathcal{O}_q[\overline{N_{w^{-\epsilon}}}].$$

Since $\mathbf{U}_q^-(w,\epsilon)$ is compatible with $\mathbf{B}^{\mathrm{up}}$ and $\iota^*_{w^{-\epsilon}}$ induces an bijection $\{G^{\mathrm{up}}(b); b \in \mathscr{B}_{w^{-\epsilon}}(\infty)\} \simeq \{\iota^*_{w^{-\epsilon}}(G^{\mathrm{up}}(b)); b \in \mathscr{B}_{w^{-\epsilon}}(\infty)\}$, it suffices to prove the corresponding assertion for the crystals; that is, $\mathscr{B}(w,\epsilon) \hookrightarrow \mathscr{B}_{w^{-\epsilon}}(\infty)$. Since we have $*(\mathscr{B}(w,\epsilon)) = \mathscr{B}(w,-\epsilon)$ and $*(\mathscr{B}_w(\infty)) = \mathscr{B}_{w^{-1}}(\infty)$, it is enough to prove the claim for the $\epsilon = 1$ case.

We prove $\mathscr{B}(w,1) \subset \mathscr{B}_{w^{-1}}(\infty)$ by the induction on $\ell = \ell(w)$. Let $\widetilde{w} = (i_1, \ldots, i_\ell) \in R(w)$ be a reduced expression. We prove that

$$b := F(\mathbf{c}, \widetilde{w}) \mod q\mathscr{L}(\infty) \in \mathscr{B}_{w^{-1}}(\infty)$$

for all $\mathbf{c} \in \mathbb{Z}_{\geq 0}^{\ell}$. For $\ell = 1$ case, by the constructions of $\mathscr{B}(s_i,\epsilon)$ and $\mathscr{B}_{s_i}(\infty)$, we have $\mathscr{B}(s_i,\epsilon) = \mathscr{B}_{s_i}(\infty)$ for all $i \in I$ and $\epsilon \in \{\pm 1\}$. For $l \geq 2$, we set $w_{\geq 2} := s_{i_2} \cdots s_{i_\ell} \in W$, $\widetilde{w}_{\geq 2} := (i_2, i_3, \ldots, i_\ell) \in R(w_{\geq 2})$, $\mathbf{c}_{\geq 2} := (c_2, c_3, \ldots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell-1}$, and

$$b_{\geq 2} := F(\mathbf{c}_{\geq 2}, \widetilde{w}_{\geq 2}) \mod q\mathscr{L}(\infty).$$

Since $F(\mathbf{c}, \widetilde{w}) = f_{i_1}^{(c_1)} T_{i_1} F(\mathbf{c}_{\geq 2}, \widetilde{w}_{\geq 2})$ and $F(\mathbf{c}_{\geq 2}, \widetilde{w}_{\geq 2}) \in \mathscr{L}(\infty) \cap T_{i_1}^{-1}\mathbf{U}_q^-(\mathfrak{g})$, we have $b = \widetilde{f}_{i_1}^{c_1} \widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2}$ by Proposition 4.15. We have $b_{\geq 2} \in \mathscr{B}_{w_{\geq 2}^{-1}}(\infty)$ by the induction hypothesis, and hence $\widetilde{e}_{i_1}^{\max} b_{\geq 2} \in \mathscr{B}_{w_{\geq 2}^{-1}}(\infty)$ by Lemma 5.10 (2).

We consider the image of $b$ under the Kashiwara embedding $\Psi_{i_1} : \mathscr{B}(\infty) \to \mathscr{B}(\infty) \otimes \mathscr{B}_{i_1}$ and show that it is contained in $\mathscr{B}_{w_{\geq 2}^{-1}}(\infty) \otimes \mathscr{B}_{i_1}$. From the third displayed equation in the proof of [53, Proposition 3.4.7],* we have

$$\Psi_{i_1}(\widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2}) = \widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0).$$

Since $\Psi_{i_1}$ is a strict embedding, we have

$$\Psi_{i_1}(\widetilde{f}_{i_1} \widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2}) = \widetilde{f}_{i_1}\big(\widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0)\big).$$

If $\varphi_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2}) \leq \varepsilon_{i_1}(\widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0))$, we have $\widetilde{f}_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0)) = \widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})+1} b_{i_1}(0)$. This is contained in $\mathscr{B}_{w_{\geq 2}^{-1}}(\infty) \otimes \mathscr{B}_{i_1}$.

Suppose $\varphi_{i_1}(\widetilde{e}_{i_1}^{\max} b_{\geq 2}) > \varepsilon_{i_1}(\widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0)) = \varphi_{i_1}(b_{\geq 2})$. This means that $\varepsilon_{i_1}(b_{\geq 2}) > 0$ and

*We remark that there is a typo, but it can be fixed by using (2.38).

$$\widetilde{f}_{i_1}\big(\widetilde{e}_{i_1}^{\max} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0)\big) = \widetilde{e}_{i_1}^{\max -1} b_{\geq 2} \otimes \widetilde{f}_{i_1}^{\varphi_{i_1}(b_{\geq 2})} b_{i_1}(0).$$

This is also in $\mathscr{B}_{w_{\geq 2}^{-1}}(\infty) \otimes \mathscr{B}_{i_1}$ by Lemma 5.10(2).

Hence $\widetilde{f}_{i_1} \widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2} \in \mathscr{B}_{w^{-1}}(\infty)$. By [29, Theorem 3.3.2], we obtain $b = \widetilde{f}_{i_1}^{c_1} \widetilde{f}_{i_1}^{*\varphi_{i_1}(b_{\geq 2})} \widetilde{e}_{i_1}^{\max} b_{\geq 2} \in \mathscr{B}_{w^{-1}}(\infty)$. □

## 6. Construction of initial seed: Quantum flag minors

In this section, we give a construction of the quantum initial seed in Conjecture 1.1 which corresponds to the initial seed in [22]. We consider only the $\epsilon = -1$ case, but the other case follows by applying the $*$-involution.

### 6.1. Quantum generalized minors

*6.1.1*

We first define a quantum generalized minor. This is a $q$-analogue of a (restricted) generalized minor $D_{w\lambda} = D_{w\lambda,\lambda}$ which is defined in [22, 7.1].

DEFINITION 6.1 (QUANTUM GENERALIZED MINOR)

For $\lambda \in P_+$ and $w \in W$, let

$$\Delta_{w\lambda} = \Delta_{w\lambda,\lambda} := j_\lambda(u_{w\lambda}).$$

We call it a *quantum generalized minor*. When $\lambda$ is a fundamental weight, we call it a *quantum flag minor*.

By its definition, it is given by a matrix coefficient as

$$(\Delta_{w\lambda,\lambda}, x) = (u_{w\lambda}, x u_\lambda),$$

for $x \in \mathbf{U}_q^-(\mathfrak{g})$.

*6.1.2*

The following result for extremal vectors is well known.

LEMMA 6.2 ([49, LEMMA 8.6])

*For $\lambda, \mu \in P_+$ and $w \in W$, we have*

$$\Phi(\lambda,\mu)(u_{w(\lambda+\mu)}) = u_{w\lambda} \otimes u_{w\mu}.$$

*It follows that*

$$q_{\lambda,\mu}(u_{w\lambda} \otimes u_{w\mu}) = u_{w\lambda + w\mu}.$$

*Therefore we get*

$$q^{(w\mu-\mu,\lambda)} \Delta_{w\lambda} \Delta_{w\mu} = \Delta_{w(\lambda+\mu)}$$

*by Proposition 3.31. In particular, $\Delta_{w,\lambda}$ is strongly real for all $w \in W$ and $\lambda \in P_+$.*

*6.1.3*

We describe extremal vectors in terms of the PBW basis. This is a straightforward generalization of [7]. For $w \in W$, we fix a reduced word $\widetilde{w} = (i_1, \ldots, i_\ell) \in R(w)$. For $1 \le k \le \ell$, we define the following operations as in [22, 9.8],

$$k^- := \max(0, \{1 \le s \le k-1; i_s = i_k\}),$$
$$k_{\max} := \max\{1 \le s \le \ell; i_s = i_k\}.$$

PROPOSITION 6.3

*For $0 \le k \le \ell$, we define $\mathbf{n}_k$ by*

$$\mathbf{n}_k(j) := \begin{cases} 1 & \text{if } i_j = i_k, j \le k, \\ 0 & \text{otherwise.} \end{cases}$$

*If $i = i_k$ (here we understand that $i = i_k$ holds for all $i$ if $k = 0$), we have $F_{\epsilon=-1}(m\mathbf{n}_k; \widetilde{w}) u_{m\varpi_i} = u_{s_{i_1} \cdots s_{i_k} m\varpi_i}$ for $m \ge 1$.*

*Proof*

We follow the argument in [7, Section 2.1, Lemma]. We prove the assertion by an induction on $k$. The assertion is trivial when $k = 0$. Note that

$$F_{-1}(m\mathbf{n}_k, \widetilde{w}) = T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(f_{i_k}^{(m)}) F_{-1}(m\mathbf{n}_{k^-}, \widetilde{w}).$$

Therefore we have

$$F_{-1}(m\mathbf{n}_k, \widetilde{w}) u_{m\varpi} = T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(f_{i_k}^{(m)}) u_{s_{i_1} \cdots s_{i_{k^-}} m\varpi_i}$$

by the induction hypothesis. By (4.6), this is equal to

$$S_{i_1}^{-1} \cdots S_{i_{k-1}}^{-1}(f_{i_k}^{(m)}) S_{i_{k-1}} \cdots S_{i_1} S_{i_1}^{-1} \cdots S_{i_{k^-}}^{-1} u_{m\varpi_i}$$
$$= S_{i_1}^{-1} \cdots S_{i_{k-1}}^{-1}(f_{i_k}^{(m)}) S_{i_{k-1}} \cdots S_{i_{k^-+1}} u_{m\varpi_i}.$$

Since none of $i_{k^-+1}, \ldots, i_{k-1}$ is $i$, this is equal to

$$S_{i_1}^{-1} \cdots S_{i_{k-1}}^{-1}(f_{i_k}^{(m)}) u_{m\varpi_i}.$$

By (5.7), this is nothing but $S_{i_1}^{-1} \cdots S_{i_k}^{-1} u_{m\varpi_i}$. Therefore the assertion also holds for $k$. □

By the above proposition, we have $\pi_{m\varpi_{i_k}}(b_{-1}(m\mathbf{n}_k; \widetilde{w})) \ne 0$ for all $1 \le k \le \ell$ and $m \ge 1$. Hence $j_{m\varpi_{i_k}}(u_{s_{i_1} \cdots s_{i_k} m\varpi_{i_k}}) = G^{\mathrm{up}}(b_{-1}(m\mathbf{n}_k, \widetilde{w}))$ for all $1 \le k \le \ell$. As a special case, we obtain the following result.

COROLLARY 6.4

*Let $w \in W$, $\widetilde{w} = (i_1, \ldots, i_\ell) \in R(w)$. For $i \in I$, we set $\mathbf{n}^i$ to be $\mathbf{n}_{k_{\max}}$ with $i_k = i$. For $\lambda \in P_+$, we set $\mathbf{n}^\lambda := \sum_{i \in I} \lambda_i \mathbf{n}^i \in \mathbb{Z}_{\ge 0}^\ell$. Then we have*

$$\Delta_{w\lambda} = G^{\mathrm{up}}\big(b_{-1}(\mathbf{n}^\lambda, \widetilde{w})\big). \tag{6.5}$$

*Proof*

By Proposition 6.3, we have

$$\Delta_{wm\varpi_i} = G^{\mathrm{up}}\big(b_{-1}(m\mathbf{n}^i, \widetilde{w})\big) \tag{6.6}$$

for all $i \in I$. Then by (6.6), Lemma 6.2, and Corollary 4.32, we obtain the assertion. □

## 6.2. Commutativity relations

In this subsection, we prove that quantum generalized minors $\{\Delta_{w\lambda}\}$ $q$-commute with $G^{\mathrm{up}}(b)$ for $b \in \mathscr{B}_w(\infty)$ in the quotient $\mathcal{O}_q[\overline{N_w}]$. It means that $\Delta_{w\lambda}$ and $G^{\mathrm{up}}(b)$ $q$-commute up to $(\mathbf{U}_w^-)^\perp$. By Theorem 5.13, they literally $q$-commute when $b \in \mathscr{B}_w(w, -1)$. We denote the projection of $G^{\mathrm{up}}(b)$ to $\mathcal{O}_q[\overline{N_w}]$ also by $G^{\mathrm{up}}(b)$ for brevity.

*6.2.1*

For the proof of certain $q$-commutativity relations, we need to use the quasi $\mathcal{R}$-matrix. We recall its properties.

First we consider another coproduct $\overline{\Delta}$ defined by $(^- \otimes {}^-) \circ \Delta \circ {}^-$. We have an analogue of Lemma 2.5:

$$\overline{\Delta}(q^h) = q^h \otimes q^h, \tag{6.7a}$$

$$\overline{\Delta}(e_i) = e_i \otimes t_i + 1 \otimes e_i, \tag{6.7b}$$

$$\overline{\Delta}(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i. \tag{6.7c}$$

We consider the following completion:

$$\mathbf{U}_q^+(\mathfrak{g}) \mathbin{\widehat{\otimes}} \mathbf{U}_q^-(\mathfrak{g}) = \bigoplus_{\xi \in Q} \prod_{\xi = \xi' + \xi''} \mathbf{U}_q^+(\mathfrak{g})_{\xi'} \otimes \mathbf{U}_q^-(\mathfrak{g})_{\xi''}.$$

Note that the counit $\varepsilon$ extends to the completion. In [45, Chapter 4], Lusztig has shown that there exists a unique intertwiner $\Xi \in \mathbf{U}_q^+(\mathfrak{g}) \mathbin{\widehat{\otimes}} \mathbf{U}_q^-(\mathfrak{g})$ such that

$$\Xi \circ \Delta(x) = \overline{\Delta}(x) \circ \Xi \quad \text{for any } x \in \mathbf{U}_q(\mathfrak{g}), \tag{6.8}$$

$\varepsilon(\Xi) = 1$, and $\Xi \circ \overline{\Xi} = \overline{\Xi} \circ \Xi = 1$. We have an analogue of Lemma 2.5:

$$\overline{\Delta}(x) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} x_{(2)} t_{\mathrm{wt}\, x_{(1)}} \otimes x_{(1)}, \tag{6.9}$$

for any $x \in \mathbf{U}_q^-(\mathfrak{g})$ with $r(x) = \sum x_{(1)} \otimes x_{(2)}$. In particular, we have

$$\overline{\Delta}(x)(u_\lambda \otimes u_\mu) = \sum q^{-(\mathrm{wt}\, x_{(1)}, \mathrm{wt}\, x_{(2)})} x_{(2)} t_{\mathrm{wt}\, x_{(1)}} u_\lambda \otimes x_{(1)} u_\mu, \tag{6.10}$$

for such $x \in \mathbf{U}_q^-(\mathfrak{g})$.

*6.2.2*

PROPOSITION 6.11

*For $b \in \mathscr{B}_w(\mu)$ and $u_{w\lambda} \in \mathscr{B}_w(\lambda)$, we have the following $q$-commutation relation in $\mathcal{O}_q[\overline{N_w}]$:*

$$j_\lambda(u_{w\lambda}) G^{\mathrm{up}}\big(j_\mu(b)\big) \simeq G^{\mathrm{up}}\big(j_\mu(b)\big) j_\lambda(u_{w\lambda}). \tag{6.12}$$

*Proof*
Since we consider only the equality in the quantum closed unipotent cell $\mathcal{O}_q[\overline{N}_w]$, it is enough to check that inner products with $x \in \mathbf{U}_w^-$ are the same up to some $q$-shifts, and the $q$-shifts do not depend on choice of $x$. By Proposition 3.31, the left-hand side in (6.12) is equal to

$$(6.13) \qquad \big(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b), \Delta(x)(u_\lambda \otimes u_\mu)\big)_{\lambda,\mu},$$

where $(\cdot,\cdot)_{\lambda,\mu}$ denotes the inner product on $V(\lambda) \otimes V(\mu)$ defined by $(u \otimes u', v \otimes v')_{\lambda,\mu} := (u,v)_\lambda (u',v')_\mu$. We use the quasi $\mathcal{R}$-matrix to rewrite this as

$$(6.14) \qquad \big(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b), (\overline{\Xi} \circ \overline{\Delta}(x) \circ \Xi)(u_\lambda \otimes u_\mu)\big)_{\lambda,\mu}.$$

Since the action of the quasi $\mathcal{R}$-matrix is trivial on the highest weight vector, (6.14) is equal to

$$(6.15) \qquad \big(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b), (\overline{\Xi} \circ \overline{\Delta}(x))(u_\lambda \otimes u_\mu)\big)_{\lambda,\mu}.$$

Since the inner product has an adjoint property for $\varphi$, (6.15) is equal to

$$(6.16) \qquad \big((\varphi \otimes \varphi)(\overline{\Xi})(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b)), (\overline{\Delta}(x))(u_\lambda \otimes u_\mu)\big)_{\lambda,\mu}.$$

Note that $(\overline{\Delta}(x))(u_\lambda \otimes u_\mu)$ is contained in the tensor product of Demazure modules $V_w(\lambda) \otimes V_w(\mu)$ by Section 5.2.3. By the form of quasi $\mathcal{R}$-matrix (6.8) and the definition of $\varphi$, the nontrivial part of $(\varphi \otimes \varphi)(\overline{\Xi})(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b))$ is not contained in the tensor product $V_w(\lambda) \otimes V_w(\mu)$; therefore (6.16) is equal to

$$(6.17) \qquad \big(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b), (\overline{\Delta}(x))(u_\lambda \otimes u_\mu)\big)_{\lambda,\mu}.$$

By (6.9), (6.17) is equal to

$$\big(u_{w\lambda} \otimes G^{\mathrm{up}}_{\mu}(b), \big(\mathrm{flip} \circ \Delta(x)\big)(u_\lambda \otimes u_\mu)\big)_{\lambda,\mu}$$

and also to

$$\big(j_\lambda(u_{w\lambda}) \otimes G^{\mathrm{up}}(j_\mu b), \big(\mathrm{flip} \circ r(x)\big)_K$$

up to some $q$-shifts, where $\mathrm{flip}(P \otimes Q) := Q \otimes P$. Therefore we get

$$\big(j_\lambda(u_{w\lambda}) G^{\mathrm{up}}(j_\mu(b)), x\big)_K \simeq \big(G^{\mathrm{up}}(j_\mu(b)) j_\lambda(u_{w\lambda}), x\big)_K$$

for any $x \in \mathbf{U}_w^-$. Here we note that $q$-shifts depend only on the weights of $u_{w\lambda}$ and $j_\mu(b)$ and are independent of $x$. Then we obtain the assertion. ☐

Restricting the above equality, we obtain the following $q$-commutativity relations in $\mathcal{O}_q[N(w)]$.

COROLLARY 6.18
*For $\mathbf{c} \in \mathbf{Z}^{\ell}_{\geq 0}$, we have*

$$G^{\mathrm{up}}\big(b_{-1}(\mathbf{c}, \widetilde{w})\big)\Delta_{w\lambda} \simeq \Delta_{w\lambda} G^{\mathrm{up}}\big(b_{-1}(\mathbf{c}, \widetilde{w})\big).$$

## 6.3. Factorization of the $q$-center

In this subsection, we prove the multiplicative property of $\mathscr{B}_w(\infty)$ with respect to the quantum minors $\{\Delta_{w\lambda}\}_{\lambda \in P_+}$ in $\mathcal{O}_q[\overline{N_w}]$. This is a generalization of [9, 3.1], [8], and [34, Lemma 4.2]. This result can be considered as a $q$-analogue of [22, Lemma 15.8].

*6.3.1*

Using Corollary 3.17 and (3.18) inductively, we obtain the following lemma.

LEMMA 6.19

*Let $w \in W$ and $\widetilde{w} = (i_1, \dots, i_\ell) \in R(w)$ as above. We define*

$$\varepsilon_{\widetilde{w}}(b) := \big(\varepsilon_{i_1}(b), \varepsilon_{i_2}(\widetilde{e}_i^{\max} b), \dots, \varepsilon_{i_\ell}(\widetilde{e}_{i_{l-1}}^{\max} \cdots \widetilde{e}_{i_1}^{\max} b)\big)$$

*for $b \in \mathscr{B}(\infty)$. For $b_1, b_2 \in \mathscr{B}(\infty)$, let us write*

$$G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2) = \sum d^b_{b_1,b_2}(q) G^{\mathrm{up}}(b)$$

*with $d^b_{b_1,b_2}(q) \in \mathcal{A}$. If $d^b_{b_1,b_2}(q) \neq 0$, then $\varepsilon_{\widetilde{w}}(b) \leq \varepsilon_{\widetilde{w}}(b_1) + \varepsilon_{\widetilde{w}}(b_2)$, where $\leq$ is the lexicographic order on $\mathbb{Z}^\ell_{\geq 0}$ as in Section 4.3.5.*

*Let $b \in \mathscr{B}_w(\infty)$ with $\varepsilon_{\widetilde{w}}(b) = \varepsilon_{\widetilde{w}}(b_1) + \varepsilon_{\widetilde{w}}(b_2)$ for $b_1, b_2 \in \mathscr{B}_w(\infty)$. Then we have $d^b_{b_1,b_2}(q) = q^N$ for some $N \in \mathbb{Z}$.*

*6.3.2*

PROPOSITION 6.20

*Let $w \in W$ and $\lambda, \mu \in P_+$. For $b \in \mathscr{B}_w(\mu)$ and $u_{w\lambda} \in \mathscr{B}_w(\lambda)$, there exists $b' \in \mathscr{B}_w(\lambda + \mu)$ such that*

$$\Phi(\lambda, \mu)(b') = u_{w\lambda} \otimes b,$$

*and we have an equality in $\mathcal{O}_q[\overline{N}_w]$:*

$$\Delta_{w\lambda} G^{\mathrm{up}}\big(j_\mu(b)\big) \simeq G^{\mathrm{up}}\big(j_{\lambda+\mu}(b')\big).$$

*Proof*

Fix $\widetilde{w} = (i_1, \dots, i_\ell) \in R(w)$; we have

$$\widetilde{e}_{i_1}^{\max}(u_{w\lambda} \otimes b) = u_{w_{\geq 2}\lambda} \otimes \widetilde{e}_{i_1}^{\max} b$$

by the tensor-product rule (2.34a) for crystal operators and $\varphi_{i_1}(u_{w\lambda}) = 0$. Using this recursively, we get

$$\widetilde{e}_{i_l}^{\max} \cdots \widetilde{e}_{i_1}^{\max}(u_{w\lambda} \otimes b) = u_\lambda \otimes u_\mu.$$

In particular, there exists $b' \in \mathscr{B}_w(\lambda + \mu)$ such that $\Phi(\lambda, \mu)(b') = u_{w\lambda} \otimes b$. By Propositions 3.31 and 3.29, we have

$$q^{(\mathrm{wt}\, b - \mu, \lambda)} \Delta_{w\lambda} G^{\mathrm{up}}(j_\mu b) = G^{\mathrm{up}}\big(j_{\lambda+\mu}(b')\big) + \sum f^{b''}_{b,w\lambda}(q) G^{\mathrm{up}}(b'') \tag{6.21}$$

for some $f^{b''}_{b,w\lambda}(q) \in q\mathbb{Z}[q]$. By the second assertion of Lemma 6.19, we have $f^{j_{\lambda+\mu}(b')}_{b,w\lambda}(q) = 0$ as in [9, Section 1.8, Proposition(i)].

Applying the dual bar-involution $\sigma$, we obtain

$$(6.22)\quad \begin{aligned} &q^{-(\mathrm{wt}\, b-\mu,\lambda)+(\mathrm{wt}\, b-\mu,w\lambda-\lambda)}G^{\mathrm{up}}(j_\mu b)\Delta_{w\lambda} \\ &\quad = G^{\mathrm{up}}\big(j_{\lambda+\mu}(b')\big)+\sum f^{b''}_{b,w\lambda}(q^{-1})G^{\mathrm{up}}(b''). \end{aligned}$$

By Proposition 6.11, we have $G^{\mathrm{up}}(j_\mu b)\Delta_{w\lambda}=q^m\Delta_{w\lambda}G^{\mathrm{up}}(j_\mu b)$ for some $m\in\mathbb{Z}$ in $\mathcal{O}_q[\overline{N_w}]$. It is equal to

$$(6.23)\quad \begin{aligned} &q^{-(\mathrm{wt}\, b-\mu,\lambda)+(\mathrm{wt}\, b-\mu,w\lambda-\lambda)+m}\Delta_{w\lambda}G^{\mathrm{up}}(j_\mu b) \\ &\quad = G^{\mathrm{up}}\big(j_{\lambda+\mu}(b')\big)+\sum f^{b''}_{b,w\lambda}(q^{-1})G^{\mathrm{up}}(b''). \end{aligned}$$

Therefore we obtain $f^{b''}_{b,w\lambda}(q)=0$ for all $b''$ by comparing (6.21) and (6.23). □

Since there exists $\mu\in P_+$ such that $\pi_\mu(b)\neq 0$, we obtain the following theorem.

THEOREM 6.24
*Let $b\in\mathscr{B}_w(\infty)$ and $\lambda\in P_+$. There exists $b'\in\mathscr{B}_w(\infty)$ such that*

$$\Delta_{w\lambda}G^{\mathrm{up}}(b)\simeq G^{\mathrm{up}}(b')$$

*in $\mathcal{O}_q[\overline{N_w}]$.*

Taking $b$ from $\mathscr{B}(w,-1)$, we obtain the following theorem by Corollary 4.32.

THEOREM 6.25
*For $\mathbf{c}\in\mathbf{Z}^\ell_{\geq 0}$ and $\lambda\in P_+$, we have*

$$\Delta_{w\lambda}G^{\mathrm{up}}\big(b_{-1}(\mathbf{c},\widetilde{w})\big)\simeq G^{\mathrm{up}}\big(b_{-1}(\mathbf{c}+\mathbf{n}^\lambda,\widetilde{w})\big).$$

*6.3.3*
The following is a generalization of Caldero's result [9, Lemma 2.1, Theorem 2.2]. It follows from Theorem 6.24 by an induction on the length of $w$.

THEOREM 6.26
*Let $w\in W$, and fix $\widetilde{w}\in R(w)$. We set*

$$\Delta_{\widetilde{w},k}:=\Delta_{s_{i_1}\cdots s_{i_k}\varpi_{i_k}}$$

*for $1\leq k\leq\ell$. Then $\{\Delta_{\widetilde{w},k}\}_{1\leq k\leq\ell}$ forms a strongly compatible subset.*

*6.3.4*
Following [22, Section 15.5], we call $\mathbf{c}\in\mathbb{Z}^\ell_{\geq 0}$ *interval-free* if $\mathbf{c}$ satisfies the following conditions:

$$c^{(i)}:=\min\{c_k;i_k=i\}=0$$

for all $i\in I$. By definition, ${}^\varphi\mathbf{c}:=\mathbf{c}-\sum_{i\in I}c^{(i)}\mathbf{n}^i\in\mathbb{Z}^\ell_{\geq 0}$ is interval free. We have the following factorization property with respect to the extremal vectors $\{\Delta_{w\lambda}\}_{\lambda\in P_+}$.

THEOREM 6.27
*For* $\mathbf{c} \in \mathbb{Z}_{\geq 0}^{\ell}$, *we set* $\lambda(\mathbf{c}) := \sum_{i \in I} c^{(i)} \varpi_i \in P_+$. *Then we have*

$$G^{\mathrm{up}}\big(b_{-1}(\mathbf{c}, \widetilde{w})\big) \simeq G^{\mathrm{up}}\big(b_{-1}({}^{\varphi}\mathbf{c}, \widetilde{w})\big) \Delta_{w\lambda(\mathbf{c})}.$$

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Research Institute for Mathematical Science, Kyoto University, Kyoto 606-8502, Japan; ykimura@kurims.kyoto-u.ac.jp