SURVEY ARTICLE

Stability conditions and curve counting invariants on Calabi–Yau 3-folds*

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Abstract The purpose of this paper is twofold. First we give a survey on the recent developments of curve counting invariants on Calabi–Yau 3-folds, for example, Gromov–Witten theory, Donaldson–Thomas theory, and Pandharipande–Thomas theory. Next we focus on the proof of the rationality conjecture of the generating series of PT invariants and discuss its conjectural Gopakumar–Vafa form.

Contents

1. Introduction

1.1. Background

Let X be a smooth projective Calabi–Yau 3-fold, that is,

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

We are interested in the curve counting theory on X. This is an important field of study in connection with mirror symmetry: it predicts a relationship between curve counting invariants on X and a period integral on its mirror manifold $\check{X}$. So far, curve counting invariants have been computed and compared under mirror symmetry in several situations.

Now there are three kinds of curve counting theories on X.

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1. *Gromov–Witten (GW) theory:* Counting pairs,

$$(C, f), \quad f : C \to X,$$

where C is a connected nodal curve and f is a morphism with finite automorphisms. In terms of string theory, GW invariants count *world sheets*. The moduli space defining the GW theory is Kontsevich's stable map moduli space. The resulting invariants are $\mathbb{Q}$-valued.

2. *Donaldson–Thomas (DT) theory:* Counting subschemes,

$$Z \subset X,$$

with $\dim Z \leq 1$. In terms of string theory, DT invariants count *D-branes*. The moduli space defining the DT theory is the classical Hilbert scheme. The resulting invariants are $\mathbb{Z}$-valued.

3. *Pandharipande–Thomas (PT) theory:* Counting pairs,

$$(F, s), \quad s : \mathcal{O}_X \to F,$$

where F is a pure one-dimensional sheaf, and s is surjective in dimension one. The PT invariants also count D-branes, but the stability condition is different from DT theory. The moduli space defining the PT theory is identified with the moduli space of two-term complexes,

$$I^\bullet = (\mathcal{O}_X \xrightarrow{s} F) \in D^b \operatorname{Coh}(X).$$

Here $D^b \operatorname{Coh}(X)$ is the bounded derived category of coherent sheaves on X.

An equivalence between GW and DT theories was conjectured by Maulik, Nekrasov, Okounkov, and Pandharipande [30]. Also, an equivalence between DT and PT theories was conjectured by Pandharipande and Thomas [32]. They are formulated in terms of generating functions.

On the other hand, the notion of stability conditions on $D^b \operatorname{Coh}(X)$ was introduced by Bridgeland [9]. He showed that the set of stability conditions on $D^b \times \operatorname{Coh}(X)$, denoted by

$$\operatorname{Stab}(X),$$

has the structure of a complex manifold. The space $\operatorname{Stab}(X)$ is expected to be related to the stringy Kähler moduli space, which should be isomorphic to the moduli space of complex structures of the mirror $\check{X}$. An important observation by Pandharipande and Thomas [32] is that the DT/PT correspondence should be interpreted as wall-crossing phenomena in the space of stability conditions $\operatorname{Stab}(X)$. Although it is still difficult to study $\operatorname{Stab}(X)$ when X is a projective Calabi–Yau 3-fold, kinds of limiting degenerations of Bridgeland stability have been introduced in [1], [35], and [36], and DT/PT wall-crossing is also observed in these degenerated stability conditions.

In recent years, the wall-crossing formula of DT-type invariants has been established by Joyce and Song [19] and Kontsevich and Soibelman [23] in a general setting. Since then, it has turned out that a categorical approach is useful in the study of DT-type curve counting invariants. Now several applications have been obtained, for example, DT/PT correspondence and the rationality conjecture (see [10], [33], [36], [37]). One of the purposes of this paper is to give a survey of these recent developments.

As another purpose, we focus on the rationality conjecture of the generating series of PT invariants proposed in [32]. The Euler characteristic version is proved in [37], and the virtual cycle is involved in [10]. In this paper, assuming the announced result by Behrend and Getzler [6], we give it another proof by discussing it in the framework of [36]. The main idea is the same as in [37], but the argument is simplified. We also discuss a conjectural Gopakumar–Vafa form of the generating series of PT invariants and see that it is related to the multicovering formula of generalized DT invariants introduced by Joyce and Song [19]. We also give evidence of the conjectural multicovering formula when X is a certain elliptically fibered Calabi–Yau 3-fold.

1.2. Plan of the paper

In Section 2, we give a survey on stability conditions. In Section 3, we recall several curve counting invariants on Calabi–Yau 3-folds, the relevant conjectures, and results. In Section 4, we recall the notion of Hall algebras and the generalized DT invariants counting one-dimensional sheaves. In Section 5, we give a proof of the rationality of the generating series of PT invariants in the framework of [36]. In Section 6, we discuss a Gopakumar–Vafa form of the generating series of PT invariants, and the multicovering formula of generalized DT invariants.

1.3. Notation and convention

For a triangulated category $\mathcal{D}$, the shift functor was defined in [1]. For a set of objects $\mathcal{S} \subset \mathcal{D}$, we denote by $\langle \mathcal{S} \rangle_{\mathrm{tr}}$ the smallest triangulated subcategory which contains $\mathcal{S}$ and $0 \in \mathcal{D}$. Also, we denote by $\langle \mathcal{S} \rangle_{\mathrm{ex}}$ the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{S}$ and $0 \in \mathcal{D}$. The abelian category of coherent sheaves on a variety X is denoted by $\mathrm{Coh}(X)$. We say that $F \in \mathrm{Coh}(X)$ is d-dimensional if its support is d-dimensional. We always assume that the second homology group $H_2(X, \mathbb{Z})$ is torsion-free. If there is a torsion, then the arguments are applied if we replace $H_2(X, \mathbb{Z})$ by its torsion-free part. For $\beta \in H_2(X, \mathbb{Z})$, we write $\beta > 0$ if β is a class of an effective algebraic one cycle on X.

2. Stability conditions

We begin by recalling stability conditions on abelian categories and explaining typical wall-crossing phenomena.

2.1. Definitions of stability conditions

Classically there is a notion of a stability condition on vector bundles on smooth projective curves. Let C be a smooth projective curve over $\mathbb{C}$, and let E be a vector bundle on it. The slope of E is defined by

$$\mu(E) := \deg(E)/\operatorname{rank}(E).$$

DEFINITION 2.1

A vector bundle E on C is (semi)stable if for any subbundle $0 \neq F \subsetneq E$, we have

$$\mu(F) < (\leq) \mu(E).$$

We have the following properties.

- If we fix rank r and degree d, then there is a good moduli space of slope semistable vector bundles E with $\mathrm{rank}(E) = r$ and $\deg(E) = d$.
- For any vector bundle E on C, there is a filtration (Harder-Narasimhan filtration),

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

such that each subquotient $F_i = E_i/E_{i-1}$ is semistable with $\mu(F_i) > \mu(F_{i+1})$ for all i.

A stability condition on an abelian category is defined to be a direct generalization of the above classical notion. Let $\mathcal{A}$ be an abelian category, for example, the category of coherent sheaves on an algebraic variety. Recall that its Grothendieck group is defined by

$$K(\mathcal{A}) := \bigoplus_{E \in \mathcal{A}} \mathbb{Z}[E]/\sim,$$

where the equivalence relation $\sim$ is generated by

$$[E_2] \sim [E_1] + [E_3],$$

for all exact sequences $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$. We fix a finitely generated abelian group Γ together with a group homomorphism,

$$\mathrm{cl} : K(\mathcal{A}) \to \Gamma.$$

For instance, if $\mathcal{A} = \mathrm{Coh}(X)$ for a smooth projective variety X, we can take Γ to be the image of the Chern character map,

$$\mathrm{ch} : K(\mathcal{A}) \twoheadrightarrow \Gamma \subset H^*(X, \mathbb{Q}), \tag{1}$$

and $\mathrm{cl} = \mathrm{ch}$. Let $\mathbb{H} \subset \mathbb{C}$ be the subset

$$\mathbb{H} = \left\{ r \exp(\pi i \phi) : r > 0, 0 < \phi \leq 1 \right\}.$$

The following formulation of stability conditions is due to Bridgeland [9].

DEFINITION 2.2

A stability condition on $\mathcal{A}$ is a group homomorphism,

$$Z : \Gamma \to \mathbb{C},$$

satisfying the following axioms.

(i) For any nonzero object $E \in \mathcal{A}$, we have

$$Z(E) := Z\big(\mathrm{cl}(E)\big) \in \mathbb{H}.$$

In particular, the argument

$$\arg Z(E) \in (0, \pi]$$

is well-defined. An object $E \in \mathcal{A}$ is called Z-(semi)stable if for any nonzero subobject $0 \neq F \subsetneq E$, we have

$$\arg Z(F) < (\leq) \arg Z(E).$$

(ii) For any object $E \in \mathcal{A}$, there is a filtration (Harder-Narasimhan filtration),

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E,$$

such that each subquotient $F_i = E_i/E_{i-1}$ is Z-semistable with

$$\arg Z(F_1) > \arg Z(F_2) > \cdots > \arg Z(F_N).$$

Here we give some examples.

EXAMPLE 2.3

(i) Let C be a smooth projective curve over $\mathbb{C}$, and take $\mathcal{A} = \mathrm{Coh}(C)$. We set Γ to be

$$\Gamma = \mathbb{Z} \oplus \mathbb{Z},$$

and we set a group homomorphism $\mathrm{cl} : K(C) \to \Gamma$ to be

$$\mathrm{cl}(E) = \big(\mathrm{rank}(E), \deg(E)\big).$$

Let $Z : \Gamma \to \mathbb{C}$ be the map defined by

$$Z(r, d) = -d + \sqrt{-1}r.$$

Then it is easy to see that Z is a stability condition on $\mathrm{Coh}(C)$. An object $E \in \mathrm{Coh}(C)$ is Z-semistable if and only if E is a torsion sheaf or E is a semistable vector bundle in the sense of Definition 2.1.

(ii) Let A be a finite-dimensional algebra over $\mathbb{C}$, and let $\mathcal{A}$ be the abelian category of finitely generated right A-modules. There is a finite number of simple objects $S_1, S_2, \ldots, S_N$ in $\mathcal{A}$ such that

$$K(\mathcal{A}) \cong \bigoplus_{i=1}^{N} \mathbb{Z}[S_i].$$

We set $\Gamma = K(\mathcal{A})$ and $\mathrm{cl} = \mathrm{id}$. Choose elements

$$z_1, z_2, \ldots, z_N \in \mathbb{H}.$$

Then the map $Z : \Gamma \to \mathbb{C}$ defined by

$$Z\Big(\sum_i a_i[S_i]\Big) = \sum_i a_i z_i$$

is a stability condition on $\mathcal{A}$.

(iii) The following generalization of (i) is used in the sections below. Let X be a smooth projective variety over $\mathbb{C}$. We set

$$\mathrm{Coh}_{\leq 1}(X) := \big\{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \leq 1\big\}.$$

We set

$$\Gamma_0 := \mathbb{Z} \oplus H_2(X, \mathbb{Z}),$$

and we set the group homomorphism $\mathrm{cl}_0 : K(\mathrm{Coh}_{\le 1}(X)) \to \Gamma_0$ to be

$$\mathrm{cl}_0(E) := \big(\mathrm{ch}_3(E), \mathrm{ch}_2(E)\big).$$

By the Riemann–Roch theorem, $\mathrm{cl}_0(E)$ is also written as $(\chi(E), [E])$, where $[E]$ is the fundamental homology class determined by E and $\chi(E)$ is the holomorphic Euler characteristic.

Let ω be an $\mathbb{R}$-ample divisor on X. We set $Z_\omega : \Gamma_0 \to \mathbb{C}$ to be

$$Z_\omega(n, \beta) := -n + (\omega \cdot \beta)\sqrt{-1}.$$

Then Z_ω is a stability condition on $\mathrm{Coh}_{\le 1}(X)$. An object $E \in \mathrm{Coh}_{\le 1}(X)$ is Z_ω-(semi)stable if and only if E is an ω-Gieseker (semi)stable sheaf (cf. [16]). If $\dim X = 1$ and $\deg \omega = 1$, then Z_ω coincides with the stability condition constructed in (i).

2.2. Wall-crossing phenomena

Here we explain a rough idea of wall-crossing phenomena and a simple example. We set

$$\mathrm{Stab}(\mathcal{A}) := \{Z \in \Gamma_{\mathbb{C}}^{\vee} : Z \text{ is a stability condition on } \mathcal{A}\}.$$

For instance, in Example 2.3(ii), we have the identification

$$\mathrm{Stab}(\mathcal{A}) \cong \mathbb{H}^N.$$

For $v \in \Gamma$, we are interested in ‘counting invariants’,

$$\mathrm{Stab}(\mathcal{A}) \ni Z \mapsto I_v(Z) \in \mathbb{Q},$$

where $I_v(Z)$ ‘counts’ Z-semistable objects $E \in \mathcal{A}$ with $\mathrm{cl}(E) = v$. There may be several choices for the definition of $I_v(Z)$. For instance, we can consider a moduli space of Z-semistable objects $E \in \mathcal{A}$ with $\mathrm{cl}(E) = v$, denoted by $M_v(Z)$, and take $I_v(Z)$ to be

$$I_v(Z) = \chi\big(M_v(Z)\big).$$

Here $\chi(*)$ is the topological Euler characteristic. We need to check the existence of the moduli space $M_v(Z)$, but this holds in the cases given in Example 2.3.

In principle, there should be a wall and chamber structure on the space $\mathrm{Stab}(\mathcal{A})$ such that $I_v(Z)$ is constant on a chamber but jumps on a wall. The set of walls is given by a countable number of real codimension one submanifolds $\{W_\lambda\}_{\lambda \in \Lambda}$ in $\mathrm{Stab}(\mathcal{A})$, and a chamber is a connected component,

$$\mathcal{C} \subset \mathrm{Stab}(\mathcal{A}) \setminus \bigcup_{\lambda \in \Lambda} W_\lambda.$$

For instance, let us consider the algebra A given by

$$A = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$$

Let $\mathcal{A}$ be the abelian category of finitely generated right A-modules. (In other words, $\mathcal{A}$ is the category of representations of a quiver with two vertices and one arrow.) There are two simple objects in $\mathcal{A}$,

$$S_i = \mathbb{C} \cdot e_i, \quad i = 1, 2,$$

whose right A-actions are given by

$$e_i \cdot \begin{pmatrix} a_1 & a_3 \\ 0 & a_2 \end{pmatrix} = a_i e_i.$$

We take an object $E \in \mathcal{A}$, which is isomorphic to $\mathbb{C}^2$ as a $\mathbb{C}$-vector space, and the right A-action is the standard one. There is an exact sequence in $\mathcal{A}$,

$$0 \to S_2 \to E \to S_1 \to 0. \tag{2}$$

Let us identify $\mathrm{Stab}(\mathcal{A})$ with $\mathbb{H}^2$, as in Example 2.3(ii). For a stability condition

$$Z = (z_1, z_2) \in \mathrm{Stab}(\mathcal{A}) \cong \mathbb{H}^2,$$

the exact sequence (2) easily implies that

$$E \text{ is } \begin{cases} Z\text{-stable} & \text{if } \arg z_2 < \arg z_1, \\ Z\text{-semistable} & \text{if } \arg z_2 = \arg z_1, \\ \text{not } Z\text{-semistable} & \text{if } \arg z_2 > \arg z_1. \end{cases}$$

In particular for an element

$$v = \mathrm{cl}(E) = (1, 1) \in \Gamma,$$

the moduli space $M_v(Z)$ is

$$M_v(Z) = \begin{cases} \{E\} & \text{if } \arg z_2 < \arg z_1, \\ \{E\} \cup \{S_1 \oplus S_2\} & \text{if } \arg z_2 = \arg z_1, \\ \emptyset & \text{if } \arg z_2 > \arg z_1. \end{cases}$$

The 'counting invariant' $I_v(Z) = \chi(M_v(Z))$ is

$$I_v(Z) = \begin{cases} 1 & \text{if } \arg z_2 < \arg z_1, \\ 2 & \text{if } \arg z_2 = \arg z_1, \\ 0 & \text{if } \arg z_2 > \arg z_1. \end{cases}$$

Here we have observed wall-crossing phenomena of $I_v(Z)$, whose wall is given by

$$W = \big\{(z_1, z_2) \in \mathbb{H}^2 : \arg z_1 = \arg z_2\big\}.$$

2.3. Weak stability conditions

A slightly generalized notion of stability conditions is sometimes useful. For instance, if we consider stability conditions in the sense of Definition 2.2, then there is no stability condition on $\mathrm{Coh}(X)$ if $\dim X \geq 2$ (cf. [35, Lemma 2.7]). On the other hand, there are classical notions of stability conditions on $\mathrm{Coh}(X)$, such as slope stability (see [16]). The slope stability can be formulated in the language of weak stability conditions introduced in [36].

Let $\mathcal{A}$ be an abelian category. As in Section 2.1, we fix a finitely generated free abelian group Γ together with a group homomorphism $\mathrm{cl}: K(\mathcal{A}) \to \Gamma$. We also fix a filtration of Γ,

$$0 = \Gamma_{-1} \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_N = \Gamma$$

such that each subquotient Γ_i/Γ_{i-1} is a free abelian group.

DEFINITION 2.4

A weak stability condition on $\mathcal{A}$ is

$$Z = \{Z_i\}_{i=0}^{N} \in \prod_{i=0}^{N} \mathrm{Hom}_{\mathbb{Z}}(\Gamma_i/\Gamma_{i-1}, \mathbb{C})$$

such that the following conditions are satisfied.

(i) For nonzero $E \in \mathcal{A}$, take $-1 \leq i \leq N$ such that $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$. (We regard $\Gamma_{-2} = \emptyset$.) Then we have

$$Z(E) := Z_i\big([\mathrm{cl}(E)]\big) \in \mathbb{H}.$$

Here $[\mathrm{cl}(E)]$ is the class of $\mathrm{cl}(E)$ in Γ_i/Γ_{i-1}. We say that $E \in \mathcal{A}$ is Z-(semi)stable if for any exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$, we have the inequality

$$\arg Z(F) < (\leq) \arg Z(G). \tag{3}$$

(ii) There is a Harder-Narasimhan filtration for any $E \in \mathcal{A}$.

When $N = 0$, a weak stability condition is a stability condition in the sense of Definition 2.2.

REMARK 2.5

If the inequality (3) is strict, we have the following three possibilities:

$$\arg Z(F) < \arg Z(E) < \arg Z(G), \tag{4}$$

$$\arg Z(F) < \arg Z(E) = \arg Z(G), \tag{5}$$

$$\arg Z(F) = \arg Z(E) < \arg Z(G). \tag{6}$$

If $N = 0$, that is, Z is a stability condition, then only the inequality (4) is possible. On the other hand when $N > 0$, the inequalities (5) and (6) are also possible.

Here we give some examples.

EXAMPLE 2.6

(i) Let X be a d-dimensional smooth projective variety, and let $\mathcal{A} = \mathrm{Coh}(X)$. Take $\Gamma = \mathrm{Im}\,\mathrm{ch}$, take $\mathrm{cl} = \mathrm{ch}$ as in (1), and take a filtration

$$\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_d,$$

given by

$$\Gamma_i = \Gamma \cap H^{\geq 2d-2i}(X, \mathbb{Q}).$$

Choose

$$0 < \phi_d < \phi_{d-1} < \cdots < \phi_0 < 1$$

and an ample divisor ω on X. Set $Z_i : \Gamma_i/\Gamma_{i-1} \to \mathbb{C}$ to be

$$Z_i(v) = \exp(\sqrt{-1}\pi\phi_i) \int_X v \cdot \omega^i.$$

Then $Z = \{Z_i\}_{i=0}^d$ is a weak stability condition on $\mathrm{Coh}(X)$. In this case, $E \in \mathrm{Coh}(X)$ is Z-semistable if and only if it is pure sheaf, that is, there is no $0 \neq F \subset E$ with $\dim \mathrm{Supp}(F) < \dim \mathrm{Supp}(E)$.

(ii) Let X be a smooth projective surface, and take Γ and cl as above. We set $\Gamma_0 \subset \Gamma_1 = \Gamma$ to be

$$\Gamma_0 = \Gamma \cap H^4(X, \mathbb{Q}),$$

hence

$$\Gamma_1/\Gamma_0 = \Gamma \cap (H^0 \oplus H^2).$$

We set $Z_i : \Gamma_i/\Gamma_{i-1} \to \mathbb{C}$ to be

$$Z_0(n) = -n,$$
$$Z_1(r, D) = -D \cdot \omega + \sqrt{-1}r.$$

Then $Z = \{Z_i\}_{i=0}^1$ is a weak stability condition on $\mathrm{Coh}(X)$. An object $E \in \mathrm{Coh}(X)$ is Z-semistable if and only if E is a torsion sheaf or an ω-slope semistable sheaf (cf. [16]).

In [36], the space of weak stability conditions on triangulated categories was introduced. Namely a weak stability condition on a triangulated category $\mathcal{D}$ is a pair $(Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ and Z is a weak stability condition on $\mathcal{A}$. We denote by

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}) \tag{7}$$

the set of weak stability conditions on $\mathcal{D}$ satisfying some good properties, that is, local finiteness and support properties (see [36, Section 2] for the details on these properties). Using the same argument by Bridgeland [9, Theorem 7.1], it was proved in [36, Theorem 2.15] that the set (7) has a natural topology and that each connected component is a complex manifold.

3. Curve counting invariants on Calabi–Yau 3-folds

In this section, we recall several curve counting theories on Calabi–Yau 3-folds, conjectures, and the results. In what follows, we call a smooth projective complex

3-fold *Calabi–Yau* if it satisfies the following condition:

$$\bigwedge^3 T_X^\vee \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

For instance, the quintic 3-fold,

$$X = \{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{P}^4,$$

is a famous example of a Calabi–Yau 3-fold.

3.1. Gromov–Witten theory

Let X be a smooth projective Calabi–Yau 3-fold, and let C be a connected one-dimensional reduced $\mathbb{C}$-scheme with at worst nodal singularities. A morphism of schemes

$$f : C \to X$$

is a *stable map* if the set of isomorphisms $\phi : C \xrightarrow{\sim} C$ satisfying $f \circ \phi = f$ is a finite set. This condition is equivalent to one of the following conditions.

- For any ample line bundle $\mathcal{L}$ on X, the line bundle $\omega_C \otimes f^*\mathcal{L}^{\otimes 3}$ is an ample line bundle on C. Here ω_C is the dualizing sheaf of C.
- If $C' \subset C$ is an irreducible component such that $f(C')$ is a point, then

$$2g(C') + \sharp\big(C' \cap (\overline{C \setminus C'})\big) \ge 3.$$

Here $g(\cdot)$ is the arithmetic genus. The moduli space of such maps is constructed after we fix the following numerical data,

$$g \in \mathbb{Z}_{\ge 0}, \qquad \beta \in H_2(X, \mathbb{Z}).$$

We call a stable map (C, f) *type* (g, β) if $g(C) = g$ and the map f satisfies $f_*[C] = \beta$. The moduli space of stable maps (C, f) of type (g, β) is denoted by

$$\overline{M}_g(X, \beta). \tag{8}$$

The moduli space (8) is a Deligne–Mumford stack of finite type over $\mathbb{C}$ (see [22]). However, the space (8) may be singular, and its dimension may be different from its expected dimension. In fact, the tangent space and the obstruction space of the space of maps $f : C \to X$ for a fixed C are given by

$$H^0(C, f^*T_X), \qquad H^1(C, f^*T_X),$$

respectively. Hence the expected dimension of the space (8) is

$$\begin{aligned}
&\chi(C, f^*T_X) + \dim \overline{M}_g \\
&= \frac{3}{2}\deg T_C + 3g - 3 \\
&= 0.
\end{aligned}$$

Here $\overline{M}_g$ is the moduli space of genus g stable curves. Here we have used the Riemann–Roch theorem on C and the Calabi–Yau assumption of X.

Now there is a way to construct the zero-dimensional virtual fundamental cycle on (8) via perfect obstruction theory (see [4], [28]). By definition, a *perfect obstruction theory* on a scheme (or Deligne–Mumford stack) M is a morphism in the derived category of coherent sheaves $D^b \operatorname{Coh}(M)$,

$$h : E^\bullet \to L_M, \tag{9}$$

where $E^\bullet$ is a complex of vector bundles on M concentrated on $[-1,0]$ and L_M is the cotangent complex of M. The morphism h should satisfy that h^0 is an isomorphism and h^{-1} is surjective. Given such a morphism (9), we are able to construct the virtual fundamental cycle,

$$[M]^{\mathrm{vir}} \in A_{\operatorname{rank} E^0 - \operatorname{rank} E^{-1}}(M).$$

Here $A_*(M)$ is the Chow group of M. Roughly speaking, the cycle $[M]^{\mathrm{vir}}$ is constructed by taking the intersection of the intrinsic normal cone and the zero-section in the vector bundle stack $[(E^{-1})^\vee/(E^0)^\vee]$ (see [4], [28] for the details).

By [4] and [28], there is a perfect obstruction theory on the moduli space (8). The resulting virtual fundamental cycle is denoted by

$$[\overline{M}_g(X,\beta)]^{\mathrm{vir}} \in A_0\big(\overline{M}_g(X,\beta),\mathbb{Q}\big).$$

Integrating the virtual cycle, we obtain the GW invariant.

DEFINITION 3.1

The *Gromov–Witten (GW) invariant* is defined by

$$N_{g,\beta}^{\mathrm{GW}} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

REMARK 3.2

Since $\overline{M}_g(X,\beta)$ is not a scheme but a Deligne–Mumford stack, the resulting invariant $N_{g,\beta}^{\mathrm{GW}}$ is not an integer in general.

One of the important examples is a contribution of multiple covers to a fixed super-rigid rational curve.

EXAMPLE 3.3

Let

$$f : X \to Y$$

be a birational contraction which contracts a smooth super rigid rational curve $C \subset X$, that is, let

$$N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

In this case, the computation of $N^{\mathrm{GW}}_{g,d[C]}$ can be reduced to a certain integration over the space $\overline{M}_g(\mathbb{P}^1, d)$. We have the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathbb{P}^1, \\ \overline{M}_g(\mathbb{P}^1, d) & \end{tikzcd}$$

where π is the universal curve and ϕ is the universal morphism. Then we have

$$N^{\mathrm{GW}}_{g,d[C]} = \int_{[\overline{M}_g(\mathbb{P}^1,d)]^{\mathrm{vir}}} c_{\mathrm{top}}\big(R^1\pi_*\phi^*\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\big). \tag{10}$$

The invariants (10) are computed in [12],

$$N^{\mathrm{GW}}_{0,d[C]} = \frac{1}{d^3}, \qquad N_{1,d[C]} = \frac{1}{12d},$$
$$N^{\mathrm{GW}}_{g,d[C]} = \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!}, \quad g \geq 2.$$

Here B_{2g} is the $2g$th Bernoulli number.

3.2. Donaldson–Thomas theory

Another curve counting invariant on a Calabi–Yau 3-fold X is defined by the integration of the virtual fundamental cycle on the moduli space of subschemes,

$$Z \subset X, \tag{11}$$

satisfying $\dim Z \leq 1$. Given numerical data,

$$n \in \mathbb{Z}, \qquad \beta \in H_2(X, \mathbb{Z}),$$

the relevant moduli space is the classical Hilbert scheme,

$$\mathrm{Hilb}_n(X, \beta), \tag{12}$$

which parameterizes subschemes (11) satisfying

$$\chi(\mathcal{O}_Z) = n, \quad [Z] = \beta. \tag{13}$$

Recall that the moduli space (12) is a projective scheme.

The moduli space (12) is also interpreted as a moduli space of rank-one torsion-free sheaves on X with a trivial first Chern class. Namely, if I is a torsion-free sheaf of rank one, then I fits into the exact sequence

$$0 \to I \to I^{\vee\vee} \to F \to 0$$

such that F is a one- or zero-dimensional sheaf. It can be shown that $I^{\vee\vee}$ is a line bundle on X and hence isomorphic to $\mathcal{O}_X$ if its first Chern class is zero. Hence I is isomorphic to I_Z, the ideal sheaf of a subscheme $Z \subset X$ with $\dim Z \leq 1$. The condition (13) is equivalent to the following condition on the Chern character,

$$\mathrm{ch}(I_Z) = (1, 0, -\beta, -n) \tag{14}$$

$$\in H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \oplus H^6(X,\mathbb{Z}). \tag{15}$$

Here we have regarded β and n as elements of $H^4(X,\mathbb{Z})$ and $H^6(X,\mathbb{Z})$ by the Poincaré duality. As a summary, there is a one-to-one correspondence between subschemes (11) satisfying (13) and torsion-free sheaves I on X satisfying (14), via $Z \mapsto I_Z$.

If we regard the space (12) as a moduli space of rank-one torsion-free sheaves, the deformation theory of coherent sheaves implies that the spaces

$$\mathrm{Ext}^1_X(I_Z, I_Z), \qquad \mathrm{Ext}^2_X(I_Z, I_Z)$$

are the tangent space and the obstruction space at the point $[Z] \in \mathrm{Hilb}_n(X,\beta)$, respectively. Since X is a Calabi–Yau 3-fold, the Serre duality implies that

$$\mathrm{Ext}^2_X(I_Z, I_Z) \cong \mathrm{Ext}^1_X(I_Z, I_Z)^\vee.$$

In particular, the expected dimension of the space (12) is

$$\dim \mathrm{Ext}^1_X(I_Z, I_Z) - \dim \mathrm{Ext}^2_X(I_Z, I_Z) = 0.$$

In fact there is a perfect obstruction theory on $\mathrm{Hilb}_n(X,\beta)$ (see [34]),

$$E^\bullet \to L_{\mathrm{Hilb}_n(X,\beta)},$$

satisfying

$$E^\bullet \cong E^{\bullet\vee}[1]. \tag{16}$$

A perfect obstruction theory satisfying the symmetry (16) is called a *perfect symmetric obstruction theory*. We have the associated virtual fundamental cycle,

$$[\mathrm{Hilb}_n(X,\beta)]^{\mathrm{vir}} \in A_0\big(\mathrm{Hilb}_n(X,\beta), \mathbb{Z}\big).$$

The DT invariant is defined by the integration over the virtual fundamental cycle.

DEFINITION 3.4

The *Donaldson–Thomas (DT) invariant* is defined by

$$I_{n,\beta} = \int_{[\mathrm{Hilb}_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}. \tag{17}$$

So far, $I_{n,\beta}$ has been computed in several examples in terms of generating functions.

EXAMPLE 3.5

(i) In the case $\beta = 0$, the generating series of $I_{n,0}$ was computed by Li [27], Behrend and Fantechi [5], and Levine and Pandharipande [26],

$$\sum_{n\in\mathbb{Z}} I_{n,0} q^n = M(-q)^{\chi(X)}.$$

Here $M(q)$ is the MacMahon function,

$$\begin{aligned} M(q) &= \prod_{k\geq 1} \frac{1}{(1-q^k)^k} \\ &= 1+q+3q^2+6q^3+\cdots . \end{aligned}$$

(ii) Let $C \subset X$ be a super rigid rational curve as in Example 3.3. Then the invariant $I_{n,d[C]}$ was computed by Behrend and Bryan [3],

$$\sum_{n,d} I_{n,d[C]} q^n t^d = M(-q)^{\chi(X)} \prod_{k\geq 1} \bigl(1-(-q)^k t\bigr)^k .$$

3.3. DT theory via Behrend function

The integration (17) is usually difficult to compute. On the other hand, Behrend [2] showed that the invariant (17) is also obtained as a certain weighted Euler characteristic of a certain constructible function on $\mathrm{Hilb}_n(X,\beta)$. In many situations, computations of weighted Euler characteristic are easier than computations of virtual fundamental cycles.

In fact, for any $\mathbb{C}$-scheme M, Behrend [2] constructed a canonical constructible function,

$$\nu_M : M \to \mathbb{Z},$$

satisfying the following properties.

- If $\pi : M_1 \to M_2$ is a smooth morphism with relative dimension d, we have

$$\nu_{M_1} = (-1)^d \pi^* \nu_{M_2} .$$

- For $p \in M$, suppose that there is an analytic open neighborhood $p \in U \subset M$, a complex manifold V, and a holomorphic function $f : V \to \mathbb{C}$ such that $U \cong \{df = 0\}$. Then we have

$$\nu(p) = (-1)^{\dim V} \bigl(1-\chi(M_p(f))\bigr). \tag{18}$$

Here $M_p(f)$ is the Milnor fiber of f at $p \in V$.

- If M has a symmetric perfect obstruction theory, we have

$$\begin{aligned} \int_{[M]^{\mathrm{vir}}} 1 &= \int_M \nu_M \, d\chi \\ &:= \sum_{k\in\mathbb{Z}} k\chi\bigl(\nu^{-1}(k)\bigr). \end{aligned} \tag{19}$$

Here the Milnor fiber $M_p(f)$ is defined as follows. Let $p \in V' \subset V$ be an analytic small neighborhood, and fix a norm $\|\cdot\|$ on V'. Then for $0 < \varepsilon \ll \delta \ll 1$, the topological type of the space

$$\bigl\{ z \in V' : \|z-p\| \leq \delta,\ f(z) = f(p) + \epsilon \bigr\} \tag{20}$$

does not depend on ε, δ. The Milnor fiber $M_p(f)$ is defined to be the topological space (20).

By the property (19), the invariant $I_{n,\beta}$ is also obtained by

$$I_{n,\beta} = \int_{\mathrm{Hilb}_n(X,\beta)} \nu \, d\chi.$$

Here we have written $\nu_{\mathrm{Hilb}_n(X,\beta)}$ as ν for simplicity. An important fact is that the local moduli space of objects in $\mathrm{Coh}(X)$ is analytically locally written as a critical locus of some holomorphic function on a complex manifold up to gauge equivalence. This fact is proved in [19, Theorem 5.2] in a more general setting. In particular the function ν on $\mathrm{Hilb}_n(X,\beta)$ can be computed using the expression (18).

A rough idea of the proof of the critical locus condition in [19, Theorem 5.2] is as follows. For $E \in \mathrm{Coh}(X)$, we are interested in the deformations of E. By applying spherical twists associated to line bundles, we may assume that E is a locally free sheaf or, equivalently, a holomorphic vector bundle (see [19, Corollary 8.5]). Let

$$\overline{\partial} : E \to E \otimes \Omega^{0,1}$$

be the $\overline{\partial}$-connection which determines a holomorphic structure of E, where $\Omega^{0,1}$ is the sheaf of $(0,1)$-forms of X. Then giving a deformation of E is equivalent to giving a deformation of $\overline{\partial}$ up to gauge equivalence. This is equivalent to giving

$$A \in A^{0,1}\big(X, \mathcal{E}\mathrm{nd}(E)\big),$$

where $A^{0,1}(X, \mathcal{E}\mathrm{nd}(E))$ is the space of $\mathcal{E}\mathrm{nd}(E)$-valued $(0,1)$-forms satisfying

$$(\overline{\partial} + A)^2 = 0, \tag{21}$$

up to gauge equivalence. The equation (21) is equivalent to

$$\overline{\partial} A + A \wedge A = 0.$$

Let CS be the *holomorphic Chern–Simons function*,

$$\mathrm{CS} : A^{0,1}\big(X, \mathcal{E}\mathrm{nd}(E)\big) \to \mathbb{C},$$

defined by

$$\mathrm{CS}(A) = \int_X \Big(\frac{1}{2}\overline{\partial} A \wedge A + \frac{1}{3} A \wedge A \wedge A\Big) \wedge \sigma_X,$$

where σ_X is a nowhere-vanishing holomorphic 3-form on X (see [34]). Then $A \in A^{0,1}(X, \mathcal{E}\mathrm{nd}(E))$ satisfies the equation (21) if and only if A is a critical locus of the function CS. Therefore the local moduli space of E is written as

$$\{d\mathrm{CS} = 0\}/G,$$

where G is the group of isomorphisms of E as a C^∞-vector bundle, that is, the local moduli space of objects in $\mathrm{Coh}(X)$ is written as a critical locus up to gauge equivalence.

However, $A^{0,1}(X, \mathcal{E}\mathrm{nd}(E))$ is an infinite-dimensional vector space, and we need to find a suitable finite-dimensional vector subspace of $A^{0,1}(X, \mathcal{E}\mathrm{nd}(E))$. This was worked out in [19, Theorem 5.2] by using the Hodge theory. Namely, the space of harmonic forms U on $A^{0,1}(X, \mathcal{E}\mathrm{nd}(E))$ is finite-dimensional, satisfying $U \cong \mathrm{Ext}^1(E,E)$, and we restrict CS to U (for the details, see [19, Theorem 5.2]).

EXAMPLE 3.6

(i) Suppose that $\mathrm{Hilb}_n(X,\beta)$ is nonsingular of dimension d. By the property (18), the Behrend function on $\mathrm{Hilb}_n(X,\beta)$ coincides with $(-1)^d$. Therefore we have

$$I_{n,\beta} = (-1)^d \chi\big(\mathrm{Hilb}_n(X,\beta)\big).$$

(ii) Suppose that $\mathrm{Hilb}_n(X,\beta)$ is isomorphic to the spectrum of $\mathbb{C}[z]/z^k$ for some $k \geq 1$. (For instance, the local moduli space of a rigid rational curve $C \subset X$ with $N_{C/X} = \mathcal{O}_C \oplus \mathcal{O}_C(-2)$ is written as the spectrum of $\mathbb{C}[z]/z^k$ for some $k \geq 1$.) Then $\mathrm{Hilb}_n(X,\beta)$ is written as $\{df = 0\}$, where f is

$$f : \mathbb{C} \ni z \mapsto z^{k+1} \in \mathbb{C}.$$

The Milnor fiber of f at $0 \in \mathbb{C}$ is $(k+1)$-points; hence we have

$$I_{n,\beta} = \nu(0) = k.$$

3.4. GW/DT correspondence

As we mention above, a GW invariant is not necessarily an integer, while a DT invariant is always an integer. Although both theories seem different, Maulik, Nekrasov, Okounkov, and Pandharipande [30] proposed a conjecture on a certain relationship between GW and DT theories. The conjecture is formulated in terms of generating functions, and it also implies a hidden integrality of GW invariants.

Let us introduce the generating functions. The generating function of the GW side is

$$\mathrm{GW}(X) = \sum_{g \geq 0, \beta > 0} N_{g,\beta}^{\mathrm{GW}} \lambda^{2g-2} t^{\beta}.$$

Here $\beta > 0$ means that β is a homology class of a nonzero effective one-cycle on X. Similarly the generating function of the DT side is

$$\mathrm{DT}(X) = \sum_{n \in \mathbb{Z}, \beta \geq 0} I_{n,\beta} q^n t^{\beta}.$$

The series $\mathrm{DT}(X)$ can be written as

$$\mathrm{DT}(X) = \sum_{\beta \geq 0} \mathrm{DT}_{\beta}(X) t^{\beta},$$

where $\mathrm{DT}_{\beta}(X)$ is a Laurent series of q. (It is easy to check that $\mathrm{Hilb}_n(X,\beta) = \emptyset$; hence $I_{n,\beta} = 0$, for $n \ll 0$.) The term $\mathrm{DT}_0(X)$ is a contribution of zero-dimensional subschemes and does not contribute to curve counting on X. The *reduced DT series* are defined by

$$\mathrm{DT}'(X) = \frac{\mathrm{DT}(X)}{\mathrm{DT}_0(X)}, \qquad \mathrm{DT}'_{\beta}(X) = \frac{\mathrm{DT}_{\beta}(X)}{\mathrm{DT}_0(X)}. \tag{22}$$

Note that $\mathrm{DT}_0(X)$ is given by the power of the MacMahon function by Example 3.5(i).

CONJECTURE 3.7 ([30, CONJECTURES 2, 3])

(i) (Rationality conjecture) *The Laurent series* $\mathrm{DT}'_\beta(X)$ *is the Laurent expansion of a rational function of* q*, invariant under* $q \leftrightarrow 1/q$.

(ii) (GW/DT correspondence) *By the variable change* $q = -e^{i\lambda}$*, we have the equality of the generating series,*

$$\exp \mathrm{GW}(X) = \mathrm{DT}'(X).$$

Here we need some explanation on the above conjecture. The series $\mathrm{DT}'_\beta(X)$ is a priori a Laurent series of q, and it is not obvious whether it converges or not near $q = 0$. The rationality conjecture asserts that $\mathrm{DT}'_\beta(X)$ actually converges near $q = 0$, and moreover, it can be analytically continued to give a meromorphic function (in fact, a rational function) on the q-plane. The invariance under $q \leftrightarrow 1/q$ implies that the above analytic continuation satisfies the automorphic property with respect to the transformation $q \leftrightarrow 1/q$. For instance in the situation of Example 3.3, the series $\mathrm{DT}'_{[C]}(X)$ is

$$\begin{aligned} \mathrm{DT}'_{[C]}(X) &= q - 2q^2 + 3q^3 - \cdots \\ &= \frac{q}{(1+q)^2}. \end{aligned} \tag{23}$$

The rational function (23) is invariant under $q \leftrightarrow 1/q$.

If we assume the rationality conjecture, we can expand $\mathrm{DT}'(X)$ near $q = -1$ and write it with the λ-variable via $q = -e^{i\lambda}$. The invariance of $\mathrm{DT}'_\beta(X)$ under $q \leftrightarrow 1/q$ implies that i is not involved in the λ-expansion. The GW/DT correspondence asserts that the coefficients of the above expansion are described in terms of GW invariants.

So far the above conjecture has been checked in several situations. For instance the GW/DT correspondence for a local $(-1,-1)$-curve can be checked from Examples 3.3 and 3.5, as discussed in [3]. Also, GW/DT correspondences for toric Calabi–Yau 3-folds and local curves are proved in [30] and [31], respectively, by using torus localization and degeneration formulas. On the other hand, at this moment, these arguments are applied to the above specific examples and not to arbitrary Calabi–Yau 3-folds. We have few tools in approaching Conjecture 3.7 in a general setting, except the recent progress of a *wall-crossing formula* of DT-type invariants. This was established by Joyce and Song [19] and Kontsevich and Soibelman [23] and is an effective tool in studying DT-type curve counting invariants for arbitrary Calabi–Yau 3-folds. So far, several applications have been given, including Conjecture 3.7(i).

A rough idea of the application of the wall-crossing formula is as follows. Recall that the moduli space $\mathrm{Hilb}_n(X, \beta)$ is interpreted as a moduli space of torsion-free rank-one sheaves on X. This is nothing but the moduli space of stable objects on $\mathrm{Coh}(X)$ with respect to weak stability conditions in Example 2.6(i).

One may try to change weak stability conditions on $\mathrm{Coh}(X)$, construct other DT-type invariants counting stable objects, and see wall-crossing phenomena as discussed in Section 2.2. However, we can easily see that there is no interesting wall-crossing phenomena with respect to weak stability conditions constructed in Example 2.6(i). Instead we can study (weak) stability conditions on another abelian subcategory in the derived category of coherent sheaves $D^b\,\mathrm{Coh}(X)$, for example, the heart of a bounded t-structure on $D^b\,\mathrm{Coh}(X)$. Then we can construct DT-type invariants counting stable objects in the derived category, and the wall-crossing formula describes how these invariants vary under a change of (weak) stability conditions. If we choose some specific (weak) stability condition, then the generating series sometimes becomes simpler than the original DT series, thus giving some nontrivial result to the DT series.

As mentioned, an important point is that the wall-crossing formula is applied for any Calabi–Yau 3-fold and not restricted to specific examples, for example, toric Calabi–Yau 3-folds. Using this new kind of technology, Conjecture 3.7(i) is now solved.* We discuss this more in Section 3.6 below.

3.5. Pandharipande–Thomas theory

Another application of the wall-crossing formula is the so-called DT/PT correspondence, that is the correspondence between DT invariants and invariants counting stable pairs (see [32]). The notion of stable pairs is introduced by Pandharipande and Thomas [32] in order to give a geometric understanding of the reduced DT theory (22). By definition, a *stable pair* on a Calabi–Yau 3-fold X is a pair

$$(F, s),$$

where F is a coherent sheaf on X and $s : \mathcal{O}_X \to F$ is a morphism satisfying the following.

- F is a pure one-dimensional sheaf; that is, there is no zero-dimensional subsheaf in F.
- The cokernel of s is a zero-dimensional sheaf.

For instance, let $C \subset X$ be a smooth curve, and let $D \subset C$ be a divisor on C. We set $F = \mathcal{O}_C(D)$ and define the morphism s to be the composition

$$s : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_C \hookrightarrow \mathcal{O}_C(D).$$

Then the pair (F, s) is a stable pair. As the above example indicates, roughly speaking, a stable pair is a pair of a curve on X and an effective divisor on it.

Note that if $Z \subset X$ is a subscheme giving a point in $\mathrm{Hilb}_n(X, \beta)$, we have a pair

$$(\mathcal{O}_Z, s), \quad s : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z,$$

*We need the result of [6], which is not yet written at this moment.

where s is a natural surjection. The pair $(\mathcal{O}_Z, s)$ fails to be a stable pair if and only if $\mathcal{O}_Z$ contains a zero-dimensional subsheaf. On the other hand, a stable pair (F, s) determines a point in $\mathrm{Hilb}_n(X, \beta)$ if and only if s is surjective.

Similarly to the DT theory, we consider the moduli space of stable pairs (F, s) satisfying

$$[F] = \beta, \quad \chi(F) = n.$$

Here $[F]$ is the fundamental homology class determined by the one-dimensional sheaf F. The resulting moduli space is denoted by

$$P_n(X, \beta). \tag{24}$$

The moduli space (24) was proved to be a projective scheme in [32]. Moreover the space (24) is interpreted as a moduli space of two-term complexes,

$$I^\bullet = \cdots \to 0 \to \mathcal{O}_X \xrightarrow{s} F \to 0 \to \cdots, \tag{25}$$

in the derived category of coherent sheaves, that is,

$$I^\bullet \in D^b \operatorname{Coh}(X).$$

The deformation theory of objects in the derived category yields that the spaces

$$\operatorname{Ext}^1_X(I^\bullet, I^\bullet), \qquad \operatorname{Ext}^2_X(I^\bullet, I^\bullet)$$

are the tangent space and the obstruction space, respectively, which are dual by Serre duality. Similarly to the DT theory, the above deformation theory provides a perfect symmetric obstruction theory on the space (24), hence the zero-dimensional virtual cycle.

DEFINITION 3.8

The *Pandharipande–Thomas (PT) invariant* is defined by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

As in the DT case, the invariant $P_{n,\beta}$ is also defined by

$$P_{n,\beta} = \int_{P_n(X,\beta)} \nu \, d\chi,$$

for the Behrend function

$$\nu : P_n(X, \beta) \to \mathbb{Z}.$$

EXAMPLE 3.9

Let $C \cong \mathbb{P}^1 \subset X$ be a super rigid rational curve as in Example 3.3. Then (F, s) is a stable pair with $[F] = [C]$ and $\chi(F) = n$ if and only if

$$F = \mathcal{O}_C(n-1), \qquad s \in H^0\big(C, \mathcal{O}_C(n-1)\big) \setminus \{0\}.$$

Hence we have

$$P_n(X,[C]) \cong \mathbb{P}\big(H^0(C,\mathcal{O}_C(n-1))\big) \\ \cong \mathbb{P}^{n-1}.$$

Therefore we have

$$P_{n,[C]} = (-1)^{\dim P_n(X,[C])}\chi\big(P_n(X,[C])\big) \\ = (-1)^{n-1}n.$$

The generating series is

$$\sum_{n\in\mathbb{Z}} P_{n,[C]}q^n = q - 2q^2 + 3q^3 - \cdots.$$

Note that the above series coincides with $\mathrm{DT}'_{[C]}(X)$ by (23).

Similarly to the DT theory, we consider the generating series,

$$\mathrm{PT}(X) = \sum_{n\in\mathbb{Z},\beta\geq 0} P_{n,\beta}q^n t^\beta \\ = 1 + \sum_{\beta>0}\mathrm{PT}_\beta(X),$$

where $\mathrm{PT}_\beta(X)$ is a Laurent series of q. In [32, Conjecture 3.3], Pandharipande and Thomas proposed the following conjecture.

CONJECTURE 3.10

We have the equality of the generating series,

$$\mathrm{DT}'_\beta(X) = \mathrm{PT}_\beta(X). \tag{26}$$

Note that we have already observed the formula (26) in Example 3.9 when the curve class is a class of a super rigid rational curve.

Similarly to Conjecture 3.7(i), the formula (26) is also a consequence of the wall-crossing formula. A rough idea is as follows. Suppose that there is an abelian subcategory $\mathcal{A}$ in $D^b\,\mathrm{Coh}(X)$ and a stability condition σ on it such that the ideal sheaf I_Z for a one-dimensional subscheme $Z\subset X$ is a σ-stable object in $\mathcal{A}$. If there is a zero-dimensional subsheaf $Q\subset\mathcal{O}_Z$, that is, if $\mathcal{O}_X\to\mathcal{O}_Z$ is not a stable pair, then there is a sequence,

$$Q[-1]\to I_Z\to I_{Z'}, \tag{27}$$

where Z' is a one-dimensional subscheme in Z defined by $\mathcal{O}_{Z'} = \mathcal{O}_Z/Q$. Suppose that the sequence (27) is an exact sequence in $\mathcal{A}$. Then we expect that we can deform a stability condition σ to another stability condition τ such that the sequence (27) destabilizes I_Z with respect to τ. Instead, if we take an exact sequence in $\mathcal{A}$,

$$I_{Z'}\to E\to Q[-1],$$

then the object E may be τ-stable. Such an object E is isomorphic to a two-term complex,

$$E \cong (\mathcal{O}_X \xrightarrow{s} F),$$

for a one-dimensional sheaf F, and one may expect that (F, s) is a stable pair. If the above story is correct, then σ corresponds to the DT theory, τ corresponds to the PT theory, and the relationship between these theories should be described by the wall-crossing formula.

3.6. Product formula of the generating series

In this subsection, we discuss the result obtained by applying the wall-crossing formula.

THEOREM 3.11 ([37, THEOREM 4.7], [36, THEOREM 3.14], [10, THEOREM 1.1])

For each $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, there are invariants

$$N_{n,\beta} \in \mathbb{Q}, \qquad L_{n,\beta} \in \mathbb{Q}$$

satisfying the following:

- *there is $d \in \mathbb{Z}_{>0}$ such that $N_{n,\beta} = N_{n',\beta}$ if $n \pm n' \in d\mathbb{Z}$ and $\beta \neq 0$,*
- *$L_{n,\beta} = L_{-n,\beta}$, and $L_{n,\beta} = 0$ for $|n| \gg 0$,*

such that we have the following infinite product expansion formulas,

$$\mathrm{PT}(X) = \prod_{n>0,\beta>0} \exp\big((-1)^{n-1} n N_{n,\beta} q^n t^\beta\big) \Big(\sum_{n,\beta} L_{n,\beta} q^n t^\beta\Big), \tag{28}$$

$$\mathrm{DT}(X) = \prod_{n>0} \exp\big((-1)^{n-1} n N_{n,0} q^n\big)\, \mathrm{PT}(X). \tag{29}$$

We explain how to deduce the formula (28) via wall-crossing in Section 5.

REMARK 3.12

More precisely, the results in [37] and [36] are Euler characteristic versions of the corresponding results; that is, take the (nonweighted) Euler characteristic in defining the invariants $I_{n,\beta}$, $P_{n,\beta}$. As discussed in the arXiv version of [36, Theorem 8.11], the formulas (28) and (29) can be proved by combining the work of Joyce and Song [19] and Behrend and Getzler's announced result [6]. The latter result is the derived category version of [19, Theorem 5.3]; that is, the moduli stack of certain objects in the derived category is locally written as a critical locus of some holomorphic function up to gauge action. The precise statement was formulated in [39, Conjecture 4.3]. On the other hand, in [10], Bridgeland proved Theorem 3.11 without relying on [6], using arguments different from ours.

The invariants $N_{n,\beta}$ and $L_{n,\beta}$ are also interpreted as counting invariants of certain objects in the derived category. Roughly speaking:

- let ω be an $\mathbb{R}$-ample divisor, and let Z_ω be the stability condition on $\mathrm{Coh}_{\leq 1}(X)$ constructed in Example 2.3(iii); in the notation of Example 2.3(iii), the invariant $N_{n,\beta}$ counts Z_ω-semistable objects $E \in \mathrm{Coh}_{\leq 1}(X)$ satisfying

$$\mathrm{cl}_0(E) = (n, \beta) \in \Gamma_0;$$

- the invariant $L_{n,\beta}$ counts certain semistable objects in the derived category $E \in D^b \mathrm{Coh}(X)$ satisfying

$$\begin{aligned}\mathrm{ch}(E) &= (1, 0, -\beta, -n) \\ &\in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \oplus H^6(X, \mathbb{Z});\end{aligned}$$

the relevant stability condition is self-dual with respect to the derived dual.

In order to define $N_{n,\beta}$, we need to choose an $\mathbb{R}$-ample divisor ω, but it can be shown that $N_{n,\beta}$ does not depend on ω (cf. Lemma 4.8). The self-duality in defining $L_{n,\beta}$ means that if E is (semi)stable, then its derived dual,

$$\mathbf{R}\mathcal{H}om(E, \mathcal{O}_X) \in D^b \mathrm{Coh}(X),$$

is also (semi)stable. The equality $L_{n,\beta} = L_{-n,\beta}$ is a consequence of the self-duality.

In some cases, the invariants $N_{n,\beta}$ and $L_{n,\beta}$ are defined in a way similar to DT or PT invariants. Let us take $n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z})$, and an ample $\mathbb{R}$-divisor ω on X. Let $M_{n,\beta}(\omega)$ be the moduli space of Z_ω-semistable objects $E \in \mathrm{Coh}_{\leq 1}(X)$ satisfying $\mathrm{cl}_0(E) = (n, \beta)$, in the notation of Example 2.3(iii). If n and β are coprime and ω is in a general position of the ample cone, then any Z_ω-semistable sheaf $E \in \mathrm{Coh}_{\leq 1}(X)$ is Z_ω-stable, and the moduli space $M_{n,\beta}(\omega)$ is a projective scheme with a symmetric perfect obstruction theory. The invariant $N_{n,\beta}(\omega)$ is defined by

$$N_{n,\beta}(\omega) := \int_{[M_{n,\beta}(\omega)]^{\mathrm{vir}}} 1 = \int_{M_{n,\beta}(\omega)} \nu \, d\chi.$$

Here ν is the Behrend function on $M_{n,\beta}(\omega)$. We show in Lemma 4.8 that $N_{n,\beta}(\omega)$ is independent of ω, so we can write it as $N_{n,\beta}$.

On the other hand, if n and β are not coprime, then the Z_ω-semistable sheaf may not be Z_ω-stable, and there is no fine moduli space $M_{n,\beta}(\omega)$ in this case. Instead we should work with the moduli stack of Z_ω-semistable objects, denoted by $\mathcal{M}_{n,\beta}(\omega)$. The moduli stack $\mathcal{M}_{n,\beta}(\omega)$ is known to be an Artin stack of finite type over $\mathbb{C}$. However, it is not obvious how to define counting invariants via $\mathcal{M}_{n,\beta}(\omega)$, since at this moment there is no reasonable notion of perfect obstruction theories nor virtual fundamental cycles on Artin stacks. Also, it is not obvious how to define the weighted Euler characteristic of $\mathcal{M}_{n,\beta}(\omega)$, weighted by the Behrend function. The only known way (at this moment) to do this is to introduce the *logarithm* of the moduli stack $\mathcal{M}_{n,\beta}(\omega)$ in the Hall algebra and integrate it. We discuss this construction in Section 4.

As a corollary of Theorem 3.11, we have the following result.

COROLLARY 3.13 ([37, THEOREM 4.7], [36, THEOREM 3.14], [10, THEOREM 1.1])
Conjecture 3.7(i) and Conjecture 3.10 are true.

Proof
The property of $N_{n,\beta}$ easily implies that the series

$$\sum_{n>0}(-1)^{n-1}nN_{n,\beta}q^n \tag{30}$$

is the Laurent expansion of a rational function of q, invariant under $q \leftrightarrow 1/q$ (cf. [37, Lemma 4.6]). Then Conjecture 3.7(i) follows from the rationality of (30) and the property of $L_{n,\beta}$.

As for Conjecture 3.10, the formula (29) in particular implies that

$$\mathrm{DT}_0(X) = \prod_{n>0}\exp\bigl((-1)^{n-1}nN_{n,0}q^n\bigr).$$

Hence the formula (26) follows. □

4. Hall algebras and generalized Donaldson–Thomas invariants

In Section 3.6, we discuss the invariants $N_{n,\beta}$ and $L_{n,\beta}$, which count certain objects in the derived category $D^b\operatorname{Coh}(X)$. As we discuss there, the definition of these invariants is not obvious if there is a strictly semistable object. In this section, we introduce a (stack-theoretic) Hall algebra of coherent sheaves and explain how to construct $N_{n,\beta}$ via that algebra. The construction is due to Joyce and Song [19] and is called the *generalized Donaldson–Thomas invariant*. (The invariant $L_{n,\beta}$ can be similarly constructed, and we discuss it in Section 5.)

4.1. Grothendieck groups of varieties

We recall the notion of Grothendieck groups of varieties. Let S be a variety over $\mathbb{C}$. We define the group $K(\mathrm{Var}/S)$ to be the group generated by isomorphism classes of symbols

$$[\rho : Y \to S],$$

where $\rho : Y \to S$ is an S-variety of finite type over $\mathbb{C}$, and two symbols $[\rho_i : Y_i \to S]$ for $i = 1, 2$ are isomorphic if there is an isomorphism $Y_1 \xrightarrow{\sim} Y_2$ preserving the morphisms ρ_i. The relation is generated by

$$[\rho : Y \to S] \sim [\rho|_V : V \to S] + [\rho|_U : U \to S],$$

where $V \subset Y$ is a closed subvariety and $U := Y \setminus V$. If $S = \operatorname{Spec}\mathbb{C}$, we write $K(\mathrm{Var}/S)$ as $K(\mathrm{Var}/\mathbb{C})$ for simplicity.

The structure of the group $K(\mathrm{Var}/\mathbb{C})$ was studied in [7]. This is generated by smooth projective varieties $[Y]$ with the relation given by

$$[\widehat{Y}] - [E] \sim [Y] - [C], \tag{31}$$

where $C \subset Y$ is a smooth subvariety, $\widehat{Y} \to Y$ is a blowup at C, and $E \subset \widehat{Y}$ is the exceptional divisor.

Several interesting invariants of varieties can be extended to invariants of elements in $K(\mathrm{Var}/\mathbb{C})$, using the above description of the generators and relations. For instance for a smooth projective variety Y, its Poincaré polynomial is defined by

$$P_t(Y) = \sum_{i=0}^{2\dim Y} (-1)^i \dim H^i(Y,\mathbb{C}) t^i. \tag{32}$$

The polynomial $P_t(\cdot)$ is compatible with respect to the relation (31); hence there is a map,

$$P_t : K(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[t], \tag{33}$$

such that $P_t([Y])$ coincides with (32) if Y is smooth and projective.

4.2. Grothendieck groups of stacks

The notion of a Grothendieck group of varieties can be generalized to that of Artin stacks. For an introduction to stacks, the reader can consult [25].

Let $\mathcal{S}$ be an Artin stack, locally of finite type over $\mathbb{C}$. We define the $\mathbb{Q}$-vector space $K(\mathrm{St}/\mathcal{S})$ to be generated by isomorphism classes of symbols

$$[\rho : \mathcal{Y} \to \mathcal{S}],$$

where $\mathcal{Y}$ is an Artin stack of finite type over $\mathbb{C}$, ρ is a 1-morphism, and two symbols $[\rho_i : \mathcal{Y}_i \to \mathcal{S}]$ for $i = 1, 2$ are *isomorphic* if there is a 1-isomorphism of stacks $f : \mathcal{Y}_1 \xrightarrow{\sim} \mathcal{Y}_2$ with a 2-isomorphism $\rho_2 \circ f \cong \rho_1$. For a technical reason, we assume that $\mathcal{Y}$ has affine geometric stabilizers; that is, for any $\mathbb{C}$-valued point $y \in \mathcal{Y}(\mathbb{C})$, the automorphism group $\mathrm{Aut}(k(y))$ is an affine algebraic group. The relation is generated by

$$[\rho : \mathcal{Y} \to \mathcal{S}] \sim [\rho|_{\mathcal{V}} : \mathcal{V} \to \mathcal{S}] + [\rho|_{\mathcal{U}} : \mathcal{U} \to \mathcal{S}],$$

where $\mathcal{V} \subset \mathcal{Y}$ is a closed substack and $\mathcal{U} := \mathcal{Y} \setminus \mathcal{V}$.

Let P_t be the map defined in Lemma 4.1. The following result was proved in [18, Theorem 4.10].

LEMMA 4.1

There is a map,

$$P_t : K(\mathrm{St}/\mathcal{S}) \to \mathbb{Q}(t),$$

such that we have

$$P_t\big([\rho : [Y/\mathrm{GL}_m(\mathbb{C})] \to \mathcal{S}]\big) = \frac{P_t([Y])}{P_t([\mathrm{GL}_m(\mathbb{C})])}.$$

Here Y is a quasi-projective variety on which $\mathrm{GL}_m(\mathbb{C})$ acts.

Proof

We sketch an outline of the proof. By the assumption that $\mathcal{Y}$ has affine geometric stabilizers, we can apply Kresch's result [24, Proposition 3.5.9] to show that any element $u \in K(\mathrm{St}/\mathcal{S})$ is written as a finite sum

$$\sum_{i=1}^{k} \left[\rho_i : [Y_i/\operatorname{GL}_{m_i}(\mathbb{C})] \to \mathcal{S}\right], \tag{34}$$

where Y_i is a quasi-projective variety on which $\operatorname{GL}_{m_i}(\mathbb{C})$ acts. Then we set $P_t(u)$ to be

$$P_t(u) = \sum_{i=1}^{k} \frac{P_t([Y_i])}{P_t([\operatorname{GL}_{m_i}(\mathbb{C})])}.$$

The proof given in [18, Theorem 4.10] shows that $P_t(u)$ does not depend on the expression (34). □

REMARK 4.2

More precisely, it is proved in [18, Theorem 4.10] that the map P_t in Lemma 4.1 satisfies

$$P_t\left(\left[\rho : [Y/G] \to \mathcal{S}\right]\right) = \frac{P_t([Y])}{P_t([G])}.$$

Here Y is a quasi-projective variety and G is a special algebraic group acting on Y, where an algebraic group G is called special if any principal G-bundle is Zariski locally trivial. For instance $\operatorname{GL}_m(\mathbb{C})$, $(\mathbb{C}^*)^k$ are special algebraic groups.

On the other hand, the finite group $\mathbb{Z}/k\mathbb{Z}$ is not special as $\mathbb{C}^* \ni z \mapsto z^k \in \mathbb{C}^*$ is not Zariski locally trivial. For instance, let us consider an element of the form $[\rho : [\operatorname{Spec}\mathbb{C}/G] \to \mathcal{S}]$ for $G = \mathbb{Z}/k\mathbb{Z}$. Then we have

$$[\operatorname{Spec}\mathbb{C}/G] \cong [\mathbb{C}^*/\mathbb{C}^*],$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^*$ by $g \cdot z = g^k z$. Therefore we have

$$\begin{aligned} P_t\left(\left[[\operatorname{Spec}\mathbb{C}/G] \xrightarrow{\rho} \mathcal{S}\right]\right) &= \frac{P_t(\mathbb{C}^*)}{P_t(\mathbb{C}^*)} \\ &= 1. \end{aligned}$$

We need the notions of pushforward and pullback for the groups $K(\mathrm{St}/\mathcal{S})$. Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a morphism of stacks. Then we have the pushforward,

$$f_* : K(\mathrm{St}/\mathcal{S}_1) \to K(\mathrm{St}/\mathcal{S}_2),$$

defined by

$$f_*[\rho : \mathcal{Y} \to \mathcal{S}_1] = [f \circ \rho : \mathcal{Y} \to \mathcal{S}_2].$$

Moreover, if f is of finite type, then we have the pullback,

$$f^* : K(\mathrm{St}/\mathcal{S}_2) \to K(\mathrm{St}/\mathcal{S}_1),$$

defined by

$$f^*[\rho : \mathcal{Y} \to \mathcal{S}_2] = [f^*\rho : \mathcal{Y} \times_{\mathcal{S}_2} \mathcal{S}_1 \to \mathcal{S}_1].$$

4.3. Hall algebras of coherent sheaves

For a smooth projective variety X over $\mathbb{C}$, we denote by $\mathcal{M}$ the moduli stack of coherent sheaves on X. Namely, $\mathcal{M}$ is a 2-functor,

$$\mathcal{M}: (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{groupoid}), \tag{35}$$

which sends a $\mathbb{C}$-scheme S to the groupoid whose objects consist of flat families of coherent sheaves over S,

$$\mathcal{E} \in \mathrm{Coh}(X \times S).$$

It is well known that $\mathcal{M}$ is an Artin stack which is locally of finite type over $\mathbb{C}$.

DEFINITION 4.3

We define the $\mathbb{Q}$-vector space $H(X)$ to be

$$H(X) := K(\mathrm{St}/\mathcal{M}).$$

We introduce the $*$-product on the $\mathbb{Q}$-vector space $H(X)$. Let $\mathcal{E}x$ be the 2-functor,

$$\mathcal{E}x: (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{groupoid}),$$

which sends a $\mathbb{C}$-scheme S to the groupoid whose objects consist of exact sequences in $\mathrm{Coh}(X \times S)$,

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0, \tag{36}$$

such that each $\mathcal{E}_i$ is flat over S. The stack $\mathcal{E}x$ is also an Artin stack locally of finite type over $\mathbb{C}$. There are 1-morphisms,

$$p_i : \mathcal{E}x \to \mathcal{M}, \quad i = 1, 2, 3,$$

which send an exact sequence (36) to the object $\mathcal{E}_i$. In particular we have the diagram

$$\begin{tikzcd} \mathcal{E}x \arrow[r, "p_2"] \arrow[d, "(p_1, p_3)"'] & \mathcal{M} \\ \mathcal{M} \times \mathcal{M} & \end{tikzcd}.$$

Also, we define the map

$$\iota : H(X) \otimes H(X) \to K(\mathrm{St}/\mathcal{M} \times \mathcal{M})$$

as follows:

$$\iota([\mathcal{Y}_1 \stackrel{\rho_1}{\to} \mathcal{M}] \otimes [\mathcal{Y}_1 \stackrel{\rho_1}{\to} \mathcal{M}]) = [\mathcal{Y}_1 \times \mathcal{Y}_2 \stackrel{\rho_1 \times \rho_2}{\to} \mathcal{M} \times \mathcal{M}].$$

We define the $*$-product on $H(X)$ to be

$$* = p_{2*}(p_1, p_3)^* \iota : H(X) \otimes H(X) \to H(X). \tag{37}$$

The following result was proved in [17].

THEOREM 4.4 ([17, THEOREM 5.2])

We have that $(H(X),*)$ *is an associative algebra with unit given by* $\delta_0 = [\operatorname{Spec}\mathbb{C} \xrightarrow{\rho} \mathcal{M}]$. *Here* $\rho(\cdot) = 0 \in \operatorname{Coh}(X)$.

Let us look at the $*$-product for "delta-functions," corresponding to objects E_1, $E_2 \in \operatorname{Coh}(X)$. Namely, for an object $E \in \operatorname{Coh}(X)$, we set

$$\delta_E = [\rho_E : \operatorname{Spec}\mathbb{C} \to \mathcal{M}], \quad \rho_E(\cdot) = E.$$

The $*$-product $\delta_{E_1} * \delta_{E_2}$ can be written as

$$\delta_{E_1} * \delta_{E_2} = \Big[\rho : \Big[\frac{\operatorname{Ext}^1(E_2, E_1)}{\operatorname{Hom}(E_2, E_1)}\Big] \to \mathcal{M}\Big]. \tag{38}$$

Here ρ is a map sending an element $u \in \operatorname{Ext}^1(E_2, E_1)$ to the object $E_3 \in \operatorname{Coh}(X)$, which fits into the exact sequence,

$$0 \to E_1 \to E_3 \to E_2 \to 0, \tag{39}$$

with extension class u. The vector space $\operatorname{Hom}(E_2, E_1)$ acts on $\operatorname{Ext}^1(E_2, E_1)$ trivially. In fact the $\mathbb{C}$-valued points of the fiber product,

$$(\mathcal{M} \times \mathcal{M}) \times_{(\rho_{E_1} \times \rho_{E_2})} \operatorname{Spec}\mathbb{C}, \tag{40}$$

bijectively correspond to the exact sequences (39) and hence to elements in $\operatorname{Ext}^1(E_2, E_1)$. Given such an extension, the group of the automorphisms of the stack (40) at the $\mathbb{C}$-valued point (39) is the kernel of the natural map,

$$\operatorname{Aut}(0 \to E_1 \to E_3 \to E_2 \to 0) \to \operatorname{Aut}(E_1) \times \operatorname{Aut}(E_2),$$

which is isomorphic to $\operatorname{Hom}(E_2, E_1)$. Hence we have the description (38).

4.4. Semistable one- or zero-dimensional sheaves

In this subsection, we assume that X is a smooth projective Calabi–Yau 3-fold over $\mathbb{C}$. Let ω be an $\mathbb{R}$-ample divisor on X. Recall that we constructed a stability condition Z_ω on the subcategory

$$\operatorname{Coh}_{\le 1}(X) \subset \operatorname{Coh}(X),$$

in Example 2.3. Given an element $(n, \beta) \in \mathbb{Z} \oplus H_2(X, \mathbb{Z})$, we have the substack

$$\mathcal{M}_{n,\beta}(\omega) \subset \mathcal{M}, \tag{41}$$

which parameterizes Z_ω-semistable $E \in \operatorname{Coh}_{\le 1}(X)$ satisfying

$$\big(\chi(E), [E]\big) = (n, \beta). \tag{42}$$

The substack (41) is known to be an open substack of $\mathcal{M}$, which is of finite type over $\mathbb{C}$. Furthermore, suppose that β and n are coprime and ω is in a general position in the ample cone. Then any Z_ω-semistable object $E \in \operatorname{Coh}_{\le 1}(X)$ satisfying (42) is Z_ω-stable, and the stack $\mathcal{M}_{n,\beta}(\omega)$ is a $\mathbb{C}^*$-gerbe over a projective scheme $M_{n,\beta}(\omega)$, that is,

$$\mathcal{M}_{n,\beta}(\omega) \cong [M_{n,\beta}(\omega)/\mathbb{C}^*]. \tag{43}$$

Here $\mathbb{C}^*$ acts on $M_{n,\beta}(\omega)$ trivially. The substack (41) defines the element of $H(X)$,

$$\delta_{n,\beta}(\omega) = [\mathcal{M}_{n,\beta}(\omega) \hookrightarrow \mathcal{M}] \in H(X).$$

Recall that we constructed a map,

$$P_t : H(X) \to \mathbb{Q}(t),$$

in Lemma 4.1. Applying P_t to $\delta_{n,\beta}(\omega)$, we obtain the element

$$P_t\big(\delta_{n,\beta}(\omega)\big) \in \mathbb{Q}(t),$$

which is interpreted as a *Poincaré polynomial* of the moduli stack $\mathcal{M}_{n,\beta}(\omega)$.

Suppose that $\mathcal{M}_{n,\beta}(\omega)$ is written as (43). Then we have

$$\begin{aligned} (t^2-1)P_t\big(\delta_{n,\beta}(\omega)\big) &= P_t(\mathbb{C}^*)P_t\big(\delta_{n,\beta}(\omega)\big) \\ &= P_t\big(M_{n,\beta}(\omega)\big). \end{aligned} \tag{44}$$

Hence we can substitute $t = 1$ into (44) and obtain

$$\lim_{t\to 1}(t^2-1)P_t\big(\delta_{n,\beta}(\omega)\big) = \chi\big(M_{n,\beta}(\omega)\big). \tag{45}$$

However, if n and β are not coprime, then $\mathcal{M}_{n,\beta}(\omega)$ is not necessarily written as (43). In this case, as the following example indicates, the rational function (44) may have a pole at $t = 1$, so the limit (45) does not make sense.

EXAMPLE 4.5

Let $C \cong \mathbb{P}^1 \subset X$ be a super-rigid rational curve as in Example 3.3. Then we have

$$\mathcal{M}_{0,k[C]}(\omega) \cong [\operatorname{Spec}\mathbb{C}/\operatorname{GL}_k(\mathbb{C})],$$

whose closed points correspond to $\mathcal{O}_C(-1)^{\oplus k}$. Therefore using [18, Lemma 4.6], we have

$$\begin{aligned} (t^2-1)P_t\big(\delta_{0,k[C]}(\omega)\big) &= (t^2-1)\frac{1}{P_t(\operatorname{GL}_k(\mathbb{C}))} \\ &= \frac{t^{k^2-k}}{t^2(t^4-1)\cdots(t^{2k}-1)}, \end{aligned}$$

and the limit $t \to 1$ does not exist when $k \geq 2$.

Instead, we take the 'logarithm' of $\delta_{n,\beta}(\omega)$ in $H(X)$.

DEFINITION 4.6

We define $\epsilon_{n,\beta}(\omega) \in H(X)$ to be

$$\epsilon_{n,\beta}(\omega) = \sum_{\substack{l\geq 1, n_i\in\mathbb{Z}, \beta_i\in H_2(X,\mathbb{Z}), 1\leq i\leq l, \\ n_1+\cdots+n_l=n, \beta_1+\cdots+\beta_l=\beta, \\ \arg Z_\omega(n_i,\beta_i)=\arg Z_\omega(n,\beta)}} \frac{(-1)^{l-1}}{l}\delta_{n_1,\beta_1}(\omega) * \cdots * \delta_{n_l,\beta_l}(\omega). \tag{46}$$

Namely, for each ray $l \subset \mathbb{H}$, if we set

$$\delta_l(\omega) = 1 + \sum_{Z_\omega(n,\beta)\in l} \delta_{n,\beta}(\omega),$$

$$\epsilon_l(\omega) = \sum_{Z_\omega(n,\beta)\in l} \epsilon_{n,\beta}(\omega),$$

then we have

$$\epsilon_l(\omega) = \log \delta_l(\omega).$$

It was shown in [18, Section 6.2] that the function $(t^2-1)P_t(\epsilon_{n,\beta}(\omega))$ has the limit $t \to 1$; hence we obtain the invariant

$$\widehat{N}_{n,\beta}(\omega) = \lim_{t\to 1}(t^2-1)P_t\big(\epsilon_{n,\beta}(\omega)\big) \in \mathbb{Q}.$$

The invariant $\widehat{N}_{n,\beta}(\omega)$ is interpreted as an *Euler characteristic* of the moduli stack $\mathcal{M}_{n,\beta}(\omega)$.

4.5. Invariants $N_{n,\beta}$

The invariant $\widehat{N}_{n,\beta}(\omega)$ is interpreted as an unweighted Euler characteristic of $\mathcal{M}_{n,\beta}(\omega)$, and we need to involve the Behrend function in order to construct DT-type invariants. It is easy to extend the notion of the Behrend function to the locally constructible function on the Artin stack $\mathcal{M}$,

$$\nu_{\mathcal{M}} : \mathcal{M} \to \mathbb{Z},$$

so that if $M \to \mathcal{M}$ is any atlas of relative dimension d, then $\nu_{\mathcal{M}} = (-1)^d \nu_M$ (cf. [19, Proposition 4.4]). We define the map

$$\nu\cdot : H(X) \to H(X) \tag{47}$$

by sending an element $[\rho : \mathcal{Y} \to \mathcal{M}]$ to the element

$$\sum_{i\in\mathbb{Z}} i[\rho|_{\mathcal{Y}_i} : \mathcal{Y}_i \to \mathcal{M}],$$

where $\mathcal{Y}_i = (\nu_{\mathcal{M}} \circ \rho)^{-1}(i)$.

DEFINITION 4.7

We define $N_{n,\beta}(\omega)$ to be

$$N_{n,\beta}(\omega) = \lim_{t\to 1}(t^2-1)P_t\big(-\nu \cdot \epsilon_{n,\beta}(\omega)\big) \in \mathbb{Q}.$$

Again the existence of the limit $t \to 1$ was proved in [18, Section 6.2]. A priori, the invariant $N_{n,\beta}(\omega)$ is defined after we choose a polarization ω. However, we have the following.

LEMMA 4.8

The invariant $N_{n,\beta}(\omega)$ does not depend on a choice of ω.

Proof
The result was proved in [19, Theorem 6.16]. □

In what follows, we set

$$N_{n,\beta} := N_{n,\beta}(\omega),$$

for some ample divisor ω on X.

EXAMPLE 4.9

(i) Suppose that n and β are coprime and that ω is in a general position. Then $\mathcal{M}_{n,\beta}(\omega)$ is written as (43) for a projective scheme $M_{n,\beta}(\omega)$. Let ν_M be the Behrend function on $M_{n,\beta}(\omega)$. Then we have

$$\epsilon_{n,\beta}(\omega) = \delta_{n,\beta}(\omega), \qquad \nu_{\mathcal{M}}|_{\mathcal{M}_{n,\beta}(\omega)} = -\nu_M.$$

Hence we have

$$\begin{aligned} N_{n,\beta} &= \int_{M_{n,\beta}(\omega)} \nu_M \, d\chi \\ &= \int_{[M_{n,\beta}(\omega)]^{\mathrm{vir}}} 1. \end{aligned}$$

(ii) In the situation of Example 4.5, we have

$$\delta_{0,[C]}(\omega) = \left[\frac{\operatorname{Spec}\mathbb{C}}{\mathbb{C}^*}\right], \qquad \delta_{0,2[C]}(\omega) = \left[\frac{\operatorname{Spec}\mathbb{C}}{\mathrm{GL}_2(\mathbb{C})}\right].$$

Therefore we have

$$\begin{aligned} \epsilon_{0,2[C]}(\omega) &= \delta_{0,2[C]}(\omega) - \frac{1}{2}\delta_{0,[C]}(\omega) * \delta_{0,[C]}(\omega) \\ &= \left[\frac{\operatorname{Spec}\mathbb{C}}{\mathrm{GL}_2(\mathbb{C})} \to \mathcal{M}\right] - \frac{1}{2}\left[\frac{\operatorname{Spec}\mathbb{C}}{\mathbb{C}^*} \to \mathcal{M}\right] * \left[\frac{\operatorname{Spec}\mathbb{C}}{\mathbb{C}^*} \to \mathcal{M}\right] \\ &= \left[\frac{\operatorname{Spec}\mathbb{C}}{\mathrm{GL}_2(\mathbb{C})} \to \mathcal{M}\right] - \frac{1}{2}\left[\frac{\operatorname{Spec}\mathbb{C}}{\mathbb{A}^1 \rtimes (\mathbb{C}^*)^2} \to \mathcal{M}\right]. \end{aligned}$$

The Behrend function $\nu_{\mathcal{M}}$ is 1 on $\mathcal{O}_C(-1)^{\oplus 2}$; hence we have

$$\begin{aligned} &(t^2-1)P_t\big(-\nu \cdot \epsilon_{0,2[C]}(\omega)\big) \\ &\quad = (t^2-1)\left\{-\frac{1}{t^2(t^2-1)(t^4-1)} + \frac{1}{2t^2(t^2-1)^2}\right\} \\ &\quad = \frac{1}{2t^2(t^2+1)}. \end{aligned}$$

By taking the limit $t \to 1$, we obtain $N_{0,2[C]} = 1/4$. In general, it can be proved that (cf. [19, Example 6.2])

$$N_{0,k[C]} = \frac{1}{k^2}.$$

(iii) Let us consider the case $\beta = 0$. In this case, $\mathcal{M}_{n,0}(\omega)$ is a moduli stack of length n zero-dimensional sheaves. Explicitly $\mathcal{M}_{n,0}(\omega)$ is described as follows. Let

$\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$ be the Grothendieck Quot scheme which parameterizes quotients

$$\mathcal{O}_X^{\oplus n} \twoheadrightarrow F, \tag{48}$$

with F zero-dimensional length n sheaves. The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$ via

$$g \cdot (\mathcal{O}_X^{\oplus n} \overset{s}{\twoheadrightarrow} F) = (\mathcal{O}_X^{\oplus n} \overset{s \circ g}{\twoheadrightarrow} F), \quad g \in \mathrm{GL}_n(\mathbb{C}).$$

Let

$$U^{(n)} \subset \mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$$

be the open subscheme corresponding to quotients (48) such that the induced morphism $H^0(s) : \mathbb{C}^{\oplus n} \to H^0(F)$ is an isomorphism. The $\mathrm{GL}_n(\mathbb{C})$-action on $\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$ preserves $U^{(n)}$, and the moduli stack $\mathcal{M}_{n,0}(\omega)$ is written as

$$\mathcal{M}_{n,0}(\omega) \cong [U^{(n)} / \mathrm{GL}_n(\mathbb{C})].$$

In principle, it may be possible to calculate $N_{n,0}$ using the above description of the moduli stack. (For instance, the computation in [38, Section 5] is applied for $n = 2$.) However, at this moment, a computation of $N_{n,0}$ for $n \geq 3$ is not yet done along with this argument. Instead, we can compute $N_{n,0}$ by using the wall-crossing formula and the computation of $\mathrm{DT}_0(X)$ in Example 3.5(i). The result was given in [19, Paragraph 6.3], [23, Paragraph 6.4], and [36, Remark 5.14]:

$$N_{n,0} = -\chi(X) \sum_{k|n, k \geq 1} \frac{1}{k^2}. \tag{49}$$

5. Wall-crossing in D0-D2-D6 bound states

Let X be a smooth projective Calabi–Yau 3-fold over $\mathbb{C}$. In this section, we explain how to deduce the product formula (28) by using the wall-crossing formula. In principle, the result is obtained by combining the arguments in [37], Joyce and Song's wall-crossing formula [19], and the announced result by Behrend and Getzler [6]. However, the arguments in [37] are complicated, and we simplify the arguments by using the framework of [36].

5.1. Category of D0-D2-D6 bound states

We define the category $\mathcal{A}_X$ as follows:

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X)[-1] \rangle_{\mathrm{ex}}.$$

In [36, Lemma 3.5], it was proved that $\mathcal{A}_X$ is the heart of a bounded t-structure on $\mathcal{D}_X$,

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{tr}} \subset D^b \mathrm{Coh}(X);$$

hence, in particular, $\mathcal{A}_X$ is an abelian category. The triangulated category $\mathcal{D}_X$ is called the category of *D0-D2-D6 bound states.*

The heart $\mathcal{A}_X$ has properties which are required in discussing DT/PT correspondence in Section 3.5. For instance, if we consider an ideal sheaf I_Z for a subscheme $Z \subset X$ with $\dim Z \leq 1$, we have the distinguished triangle,

$$\mathcal{O}_Z[-1] \to I_Z \to \mathcal{O}_X. \tag{50}$$

Since $\mathcal{O}_Z[-1]$ and $\mathcal{O}_X$ are objects in $\mathcal{A}_X$, it follows that $I_Z \in \mathcal{A}_X$ and the sequence (50) is an exact sequence in $\mathcal{A}_X$. Also, for a stable pair (F, s), let $I^\bullet = (\mathcal{O}_X \xrightarrow{s} F)$ be the associated two-term complex with $\mathcal{O}_X$ located in degree zero and F in degree one. Then $I^\bullet$ fits into the distinguished triangle,

$$F[-1] \to I^\bullet \to \mathcal{O}_X. \tag{51}$$

By the same argument as above, we have $I^\bullet \in \mathcal{A}_X$, and the sequence (51) is an exact sequence in $\mathcal{A}_X$. As the above argument indicates, the heart $\mathcal{A}_X$ is expected to be an important category in studying curve counting invariants on Calabi–Yau 3-folds.

5.2. Comparison with perverse coherent sheaves

In [1] and [35], the notions of *polynomial stability* and *limit stability* were introduced on the following category of *perverse coherent sheaves*,

$$\mathcal{A}^p := \langle \mathrm{Coh}_{\geq 2}(X)[1], \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{ex}}.$$

Here $\mathrm{Coh}_{\geq 2}(X)$ is the right orthogonal complement of $\mathrm{Coh}_{\leq 1}(X)$ in $\mathrm{Coh}(X)$. In this subsection, we compare $\mathcal{A}_X$ with $\mathcal{A}^p$.

Obviously we have

$$\mathcal{A}_X \subset \mathcal{A}^p[-1].$$

By [35, Lemma 2.16], there exists a torsion pair $(\mathcal{A}^p_1, \mathcal{A}^p_{1/2})$ on $\mathcal{A}^p$, defined by

$$\mathcal{A}^p_1 := \langle F[1], \mathcal{O}_x : F \text{ is pure two-dimensional}, x \in X \rangle_{\mathrm{ex}},$$
$$\mathcal{A}^p_{1/2} := \left\{ E \in \mathcal{A}^p : \mathrm{Hom}(F, E) = 0 \text{ for any } F \in \mathcal{A}^p_1 \right\}.$$

Namely, we have the following (cf. [15]).

- For any $T \in \mathcal{A}^p_1$ and $F \in \mathcal{A}^p_{1/2}$, we have $\mathrm{Hom}(T, F) = 0$.
- For any $E \in \mathcal{A}^p$, there is an exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{A}^p_1$ and $F \in \mathcal{A}^p_{1/2}$.

We set

$$\begin{aligned} \mathcal{A}_{X,1} &:= \mathcal{A}^p_1[-1] \cap \mathcal{A}_X \\ &= \langle \mathcal{O}_x[-1] : x \in X \rangle_{\mathrm{ex}} \end{aligned} \tag{52}$$

and

$$\begin{aligned} \mathcal{A}_{X,1/2} &:= \mathcal{A}^p_{1/2}[-1] \cap \mathcal{A}_X \\ &= \left\{ E \in \mathcal{A}_X : \mathrm{Hom}(\mathcal{A}_{X,1}, E) = 0 \right\}. \end{aligned} \tag{53}$$

It is easy to check that $(\mathcal{A}_{X,1}, \mathcal{A}_{X,1/2})$ is a torsion pair on $\mathcal{A}_X$, using the fact that $\mathcal{A}_X$ is Noetherian (cf. [36, Lemma 6.2]). We have the following lemma.

LEMMA 5.1
For an object $E \in \mathcal{A}^p_{1/2}[-1]$, suppose that

$$\operatorname{rank}(E) \in \{0,1\}, \quad c_1(E) = 0.$$

Then we have $E \in \mathcal{A}_{X,1/2}$.

Proof
We prove only the case of $\operatorname{rank}(E) = 1$. Take $E \in \mathcal{A}^p_{1/2}[-1]$ with $\operatorname{rank}(E) = 1$ and $c_1(E) = 0$. Then by [35, Lemma 3.2], we have the exact sequence in $\mathcal{A}^p[-1]$,

$$I_C \to E \to F[-1],$$

for some curve $C \subset X$ and $F \in \operatorname{Coh}_{\le 1}(X)$. Since $I_C, F[-1] \in \mathcal{A}_X$, we have $E \in \mathcal{A}_X$; hence $E \in \mathcal{A}_{X,1/2}$. $\square$

Below we use the following notation. For $E, F \in \mathcal{A}^p_{1/2}$, a morphism $u : E \to F$ in $\mathcal{A}^p$ is called a *strict monomorphism* if u is injective in $\mathcal{A}^p$ and $\operatorname{Cok}(u) \in \mathcal{A}^p_{1/2}$. Similarly u is called a *strict epimorphism* if u is surjective in $\mathcal{A}^p$ and $\ker(u) \in \mathcal{A}^p_{1/2}$. By replacing $(\mathcal{A}^p_i, \mathcal{A}^p)$ with $(\mathcal{A}_{X,i}, \mathcal{A}_X)$, we have the notions of strict monomorphism and strict epimorphism on $\mathcal{A}_{X,i}$.

5.3. Weak stability conditions on $\mathcal{A}_X$

In this subsection, we construct weak stability conditions on $\mathcal{A}_X$ (cf. Definition 2.3). The finitely generated free abelian group Γ is defined by

$$\begin{aligned}\Gamma &:= \mathbb{Z} \oplus H_2(X,\mathbb{Z}) \oplus \mathbb{Z} \\ &= \Gamma_0 \oplus \mathbb{Z},\end{aligned}$$

where Γ_0 is introduced in Example 2.3(iii). Below we write an element in Γ as (n, β, r) for $n \in \mathbb{Z}$, $\beta \in H_2(X,\mathbb{Z})$, and $r \in \mathbb{Z}$. For an object $E \in \mathcal{A}_X$, note that

$$\operatorname{ch}_i(E) \in H^{2i}(X,\mathbb{Z}), \tag{54}$$

since (54) is true for the generating set of objects $\mathcal{O}_X$ and $E \in \operatorname{Coh}_{\le 1}(X)[-1]$. Therefore the Chern characters define the group homomorphism,

$$\mathrm{cl} : K(\mathcal{A}_X) \to \Gamma,$$

given by

$$\mathrm{cl}(E) = \big(\operatorname{ch}_3(E), \operatorname{ch}_2(E), \operatorname{ch}_0(E)\big).$$

Here we have identified $H^0(X,\mathbb{Z})$ and $H^6(X,\mathbb{Z})$ with $\mathbb{Z}$, and $H^2(X,\mathbb{Z})$ with $H_2(X,\mathbb{Z})$ via Poincaré duality. We take the following 2-step filtration in Γ,

$$0 = \Gamma_{-1} \subsetneq \Gamma_0 \subsetneq \Gamma_1 = \Gamma,$$

where Γ_0 is given in Example 2.3, and the embedding $\Gamma_0 \hookrightarrow \Gamma$ is given by $(n,\beta) \mapsto (n,\beta,0)$. Hence each subquotient is given by

$$\Gamma_0/\Gamma_{-1} = \mathbb{Z} \oplus H_2(X,\mathbb{Z}),$$
$$\Gamma_1/\Gamma_0 = \mathbb{Z}.$$

Given the following data,

$$\omega \in H^2(X,\mathbb{Q}), \quad 0 < \theta < 1, \tag{55}$$

where ω is an ample class, we construct

$$Z_{\omega,\theta} = \{Z_{\omega,\theta,i}\}_{i=0}^{1} \in \prod_{i=0}^{1} \operatorname{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C}) \tag{56}$$

as follows:

$$Z_{\omega,\theta,0}(n,\beta) = n - (\omega \cdot \beta)\sqrt{-1},$$
$$Z_{\omega,\theta,1}(r) = r \exp(i\pi\theta).$$

Here $(n,\beta) \in \mathbb{Z} \oplus H_2(X,\mathbb{Z})$ and $r \in \mathbb{Z}$. We have the following lemma.

LEMMA 5.2

The system of group homomorphisms (56) is a weak stability condition on $\mathcal{A}_X$.

Proof

For an object $E \in \mathcal{A}_X$, let us take $i \in \{0,1\}$ so that $\mathrm{cl}(E) \in \Gamma_i \setminus \Gamma_{i-1}$ is satisfied. If $i = 1$, then

$$Z_{\omega,\theta}(E) \in \mathbb{R}_{>0} \exp(i\pi\theta) \subset \mathbb{H}.$$

Also, if $i = 0$, then $E \in \mathrm{Coh}_{\le 1}(X)[-1]$ and

$$Z_{\omega,\theta}(E) = Z_\omega(E[1]) \in \mathbb{H},$$

where Z_ω is defined in Example 2.3(iii). Therefore condition (i) in Definition 2.4 is satisfied.

We check condition (ii) in Definition 2.4. Let $(\mathcal{A}_{X,1}, \mathcal{A}_{X,1/2})$ be the torsion pair of $\mathcal{A}_X$, given by (52) and (53). For any $E \in \mathcal{A}_X$, there is an exact sequence in $\mathcal{A}_X$,

$$0 \to T \to E \to F \to 0, \tag{57}$$

with $T \in \mathcal{A}_{X,1}$ and $F \in \mathcal{A}_{X,1/2}$. By [35, Lemma 2.19], the categories $\mathcal{A}_{X,1}$ and $\mathcal{A}_{X,1/2}$ are finite length (i.e., Noetherian and Artinian with respect to strict epimorphism and strict monomorphism) quasi-abelian categories (see [9, Section 4] for the definition of quasi-abelian categories).

On the other hand, by the same argument as [35, Lemma 2.27], an object $E \in \mathcal{A}_X$ is $Z_{\omega,\theta}$-semistable if and only if one of the following conditions holds.

- We have $E \in \mathcal{A}_{X,1}$.

- We have $E \in \mathcal{A}_{X,1/2}$, and for any exact sequence

$$0 \to A \to E \to B \to 0$$

in $\mathcal{A}_X$ with $A, B \in \mathcal{A}_{X,1/2}$, we have

$$\arg Z_{\omega,\theta}(A) \le \arg Z_{\omega,\theta}(B). \tag{58}$$

Then for any $E \in \mathcal{A}_X$, its Harder–Narasimhan filtration is obtained by combining the sequence (57) and the Harder–Narasimhan filtration of F, where F is given by the sequence (57). The existence of the latter filtration is ensured by the fact that $Z_{\omega,\theta}$-semistable objects in $\mathcal{A}_{X,1/2}$ are characterized by the inequality (58) for exact sequences in $\mathcal{A}_{X,1/2}$, and $\mathcal{A}_{X,1/2}$ is of finite length (see the proof of [35, Theorem 2.29]). □

We remark that the abelian category $\mathcal{A}_X$ contains the subcategory

$$\mathrm{Coh}_{\le 1}(X)[-1] \subset \mathcal{A}_X,$$

which is closed under subobjects and quotients. Hence for $F \in \mathrm{Coh}_{\le 1}(X)$, the object $F[-1] \in \mathcal{A}_X$ is $Z_{\omega,\theta}$-(semi)stable if and only if F is Z_ω-(semi)stable in the sense of Example 2.3(iii).

Let

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$$

be the space of weak stability conditions on $\mathcal{D}_X$, as in (7). It is straightforward to check that the pairs $(Z_{\omega,\theta}, \mathcal{A}_X)$ satisfy the conditions required to construct the space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$, that is, local finiteness and the support property in [36, Section 2]. Therefore by applying [36, Lemma 7.1], we have the continuous morphism for a fixed ω,

$$(0,1) \ni \theta \mapsto (Z_{\omega,\theta}, \mathcal{A}_X) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X). \tag{59}$$

5.4. Comparison with μ-limit stability

Let us take

$$B + i\omega \in H^2(X, \mathbb{C})$$

with ω ample. Below we set $B = k\omega$ for $k \in \mathbb{R}$. In [37], the author introduced the notion of $\mu_{B+i\omega}$-*limit stability* on the abelian category $\mathcal{A}^p$. Suppose that an object $E \in \mathcal{A}^p[-1]$ satisfies

$$\mathrm{ch}(E) = (1, 0, -\beta, -n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6. \tag{60}$$

Then by [37, Lemma 3.8] and [37, Proposition 3.13], an object $E[1] \in \mathcal{A}^p$ is $\mu_{B+i\omega}$-limit semistable if and only if $E \in \mathcal{A}^p_{1/2}$ and the following conditions are satisfied.

- For any pure one-dimensional sheaf $0 \ne F$ which admits a strict monomorphism $F \hookrightarrow E[1]$ in $\mathcal{A}^p_{1/2}$, we have $\mathrm{ch}_3(F)/\omega\,\mathrm{ch}_2(F) \le -2k$.

• For any pure one-dimensional sheaf $0 \neq G$ which admits a strict epimorphism $E[1] \twoheadrightarrow G$ in $\mathcal{A}^p_{1/2}$, we have $\mathrm{ch}_3(G)/\omega\,\mathrm{ch}_2(G) \geq -2k$.

Now we set

$$k = \frac{1}{2\tan \pi\theta}. \tag{61}$$

Here $k = 0$ if $\theta = 1/2$. By Lemma 5.1 and the arguments in the proof of Lemma 5.2, the following lemma obviously follows.

LEMMA 5.3

Take k and θ satisfying (61). Then for an object $E \in \mathcal{A}^p[-1]$ satisfying (60), $E[1] \in \mathcal{A}^p$ is $\mu_{k\omega+i\omega}$-limit semistable in the sense of [37, Section 3] if and only if $E \in \mathcal{A}_X$ and E is $Z_{\omega,\theta}$-semistable satisfying

$$\mathrm{cl}(E) = (-n, -\beta, 1) \in \Gamma.$$

5.5. Moduli stacks of semistable objects

In this subsection, we discuss moduli stacks of semistable objects in $\mathcal{A}_X$. We denote by $\widehat{\mathcal{M}}$ the 2-functor

$$\widehat{\mathcal{M}} : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{groupoid}),$$

which sends a $\mathbb{C}$-scheme S to the groupoid whose objects consist of objects

$$\mathcal{E} \in D\big(\mathrm{Coh}(X \times S)\big)$$

such that

• the object $\mathcal{E}$ is relatively perfect over S (see [29, Definition 2.1.1]); in particular for each $s \in S$, we have the derived pullback

$$\mathcal{E}_s := \mathbf{L}i_s^* \mathcal{E} \in D^b \,\mathrm{Coh}(X); \tag{62}$$

here $i_s : X \times \{s\} \hookrightarrow X \times S$ is the inclusion;

• the object (62) satisfies

$$\mathrm{Ext}^i(\mathcal{E}_s, \mathcal{E}_s) = 0, \quad i < 0,$$

for any $s \in S$.

By the result of Lieblich [29], the 2-functor $\widehat{\mathcal{M}}$ is an Artin stack locally of finite type over $\mathbb{C}$. We note that the stack $\mathcal{M}$ considered in (35) is an open substack of $\widehat{\mathcal{M}}$.

Let $\mathcal{O}\mathrm{bj}(\mathcal{A}_X)$ be the (abstract) substack

$$\mathcal{O}\mathrm{bj}(\mathcal{A}_X) \subset \widehat{\mathcal{M}},$$

whose S-valued points consist of $\mathcal{E} \in \widehat{\mathcal{M}}(S)$ satisfying $\mathcal{E}_s \in \mathcal{A}_X$ for all $s \in S$. The stack $\mathcal{O}\mathrm{bj}(\mathcal{A}_X)$ decomposes as

$$\mathcal{O}\mathrm{bj}(\mathcal{A}_X) = \coprod_{v \in \Gamma} \mathcal{O}\mathrm{bj}^v(\mathcal{A}_X),$$

where $\mathcal{O}\mathrm{bj}^v(\mathcal{A}_X)$ is the stack of objects $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = v$. As proved in [36, Lemma 3.16], the embedding

$$\mathcal{O}\mathrm{bj}^v(\mathcal{A}_X) \subset \widehat{\mathcal{M}}$$

is an open immersion if $v = (n, \beta, r) \in \Gamma$ with $r = 0$ or $r = 1$. In particular in that case, $\mathcal{O}\mathrm{bj}^v(\mathcal{A}_X)$ is an Artin stack locally of finite type over $\mathbb{C}$. In general, $\mathcal{O}\mathrm{bj}^v(\mathcal{A}_X)$ is at least a locally constructible subset of $\widehat{\mathcal{M}}$.

Let ω and θ be as in (55). We define

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \subset \mathcal{O}\mathrm{bj}^{(-n,-\beta,1)}(\mathcal{A}_X)$$

to be the stack which parameterizes $Z_{\omega,\theta}$-semistable objects $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = (-n, -\beta, 1)$. We have the following proposition.

PROPOSITION 5.4

(i) *The stack $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$ is an Artin stack of finite type over $\mathbb{C}$.*

(ii) *If θ is sufficiently close to 1, then we have*

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \cong [P_n(X, \beta)/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts on $P_n(X, \beta)$ trivially.

(iii) *We have the isomorphism*

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta) \xrightarrow{\cong} \widehat{\mathcal{M}}_{-n,\beta}(\omega, 1 - \theta),$$

given by

$$E \mapsto \mathbf{R}\mathcal{H}\mathrm{om}(E, \mathcal{O}_X).$$

(iv) *We have*

$$\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta = 1/2) = \emptyset,$$

for $|n| \gg 0$.

Proof

By Lemma 5.3, the stack $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$ is identified with the moduli stack of $\mu_{k\omega+i\omega}$-limit semistable objects $E \in \mathcal{A}^p_{1/2}$ satisfying (60), where k is given by (61). The results of the proposition follow from the corresponding results for $\mu_{k\omega+i\omega}$-limit stability. Namely, (i) follows from [37, Proposition 3.17], (ii) follows from [37, Theorem 3.21], (iii) follows from [35, Lemma 2.28], and (iv) follows from [37, Lemma 4.4]. $\square$

5.6. Rank-one counting invariants

Using the moduli stack $\widehat{\mathcal{M}}_{n,\beta}(\omega, \theta)$, we are able to construct the invariant

$$\mathrm{DT}_{n,\beta}(\omega, \theta) \in \mathbb{Q},$$

which counts $Z_{\omega,\theta}$-semistable $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = (-n, -\beta, 1)$. Namely, suppose that any $Z_{\omega,\theta}$-semistable object $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = (-n, -\beta, 1)$ is $Z_{\omega,\theta}$-stable.

(This is true if ω and θ are chosen to be generic.) Then we have

$$\widehat{\mathcal{M}}_{n,\beta}(\omega,\theta) \cong [\widehat{M}_{n,\beta}(\omega,\theta)/\mathbb{G}_m] \tag{63}$$

for an algebraic space $\widehat{M}_{n,\beta}(\omega,\theta)$ of finite type over $\mathbb{C}$. If ν_M is the Behrend function on $\widehat{M}_{n,\beta}(\omega,\theta)$, then we can define

$$\mathrm{DT}_{n,\beta}(\omega,\theta) = \int_{\widehat{M}_{n,\beta}(\omega,\theta)} \nu_M \, d\chi.$$

On the other hand, suppose that there is a strictly $Z_{\omega,\theta}$-semistable object $E \in \mathcal{A}_X$ satisfying $\mathrm{cl}(E) = (-n,-\beta,1)$. Then the stack $\widehat{\mathcal{M}}_{n,\beta}(\omega,\theta)$ is not written in the same way as in (63), and we need to modify the definition of $\mathrm{DT}_{n,\beta}(\omega,\theta)$ using the Hall-type algebra as we discuss in Section 4. Namely, we consider

$$H(\mathcal{A}_X) := K_0\big(\mathrm{St}/\mathcal{O}\mathrm{bj}(\mathcal{A}_X)\big),$$

and the $*$-product on $H(\mathcal{A}_X)$ given in a similar way to (37), by replacing $\widehat{\mathcal{M}}$ with $\mathcal{O}\mathrm{bj}(\mathcal{A}_X)$. By Proposition 5.4, we can define the elements in $\mathcal{H}(\mathcal{A}_X)$,

$$\widehat{\delta}_{n,\beta}(\omega) = [\mathcal{M}_{n,\beta}(\omega) \stackrel{i}{\hookrightarrow} \mathcal{O}\mathrm{bj}(\mathcal{A}_X)],$$
$$\widehat{\delta}_{n,\beta}(\omega,\theta) = [\widehat{\mathcal{M}}_{n,\beta}(\omega,\theta) \hookrightarrow \mathcal{O}\mathrm{bj}(\mathcal{A}_X)],$$

where $\mathcal{M}_{n,\beta}(\omega)$ is the stack introduced in (41), and i sends $E \in \mathrm{Coh}_{\le 1}(X)$ to $E[-1] \in \mathcal{A}_X$. Its *logarithm* is defined by

$$\widehat{\epsilon}_{n,\beta}(\omega,\theta) = \sum_{\substack{l\ge 1, 1\le e\le l, (n_i,\beta_i)\in \mathbb{Z}\oplus H_2(X,\mathbb{Z}),\\ n_1+\cdots+n_l=n, \beta_1+\cdots+\beta_l=\beta\\ Z_{\omega,\theta}(-n_i,-\beta_i,0)\in \mathbb{R}_{>0}\exp(i\pi\theta), i\neq e}} \frac{(-1)^{l-1}}{l} \widehat{\delta}_{n_1,\beta_1}(\omega) * \cdots * \widehat{\delta}_{n_{e-1},\beta_{e-1}}(\omega)$$
$$* \widehat{\delta}_{n_e,\beta_e}(\omega,\theta) * \widehat{\delta}_{n_{e+1},\beta_{e+1}}(\omega) * \cdots * \widehat{\delta}_{n_l,\beta_l}(\omega).$$

Then $\mathrm{DT}_{n,\beta}(\omega,\theta) \in \mathbb{Q}$ can be defined by

$$\mathrm{DT}_{n,\beta}(\omega,\theta) = \lim_{t\to 1}(t^2-1)P_t\big(-\nu \cdot \epsilon_{n,\beta}(\omega,\theta)\big),$$

where ν is defined similarly to (47) by using the Behrend function on $\mathcal{O}\mathrm{bj}(\mathcal{A}_X)$ (see also [37, Definition 4.1], [36, Definition 4.11]). We define the invariant $L_{n,\beta} \in \mathbb{Q}$ as follows.

DEFINITION 5.5

We define $L_{n,\beta} \in \mathbb{Q}$ to be

$$L_{n,\beta} := \mathrm{DT}_{n,\beta}\Big(\omega, \theta = \frac{1}{2}\Big).$$

As a corollary of Proposition 5.4, we have the following.

COROLLARY 5.6

(i) *If θ is sufficiently close to* 1, *we have*

$$\mathrm{DT}_{n,\beta}(\omega,\theta) = P_{n,\beta}.$$

(ii) *The invariant* $L_{n,\beta}$ *satisfies*

$$L_{n,\beta} = L_{-n,\beta},$$

and they are zero for $|n| \gg 0$.

5.7. Wall-crossing formula

We define the series $\mathrm{DT}(\omega,\theta)$ by

$$\mathrm{DT}(\omega,\theta) := \sum_{n,\beta} \mathrm{DT}_{n,\beta}(\omega,\theta) q^n t^\beta. \tag{64}$$

Similarly to [36, Definition 4.11] and [39, Section 4.3], the series (64) can be defined in a certain topological vector space for $0 < \theta < 1/2$. Also, as in [36, Section 5.1], it is straightforward to check the existence of wall and chamber structure on the space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$. Therefore the following limiting series makes sense for $\phi \in (0, 1/2)$,

$$\mathrm{DT}(\omega,\phi_\pm) := \lim_{\theta \to \phi \pm 0} \mathrm{DT}(\omega,\theta).$$

Using Joyce and Song's wall-crossing formula [19] and assuming the result by Behrend and Getzler [6],* we have the following theorem (see also Remark 3.12, [39, Remark 2.32, Conjecture 4.3]).

THEOREM 5.7

For $0 < \phi < 1/2$, *we have the following formula,*

$$\mathrm{DT}(\omega,\phi_+) = \mathrm{DT}(\omega,\phi_-) \cdot \prod_{\substack{n>0,\beta>0 \\ -n+(\omega\cdot\beta)i \in \mathbb{R}_{>0} e^{i\pi\phi}}} \exp\big((-1)^{n-1} n N_{n,\beta} q^n t^\beta\big). \tag{65}$$

Proof

Let us fix ω and consider the subset

$$\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X),$$

defined by the image of the map (59). Then it is easy to check that the subspace $\mathcal{V}$ satisfies the assumptions of [36, Assumption 4.1]. Therefore the result follows from [36, Theorems 5.8, 8.10 (arXiv version)]. $\square$

As a corollary of the above theorem, we obtain the desired product expansion (28).

COROLLARY 5.8

We have the formula

$$\mathrm{PT}(X) = \prod_{n>0,\beta>0} \exp\big((-1)^{n-1} n N_{n,\beta} q^n t^\beta\big) \Big(\sum_{n,\beta} L_{n,\beta} q^n t^\beta\Big). \tag{66}$$

*The result of [6] is not yet written at the moment the author writes this manuscript.

Proof
By Corollary 5.6, we have

$$\lim_{\theta\to 1} \mathrm{DT}(\omega,\theta) = \mathrm{PT}(X).$$

On the other hand, note that if $F \in \mathrm{Coh}_{\le 1}(X)$ satisfies

$$Z_{\omega,1/2}(F[-1]) \in \mathbb{R}_{>0}\sqrt{-1},$$

then $\chi(F) = 0$. Using this fact and following the argument of [36, Theorems 5.8, 8.10], it can be checked that

$$\begin{aligned}\lim_{\theta\to 1/2} \mathrm{DT}(\omega,\theta) &= \mathrm{DT}(\omega,\theta = 1/2)\\ &= \sum_{n,\beta} L_{n,\beta} q^n t^\beta.\end{aligned}$$

Therefore applying the wall-crossing formula (65) from $\theta = 1/2$ to $\theta \to 1$, we obtain the formula (66) (see [36, Corollary 5.11] to justify this argument). □

6. Product expansion formula

In this section, we discuss a conjectural product expansion formula of the series $\mathrm{PT}(X)$ and see how it is related to our formula (66). It leads to a conjectural multicovering formula of the invariant $N_{n,\beta}$, and we give the evidence for it in a specific example.

6.1. Gopakumar–Vafa formula

For $g \ge 0$ and $\beta \in H_2(X,\mathbb{Z})$, the GW invariant $N_{g,\beta}^{\mathrm{GW}} \in \mathbb{Q}$ is not an integer in general. However, Gopakumar and Vafa [13] claimed the following integrality of $N_{g,\beta}^{\mathrm{GW}}$, based on the string duality between Type IIA string theory and M-theory.

CONJECTURE 6.1
There are integers

$$n_g^\beta \in \mathbb{Z} \quad \textit{for } g \ge 0, \qquad \beta \in H_2(X,\mathbb{Z})$$

such that we have

$$\sum_{g\ge 0,\beta>0} N_{g,\beta}^{\mathrm{GW}} \lambda^{2g-2} t^\beta = \sum_{g\ge 0,\beta>0,k\in\mathbb{Z}_{\ge 1}} \frac{n_g^\beta}{k} \left(2\sin\left(\frac{k\lambda}{2}\right)^{2g-2}\right) t^{k\beta}. \tag{67}$$

The invariant $n_g^\beta \in \mathbb{Z}$ is called a *Gopakumar–Vafa invariant*. The left-hand side of (67) can always be written as in the right-hand side of (67) for some $n_g^\beta \in \mathbb{Q}$, but the integrality of n_g^β is not obvious. The above conjecture is implied by GW/DT/PT correspondence, noting that DT or PT invariants are integers (cf. [32, Theorem 3.19]).

Now let us believe GW/DT/PT correspondence and write the GW generating series in the Gopakumar–Vafa form (67). Then the series $\mathrm{PT}(X)$ should be written as a certain conjectural formula involving n_g^β. The expected formula was formulated in [20].

CONJECTURE 6.2
There are integers

$$n_g^\beta \in \mathbb{Z} \quad \text{for } g \geq 0, \qquad \beta \in H_2(X, \mathbb{Z})$$

such that we have

$$\begin{aligned}\mathrm{PT}(X) = \prod_{\beta>0} \Bigg(& \prod_{j=1}^{\infty} \big(1-(-q)^j t^\beta\big)^{j n_0^\beta} \\ & \times \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} \big(1-(-q)^{g-1-k} t^\beta\big)^{(-1)^{k+g} n_g^\beta \binom{2g-2}{k}} \Bigg).\end{aligned} \tag{68}$$

The above conjecture is nothing but the strong rationality conjecture discussed in [32]. In what follows we discuss the relationship between formulas (66) and (68).

6.2. Multicovering formula of $N_{n,\beta}$

First let us take the logarithm of the right-hand side of (68). Then we obtain

$$\log \prod_{\beta>0} \prod_{j=1}^{\infty} \big(1-(-q)^j t^\beta\big)^{j n_0^\beta} \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} \big(1-(-q)^{g-1-k} t^\beta\big)^{(-1)^{k+g} n_g^\beta \binom{2g-2}{k}}$$

$$\begin{aligned} &= \sum_{\beta>0} \sum_{j=1}^{\infty} j n_0^\beta \log\big(1-(-q)^j t^\beta\big) \\ &\quad + \sum_{\beta>0} \sum_{g=1}^{\infty} \sum_{k=0}^{2g-2} (-1)^{k+g} n_g^\beta \binom{2g-2}{k} \log\big(1-(-q)^{g-1-k} t^\beta\big) \end{aligned} \tag{69}$$

$$\begin{aligned} &= \sum_{\beta>0} \sum_{j=1}^{\infty} j n_0^\beta \sum_{k\geq 1} \frac{(-1)^{jk-1} q^{jk}}{k} t^{k\beta} \\ &\quad + \sum_{\beta>0} \sum_{g=1}^{\infty} \sum_{a\geq 1} \frac{n_g^\beta}{a} \sum_{k=0}^{2g-2} \binom{2g-2}{k} \big\{-(-q)^a\big\}^{g-1-k} t^{a\beta}. \end{aligned} \tag{70}$$

The first term of (70) is written as

$$\sum_{\beta>0} \sum_{n=1}^{\infty} \sum_{k\geq 1, k|(\beta,n)} \frac{(-1)^{n-1} n}{k^2} n_0^{\beta/k} q^n t^\beta, \tag{71}$$

and the coefficient of t^β is an element of $q\mathbb{Q}[\![q]\!]$. As for the second term of (70), we set

$$\begin{aligned} f_g(q) &:= \sum_{k=0}^{2g-2} \binom{2g-2}{k} q^{g-1-k} \\ &= q^{1-g}(1+q)^{2g-2}. \end{aligned} \tag{72}$$

Then the second term of (70) is written as

$$\sum_{\beta>0}\sum_{g=1}^{\infty}\sum_{a\geq 1, a|\beta}\frac{n_g^{\beta/a}}{a}f_g\big(-(-q)^a\big)t^{\beta}. \tag{73}$$

Note that the coefficient of t^{β} in (73) is a polynomial of $q^{\pm 1}$ invariant under $q \leftrightarrow 1/q$.

Next taking the logarithm of (66), we obtain

$$\log \mathrm{PT}(X) = \sum_{\beta>0}\sum_{n>0}(-1)^{n-1}nN_{n,\beta}q^n t^{\beta} + \log\Big(\sum_{n,\beta}L_{n,\beta}q^n t^{\beta}\Big). \tag{74}$$

The coefficient of t^{β} in the first term of the right-hand side of (74) is an element of $q\mathbb{Q}[\![q]\!]$. We set

$$\sum_{\beta>0}L_{\beta}(q)t^{\beta} := \log\Big(\sum_{n,\beta}L_{n,\beta}q^n t^{\beta}\Big). \tag{75}$$

Then $L_{\beta}(q)$ is a polynomial of $q^{\pm 1}$ which is invariant under $q \leftrightarrow 1/q$.

For a Laurent series $F(q)$ in q, note that the decomposition

$$F(q) = F_1(q) + F_2(q),$$
$$F_1(q) \in q\mathbb{Q}[\![q]\!], \qquad F_2(q) \in \mathbb{C}[q^{\pm}1],$$

is unique if $F_2(q)$ is invariant under $q \leftrightarrow 1/q$. Hence if Conjecture 6.2 holds, the comparison of (70) with (74) gives

$$\sum_{n>0}(-1)^{n-1}nN_{n,\beta}q^n = \sum_{n=1}^{\infty}\sum_{k\geq 1, k|(\beta,n)}\frac{(-1)^{n-1}n}{k^2}n_0^{\beta/k}q^n, \tag{76}$$

$$L_{\beta}(q) = \sum_{g=1}^{\infty}\sum_{a\geq 1, a|\beta}\frac{n_g^{\beta/a}}{a}f_g\big(-(-q)^a\big). \tag{77}$$

By looking at the coefficient of q in (76), we obtain

$$N_{1,\beta} = n_{0,\beta}.$$

Then by looking at the coefficient of q^n, we obtain the following conjectural formula.

CONJECTURE 6.3

We have the following formula,

$$N_{n,\beta} = \sum_{k\geq 1, k|(n,\beta)}\frac{1}{k^2}N_{1,\beta/k}. \tag{78}$$

By the above argument, if Conjecture 6.3 is true, then $n_0^{\beta} = N_{1,\beta}$ satisfies the equation (76). Note that $N_{1,\beta}$ is an integer since the vector $(1,\beta)$ is primitive.

Also the equation (77) gives us a way to write down n_g^{β} for $g \geq 1$ in terms of $L_{n,\beta}$. Namely, if $G(q) \in \mathbb{Q}[q^{\pm 1}]$ is invariant under $q \leftrightarrow 1/q$, then there is a unique way to write $G(q)$ as

$$G(q) = \sum_{g=1}^{N} a_g f_g(q),$$

with $a_g \in \mathbb{Q}$. Hence we are able to write down n_g^{β} in terms of $L_{n,\beta}$ using the equation (77) recursively. For instance, as we show in Theorem 6.6, we have

$$n_1^{\beta} = \sum_{n} (-1)^n L_{n,\beta} - \frac{1}{2} \sum_{n_1, n_2} \sum_{\beta_1+\beta_2=\beta} (-1)^{n_1+n_2} L_{n_1,\beta_1} L_{n_2,\beta_2} + \cdots \tag{79}$$

if β is a primitive curve class. The integrality of n_g^{β} for $g \geq 1$ is not obvious from the expression of n_g^{β} in terms of $L_{n,\beta}$, as in (79). However by [32, Theorem 3.19], if $\mathrm{PT}(X)$ is once written as a product expansion (68), then the integrality of n_g^{β} follows from the integrality of $P_{n,\beta} \in \mathbb{Z}$. As a summary, we obtain the following.

THEOREM 6.4

Conjecture 6.2 is equivalent to Conjecture 6.3. In that case, we have

$$n_0^{\beta} = N_{1,\beta},$$

and there is a way to write down n_g^{β} for $g \geq 1$ in terms of $L_{n,\beta}$.

REMARK 6.5

The invariant $N_{1,\beta}$ is nothing but Katz's definition of the genus-zero Gopakumar–Vafa invariant [21].

6.3. Higher genus Gopakumar–Vafa invariants

As we observe in Theorem 6.4, if we assume Conjecture 6.2, then n_g^{β} is written in terms of $L_{n,\beta}$. The purpose of this subsection is to give its explicit formula.

For $m \geq 0$, we set $h_m(q)$ by

$$h_m(q) = \begin{cases} 1, & m = 0, \\ q^m + q^{-m}, & m \geq 1. \end{cases}$$

Let $f_g(q)$ be the function defined by (72). Then for $g \geq 1$, we have

$$f_g(q) = \sum_{m=0}^{g-1} \binom{2g-2}{g-1+m} h_m(q). \tag{80}$$

There is an inversion formula of (80). Namely, there are $c_g^{(m)} \in \mathbb{Z}$ such that

$$h_m(q) = \sum_{g=1}^{m+1} c_g^{(m)} f_g(q). \tag{81}$$

An elementary calculation shows that $c_g^{(m)}$ is given by

$$c_g^{(m)} = (-1)^{m+g-1} \left\{ \binom{m+g}{2g-1} - \binom{m+g-2}{2g-1} \right\}. \tag{82}$$

The Möbius function on $\mathbb{Z}_{\geq 1}$ is defined as follows:

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is square free,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\omega(n)$ is the number of distinct prime factors of n. Then by (77) and the Möbius inversion formula, we have

$$\sum_{g \geq 1} n_g^\beta f_g(q) = \sum_{a \geq 1, a|\beta} \frac{\mu(a)}{a} L_{\beta/a}\big(-(-q)^a\big). \tag{83}$$

If we write

$$L_\beta(q) = \sum_{n,\beta} L'_{n,\beta} q^n, \tag{84}$$

for $L'_{n,\beta} \in \mathbb{Q}$, then we have

$$\begin{aligned} (83) &= \sum_{a \geq 1, a|\beta} \frac{\mu(a)}{a} \sum_{n \in \mathbb{Z}} L'_{n,\beta/a} (-1)^{na+n} q^{na} \\ &= \sum_{a \geq 1, a|\beta} \frac{\mu(a)}{a} \sum_{n \geq 0} (-1)^{na+n} L'_{n,\beta/a} h_{na} \\ &= \sum_{n \geq 0} \sum_{a \geq 1, a|(n,\beta)} \frac{\mu(a)}{a} (-1)^{n+n/a} L'_{n/a,\beta/a} h_n \\ &= \sum_{g \geq 1} \Big(\sum_{n \geq g-1} \sum_{a \geq 1, a|(n,\beta)} \frac{\mu(a)}{a} (-1)^{n+n/a} L'_{n/a,\beta/a} c_g^{(n)} \Big) f_g(q). \end{aligned}$$

Here we have used (81) for the last equality. On the other hand, comparing (75) with (84), we have

$$L'_{n,\beta} = \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{n_1 + \cdots + n_l = n, \\ \beta_1 + \cdots + \beta_l = \beta}} \prod_{i=1}^{l} L_{n_i,\beta_i}.$$

Also using the formula (82) for $c_g^{(n)}$, we obtain the following result.

THEOREM 6.6

Suppose that Conjecture 6.2 is true. Then $n_0^\beta = N_{1,\beta}$ and n_g^β for $g \geq 1$ is given by

$$n_g^\beta = \sum_{\substack{n \geq g-1, \\ a \geq 1, a|(n,\beta)}} \sum_{\substack{l \geq 1, \\ n_1 + \cdots + n_l = n/a, \\ \beta_1 + \cdots + \beta_l = \beta/a}} \frac{\mu(a)}{al} (-1)^{l+g+n/a} \cdot \left\{ \binom{n+g}{2g-1} - \binom{n+g-2}{2g-1} \right\} \prod_{i=1}^{l} L_{n_i,\beta_i}.$$

6.4. Example: Weierstrass model

We prove Conjecture 6.2 and compute $n_{g,\beta}$ in the following specific example. Let S be a smooth projective del Pezzo surface over $\mathbb{C}$. Take general elements

$$f \in \Gamma\big(S, \mathcal{O}_S(-4K_S)\big), \qquad g \in \Gamma\big(S, \mathcal{O}_S(-6K_S)\big).$$

We construct a Calabi–Yau 3-fold with an elliptic fibration,

$$\pi : X \to S,$$

by the defining equation

$$y^2 = x^3 + fx + g,$$

in the projective bundle,

$$\mathcal{P}\mathrm{roj}\,\mathrm{Sym}^\bullet_S\big(\mathcal{O}_S \oplus \mathcal{O}_S(-2K_S) \oplus \mathcal{O}_S(-3K_S)\big) \to S.$$

Here x and y are local sections of $\mathcal{O}_S(-2K_S)$ and $\mathcal{O}_S(-3K_S)$, respectively. A Calabi–Yau 3-fold X constructed in this way is called a *Weierstrass model*. A general fiber of $\pi : X \to S$ is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve.

Let $F \subset X$ be a general fiber of π. We study the following series,

$$\mathrm{PT}(X/S) := \sum_{n,m} \mathrm{PT}_{n,m[F]}\, q^n t^m.$$

By the formula (66), we have the product expansion formula,

$$\mathrm{PT}(X/S) = \prod_{n>0,m>0} \exp\big((-1)^{n-1} n N_{n,m[F]} q^n t^m\big)\Big(\sum_{n,m} L_{n,m[F]} q^n t^m\Big). \tag{85}$$

In what follows, we omit $[F]$ in the notation for simplicity. So for instance, we write $N_{n,m[F]}$ as $N_{n,m}$.

PROPOSITION 6.7

The invariant $N_{n,m}$ satisfies the formula (78), and

$$N_{1,m} = -\chi(X).$$

Proof

Let ω_X be an ample divisor on X. Let

$$\mathcal{M}^s_{n,m}(\omega_X) \subset \mathcal{M}_{n,m}(\omega_X)$$

be the substack corresponding to Z_{ω_X}-stable objects in $\mathrm{Coh}_{\le 1}(X)$, introduced in Example 2.3(iii). Note that if $E \in \mathrm{Coh}_{\le 1}(X)$ represents a closed point of $\mathcal{M}^s_{n,m}(\omega_X)$, then E is written as

$$E \cong i_{p*}E', \tag{86}$$

for some stable sheaf E' on an elliptic fiber $\pi^{-1}(p)$ for some $p \in S$. Here $i_p : \pi^{-1}(p) \hookrightarrow X$ is the inclusion. By the classification of stable sheaves on the fibers of π given in [8], we have

$$\mathcal{M}^s_{n,m}(\omega_X) = \emptyset \quad \text{if } \gcd(n,m) > 1. \tag{87}$$

Assume that $\gcd(n,m)=1$. Let

$$Y \to S$$

be the relative moduli space of Z_{ω_X}-stable sheaves E on the fibers of $\pi : X \to S$ satisfying

$$[E]=m[F], \quad \chi(E)=n. \tag{88}$$

By the condition $\gcd(n,m)=1$ and the result of [11], the variety Y is smooth projective, irreducible, and there is a derived equivalence,

$$\Phi : D^b \operatorname{Coh}(X) \xrightarrow{\sim} D^b \operatorname{Coh}(Y), \tag{89}$$

which takes any Z_{ω_X}-stable sheaf satisfying (88) to an object of the form $\mathcal{O}_y$ for a closed point $y \in Y$. For $d \in \mathbb{Z}_{\geq 1}$, take a $\mathbb{C}$-valued point

$$[E] \in \mathcal{M}_{(dn,dm)}(\omega_X).$$

By (87), any Jordan–Hölder factor of E determines a closed point in $\mathcal{M}_{n,m}(\omega_X)$. Hence the equivalence Φ induces the isomorphism

$$\mathcal{M}_{(dn,dm)}(\omega_X) \xrightarrow{\sim} \mathcal{M}_{(d,0)}(\omega_Y).$$

Here ω_Y is an arbitrary polarization on Y. (Obviously the right-hand side does not depend on ω_Y.) Therefore we obtain

$$\begin{aligned} N_{dn,dm}(\omega_X) &= N_{d,0}(\omega_Y) \\ &= -\chi(Y) \sum_{k\geq 1, k|d} \frac{1}{k^2} \\ &= -\chi(X) \sum_{k\geq 1, k|d} \frac{1}{k^2}. \end{aligned}$$

Here the second equality follows from (49), and the last equality follows from the derived equivalence (89). Therefore we obtain the desired result. $\square$

Next we compute the invariants $L_{n,m}$.

PROPOSITION 6.8

We have $L_{n,m}=0$ for $n \neq 0$, and

$$L_{0,m}=\chi\big(\operatorname{Hilb}_m(S)\big).$$

Here $\operatorname{Hilb}_m(S)$ is the Hilbert scheme of m-points in S.

Proof

Let us take an ample divisor ω on X and a stable pair

$$s : \mathcal{O}_X \to E, \tag{90}$$

with E supported on fibers of π. By taking the Harder–Narasimhan filtration and Jordan–Hölder filtration with respect to Z_ω-stability (cf. Example 2.3(iii)), we can take a filtration of E,

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_N = E,$$

such that each $F_i = E_i/E_{i-1}$ is Z_ω-stable with

$$\arg Z_\omega(F_i) \geq \arg Z_\omega(F_{i+1}), \tag{91}$$

for all i. Note that each F_j is written as $i_{p*}F'_j$ for a stable sheaf F'_j on $\pi^{-1}(p)$ as in (86). Also the composition,

$$\mathcal{O}_X \xrightarrow{s} E \to E/E_{N-1} = F_N,$$

should be nonzero since s is surjective in dimension one. Therefore

$$\mathrm{Hom}_X(\mathcal{O}_X, F_N) \cong \mathrm{Hom}_{X_p}(\mathcal{O}_{X_p}, F_N) \neq 0,$$

which implies that

$$\arg Z_\omega(F_N) \geq \arg Z_\omega(\mathcal{O}_{X_p}) = \frac{\pi}{2}.$$

Combined with the inequality (91), we conclude that $\chi(E) \geq 0$.

The above argument shows that $P_n(X,m)$ is empty for $n < 0$; hence $P_{n,m} = 0$ for $n < 0$. By the formula (85) and the symmetry $L_{n,m} = L_{-n,m}$, we conclude that

$$L_{n,m} = 0 \quad \text{if } n \neq 0.$$

Let us compute $L_{0,m}$. By substituting $q = 0$ in the formula (85), we have

$$L_{0,m} = P_{0,m}. \tag{92}$$

Suppose that a stable pair (90) satisfies $\chi(E) = 0$. Then the above argument shows that $F_N \cong \mathcal{O}_{X_p}$, and we obtain a morphism

$$I_p \to E_{N-1},$$

which is surjective in dimension one. Here I_p is the ideal sheaf of $\pi^{-1}(p)$. Repeating the above argument, we see that

$$F_i \cong \mathcal{O}_{X_p}, \qquad \mathrm{Cok}(s) = 0, \tag{93}$$

for all i. It is easy to see that a pair (90) satisfying the property (93) is obtained by the pullback,

$$\mathcal{O}_S \twoheadrightarrow \mathcal{O}_W,$$

for a zero-dimensional subscheme $W \subset S$ of length m. Therefore we have the isomorphism

$$P_0(X,m) \cong \mathrm{Hilb}_m(S),$$

and

$$P_{0,m} = \chi\big(\mathrm{Hilb}_m(S)\big).$$

Combining this with (92), we obtain the desired result. $\square$

Combining Propositions 6.7 and 6.8, we obtain the following theorem.

THEOREM 6.9
We have the following formula,

$$\mathrm{PT}(X/S) = \prod_{m\geq 1, j\geq 1} \big(1-(-q)^j t^m\big)^{-j\chi(X)} (1-t^m)^{-\chi(S)}. \tag{94}$$

Proof
By Proposition 6.7 and Theorem 6.4, the series $\mathrm{PT}(X/S)$ is written as a Gopakumar–Vafa form (68) with n_0^m equal to $-\chi(X)$ for all $m \geq 1$. Also, Proposition 6.8 implies that

$$\begin{aligned}\sum_{n,m} L_{n,m} q^n t^m &= \sum_m L_{0,m} t^m \\ &= \sum_m \chi\big(\mathrm{Hilb}_m(S)\big) t^m \\ &= \prod_{m\geq 1} (1-t^m)^{-\chi(S)}.\end{aligned}$$

Here the last equality is Göttsche's formula [14]. Therefore we have the desired formula. □

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