On the local cohomology and support for triangulated categories

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Abstract Recently a notion of support and a construction of local cohomology functors for [TR5] compactly generated triangulated categories were introduced and studied by Benson, Iyengar, and Krause. Following their idea, we assign to any object of the category a new subset of $\operatorname{Spec}(R)$, again called the (big) support. We study this support and show that it satisfies axioms such as exactness, orthogonality, and separation. Using this support, we study the behavior of the local cohomology functors and show that these triangulated functors respect boundedness. Then we restrict our study to the categories generated by only one compact object. This condition enables us to get some nice results. Our results show that one can get a satisfactory version of the local cohomology theory in the setting of triangulated categories, compatible with the known results for the local cohomology for complexes of modules.

1. Introduction

Let $\mathcal{T}$ be a triangulated category. We say that $\mathcal{T}$ satisfies [TR5] if it has arbitrary small coproducts. An object C of $\mathcal{T}$ is compact if the functor $\operatorname{Hom}_{\mathcal{T}}(C, \,)$ preserves small coproducts. We let $\mathcal{T}^c$ denote the class of all compact objects of $\mathcal{T}$. A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a *generating set* for $\mathcal{T}$ if for each nonzero object $X \in \mathcal{T}$, there exists an object G in $\mathcal{G}$ such that $\operatorname{Hom}_{\mathcal{T}}(G, X) \neq 0$. $\mathcal{T}$ is called *compactly generated* if it has a generating set of compact objects. Throughout the paper, we assume that $\mathcal{T}$ is a [TR5] compactly generated triangulated category with the graded center $\mathcal{Z}(\mathcal{T})$.

Let R be a graded commutative Noetherian ring, and let $\phi : R \longrightarrow \mathcal{Z}(\mathcal{T})$ be a homomorphism of graded rings. This, in particular, implies that for any objects X and Y of $\mathcal{T}$, the abelian group $\operatorname{Hom}^*_{\mathcal{T}}(X, Y)$ has a structure of a graded R-module. If $X = C$ is a compact object, then the R-module $\operatorname{Hom}^*_{\mathcal{T}}(C, Y)$ is denoted by $\mathrm{H}^*_C(Y)$ and is called the cohomology of Y with respect to C.

In [1] a notion of (small) support is assigned to any object X of $\mathcal{T}$. By [1, Theorem 5.2], the support of X, denoted $\operatorname{supp}_R X$, is equal to the set $\bigcup_{C \in \mathcal{T}^c} \min \operatorname{supp}_R \mathrm{H}^*_C(X)$, where for an R-module M, $\operatorname{supp}_R M$ denotes the small

Kyoto Journal of Mathematics, Vol. 51, No. 4 (2011), 811–829
DOI 10.1215/21562261-1424866,
Received June 16, 2010. Revised January 21, 2011. Accepted February 1, 2011.
2010 Mathematics Subject Classification: Primary 18E30, 18E35, 13D45; Secondary 16E40.
The research of the authors supported in part by a grant from IPM (Nos. 87130119, 87130120, and 89130052).

(or cohomological) support of M. In this paper, we introduce a (big) support for X and show that the construction of local cohomology functors, introduced in [1], can be studied well by using this new notion of support. We study some properties of local cohomology functors and get some results that, in special cases, recover and generalize the known results about the usual local cohomology functors.

Let us be more precise. A subset ϑ of $\mathrm{Spec}(R)$, where $\mathrm{Spec}(R)$ denotes the set of graded prime ideals of R, is called *specialization closed* if any prime of $\mathrm{Spec}(R)$ containing an element of ϑ is itself contained in ϑ. For any specialization closed subset ϑ of $\mathrm{Spec}(R)$, set

$$\mathcal{T}_\vartheta = \left\{X \in \mathcal{T} \;\middle|\; \mathrm{supp}_R \, \mathrm{H}^*_C(X) \subseteq \vartheta \quad \text{for all } C \in \mathcal{T}^c\right\}.$$

By [1, Lemma 3.4], $\mathcal{T}_\vartheta$ is a localization subcategory of $\mathcal{T}$, and there exists a localization sequence

$$\begin{tikzcd} \mathcal{T}_\vartheta \arrow[r, "i"] & \mathcal{T} \arrow[l, bend right, "\Gamma_\vartheta"'] \arrow[r, "L_\vartheta"] & \mathcal{T}/\mathcal{T}_\vartheta \arrow[l, bend right, "e"'] \end{tikzcd}$$

in which Γ_ϑ is the right adjoint of the inclusion functor i, and the inclusion functor e is the right adjoint of the localization functor L_ϑ on $\mathcal{T}$. So, for any X in $\mathcal{T}$, we have an exact triangle

$$\Gamma_\vartheta X \longrightarrow X \longrightarrow L_\vartheta X \rightsquigarrow$$

in $\mathcal{T}$. $\Gamma_\vartheta X$ is called the local cohomology of X with support in ϑ.

Section 3 is devoted to the study of the quotient triangulated category $\mathcal{T}/\mathcal{T}_\vartheta$. We use the localization functor L to assign to each object X of $\mathcal{T}$ a new subset of $\mathrm{Spec}(R)$ called the (big) support, denoted $\mathrm{Supp}_R(X)$. In fact, for any prime ideal $\mathfrak{p}$ of R we consider the specialization closed subset $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}$ of $\mathrm{Spec}(R)$, and the localization functor $L_{\mathcal{Z}(\mathfrak{p})} : \mathcal{T} \to \mathcal{T}$. Then, for any $X \in \mathcal{T}$, we let $X_\mathfrak{p}$ denote the object $L_{\mathcal{Z}(\mathfrak{p})}X$. Finally, we define the (big) support of X to be the set of all $\mathfrak{p} \in \mathrm{Spec}(R)$ for $X_\mathfrak{p} \neq 0$. We study the behavior of this support in different situations. In particular, we show that we have axioms such as exactness, orthogonality, and separation for this notion of support (see Proposition 3.4). Moreover, our definition relates the vanishing of objects of $\mathcal{T}$ to the vanishing of their cohomology modules, in the sense that, for any object X of $\mathcal{T}$ we have $X = 0$ if and only if $\mathrm{Supp}_R X = \emptyset$. Finally, we show that for any object X, the cohomology of X with respect to a compact object C commutes with localization; that is, for any prime ideal $\mathfrak{p}$ of R, we have an isomorphism $\mathrm{H}^*_C(X)_\mathfrak{p} \cong \mathrm{H}^*_{C_\mathfrak{p}}(X_\mathfrak{p})$ of $R_\mathfrak{p}$-modules.

Then we study the properties of local cohomology functor Γ_ϑ in the case where $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is an $\mathbb{N}_0$-graded ring. In Section 4, we show that the right adjoint functor Γ_ϑ preserves boundedness. To show this, we assign invariants, say, cohomological dimension and cohomological grade, to any object X with respect to any graded ideal $\mathfrak{a}$ of R. We also get some results related to the vanishing of the local cohomology objects. Our results show that one can get a satisfactory version of the known results about the usual local cohomology modules in this new setting.

In Section 5, we restrict ourselves to triangulated categories that are generated by one compact object, say, C. Based on our main result in this section, for an ideal $\mathfrak{a}$ of R, a cohomologically finite object $X \in \mathcal{T}$, and for any integer i, we have $\dim_R \mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})} X) + i \leq \dim_R X + \sup_C X$. Specializing this result to the case where $\mathcal{T}$ is the derived category of a commutative Noetherian ring gives a nice formula for local cohomology of a complex X with finite cohomology, that is,

$$\dim_R\big(\mathrm{H}^i(\mathbf{R}\Gamma_{\mathfrak{a}}(X))\big) + i \leq \dim_R X + \sup_R X$$

or, more specially, for a finitely generated module M over a Noetherian ring R,

$$\dim_R H^i_{\mathfrak{a}}(M) + i \leq \dim_R M.$$

Although these results are well known and can be found in the literature (see, e.g., [3, Section 3]), the approach here is completely different and somehow interesting.

2. Preliminaries

Throughout the paper, R denotes a graded commutative Noetherian ring, and $\mathrm{Spec}(R)$ denotes the set of graded prime ideals of R. For a point $\mathfrak{p}$ in $\mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ denotes the homogeneous localization of R, which is a graded local ring.

Let M be a graded R-module. We let $M[n]$, for any integer n, denote the graded module with $M[n]^i = M^{n+i}$. For any homogeneous ideal $\mathfrak{a}$ of R the variety of $\mathfrak{a}$ is denoted by $\mathcal{V}(\mathfrak{a})$ and is the set $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$.

2.1. The cohomological support of a module

Let M be an R-module. The cohomological support (or small support) of M, denoted $\mathrm{supp}_R M$, is defined to be the set of all primes $\mathfrak{p}$ of R that appear in a minimal injective resolution of M, that is, those primes $\mathfrak{p}$ for which there exist integers i, n such that the ith term of the minimal injective resolution of M contains a direct summand isomorphic to $E(R/\mathfrak{p})[n]$. It is clear that $\mathrm{supp}_R M$ is contained in the usual support of M, $\mathrm{Supp}_R M = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

2.2. Localization sequence of functors

The sequence

$$\mathcal{T}' \xrightarrow{\quad F \quad} \mathcal{T} \xrightarrow{\quad G \quad} \mathcal{T}'' \tag{2.1}$$

of triangulated functors is called a *localization* sequence if it satisfies the following.

(i) F is fully faithful and has a right adjoint F_ρ.
(ii) G has a fully faithful right adjoint G_ρ.

(iii) For any object X of $\mathcal{T}$, $G(X)=0$ if and only if $X \cong F(X')$ for some $X' \in \mathcal{T}'$.

The reader can consult [9, Section II.2] to study the properties of a localization sequence (see also [5]). We just mention that in the above situation, for any object X of $\mathcal{T}$ there exists a triangle

$$FF_\rho(X) \longrightarrow X \longrightarrow G_\rho G(X) \rightsquigarrow$$

in $\mathcal{T}$, which is functorial in X.

2.3. Localization functor

Let $\mathcal{T}$ be a compactly generated triangulated category, and let $\mathcal{T}^{\mathrm{c}}$ denote the full subcategory formed by all compact objects. For any objects X and Y of $\mathcal{T}$, let $\mathrm{Hom}_{\mathcal{T}}(X,Y)$ denote the abelian group of morphisms, and let $\mathrm{Hom}^*_{\mathcal{T}}(X,Y)$ denote the graded abelian group $\mathrm{Hom}^*_{\mathcal{T}}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X,\Sigma^i Y)$. It clearly has a right $\mathrm{End}^*_{\mathcal{T}}(X)$-module and a left $\mathrm{End}^*_{\mathcal{T}}(Y)$-module, where for any object X of $\mathcal{T}$, $\mathrm{End}^*_{\mathcal{T}}(X) = \mathrm{Hom}^*_{\mathcal{T}}(X,X)$.

Let $Z(\mathcal{T})$ denote the graded center of $\mathcal{T}$ with $Z(\mathcal{T})^n = \{\eta : \mathrm{Id}_{\mathcal{T}} \to \Sigma^n \mid \eta\Sigma = (-1)^n \Sigma\eta\}$ for any integer n. $Z(\mathcal{T})$ is a graded commutative ring.

Throughout the paper, we fix a graded commutative Noetherian ring R and a homomorphism of graded rings $\phi : R \to Z(\mathcal{T})$. This implies that any graded abelian group $\mathrm{Hom}^*_{\mathcal{T}}(X,Y)$ is an R-module with the action induced by ϕ (for details, see [1, Section 4]). Therefore $\mathcal{T}$ becomes an R-linear triangulated category.

For any objects C and X of $\mathcal{T}$, set $\mathrm{H}^*_C(X) := \mathrm{Hom}^*_{\mathcal{T}}(C,X)$, and call this R-module the cohomology of X with respect to C.

Let $\mathcal{U}$ be a subset of $\mathrm{Spec}(R)$. The specialization closure of $\mathcal{U}$, denoted $\mathrm{cl}\,\mathcal{U}$, is defined by

$$\mathrm{cl}\,\mathcal{U} = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{q} \quad \text{for some } \mathfrak{q} \in \mathcal{U}\}.$$

A subset $\mathcal{U}$ of $\mathrm{Spec}(R)$ is called specialization closed if $\mathrm{cl}\,\mathcal{U} = \mathcal{U}$.

Let ϑ be a specialization closed subset of $\mathrm{Spec}(R)$. Set

$$\mathcal{T}_\vartheta = \{X \in \mathcal{T} \mid \mathrm{supp}_R \mathrm{H}^*_C(X) \subseteq \vartheta \quad \text{for any } C \in \mathcal{T}^{\mathrm{c}}\}.$$

By [1, Lemma 4.3], $\mathcal{T}_\vartheta$ is a localizing subcategory of $\mathcal{T}$; that is, it is closed under direct summands and small coproducts. So we have the localization functor $L_\vartheta : \mathcal{T} \to \mathcal{T}$. It induces an equivalence of categories $\mathcal{T}/\mathrm{Ker}\, L_\vartheta \cong \mathrm{Im}\, L_\vartheta$, where $\mathcal{T}/\mathrm{Ker}\, L_\vartheta$ denotes the Verdier quotient of $\mathcal{T}$ with respect to $\mathrm{Ker}\, L_\vartheta$ and $\mathrm{Im}\, L_\vartheta$ is the essential image of L_ϑ. Moreover, $L_\vartheta X = 0$ if and only if $X \in \mathcal{T}_\vartheta$. Hence we get a localization sequence of triangulated functors

$$\mathcal{T}_\vartheta \xrightarrow{\quad i \quad} \mathcal{T} \xrightarrow{\quad L_\vartheta \quad} \mathcal{T}/\mathcal{T}_\vartheta.$$

In particular, for any object X of $\mathcal{T}$, there exists a triangle

$$\Gamma_\vartheta X \longrightarrow X \longrightarrow L_\vartheta X \rightsquigarrow$$

in which Γ_ϑ is the right adjoint of the inclusion functor $\mathcal{T}_\vartheta \hookrightarrow \mathcal{T}$. $\Gamma_\vartheta X$ is then called the local cohomology of X supported on ϑ (see [1, Section 4]).

2.4. Neeman-Ravenel-Thomason localization theorem (see [8])

Let $\mathcal{T}$ be a [TR5] triangulated category, and let $\mathcal{S}$ be a localizing subcategory of $\mathcal{T}$. We say that $\mathcal{S}$ is compactly generated in $\mathcal{T}$ if $\mathcal{S}$ admits a compact generating set consisting of objects that are compact in $\mathcal{T}$. In this case, $\mathcal{S}^c = \mathcal{S} \cap \mathcal{T}^c$ and $\mathcal{T}/\mathcal{S}$ is compactly generated. Note that the Verdier quotient $\mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ preserves compactness (see [7, Chapter 4]).

2.5. Koszul object

Let r be a homogeneous element of R of degree d, and let C be an object of $\mathcal{T}$. By [1, Definition 5.10], $C/\!/r$ denotes any object that appears in the exact triangle

$$C \xrightarrow{\ r\ } \Sigma^d C \longrightarrow C/\!/r \rightsquigarrow .$$

$(C/\!/r)$ is called the Koszul object of r on C. This, for any object X of $\mathcal{T}$, gives an exact sequence of R-modules

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^*_C(X)[-d-1] &\xrightarrow{\mp r} \mathrm{H}^*_C(X)[-1] \\ \longrightarrow \mathrm{H}^*_{C/\!/r}(X) \longrightarrow \mathrm{H}^*_C(X)[-d] &\xrightarrow{\pm r} \mathrm{H}^*_C(X) \longrightarrow \cdots . \end{aligned}$$

Let $\mathbf{r} = r_1, \ldots, r_n$ be a sequence of homogeneous elements in R. The Koszul object of $\mathbf{r}$ on C, denoted $C/\!/\mathbf{r}$, is defined inductively by setting $C/\!/\mathbf{r} = C_n$, where $C_0 = C$ and $C_i = C_{i-1}/\!/r_i$ for $i \geq 1$.

When $\mathfrak{a}$ is an ideal of R, the Koszul object of $\mathfrak{a}$ on C, denoted $C/\!/\mathfrak{a}$, is defined to be any Koszul object on C, with respect to some finite sequence of generators for $\mathfrak{a}$. Of course, this object depends on the choice of the minimal generating sequence for $\mathfrak{a}$.

2.6. The (small) support of an object

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Set $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}$, and denote the composite functor $L_{\mathcal{Z}(\mathfrak{p})}\Gamma_{\mathcal{V}(\mathfrak{p})}$ by $\Gamma_\mathfrak{p}$. The support of an object X in $\mathcal{T}$ is defined in [1, Section 5] by

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \Gamma_\mathfrak{p} X \neq 0\}.$$

It is shown that $\operatorname{supp}_R X = \bigcup_{C \in \mathcal{T}^c} \min_R \mathrm{H}^*_C(X)$, where for an R-module M, $\min_R M$ denotes the set of minimal primes in its cohomological support. For more details and properties of this support, see [1, Section 5].

3. Big support

Let $\mathcal{T}$ be a compactly generated triangulated category which is R-linear, where as before R is a graded commutative Noetherian ring.

Let $\mathfrak{p}$ be a prime ideal of R. Since $\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \nsubseteq \mathfrak{p}\}$ is a specialization closed subset of $\mathrm{Spec}(R)$, there is a localization functor $L_{\mathcal{Z}(\mathfrak{p})} : \mathcal{T} \to \mathcal{T}$. For any $X \in \mathcal{T}$, let $X_{\mathfrak{p}}$ denote the object $L_{\mathcal{Z}(\mathfrak{p})}X$, and let $\mathcal{T}_{\mathfrak{p}}$ denote the essential image of $L_{\mathcal{Z}(\mathfrak{p})}$. So $\mathcal{T}_{\mathfrak{p}}$ is the full subcategory of $\mathcal{T}$ formed by all objects isomorphic to an object of the form $L_{\mathcal{Z}(\mathfrak{p})}X$ for some $X \in \mathcal{T}$. Note that $\mathcal{T}_{\mathfrak{p}}$ is the Verdier quotient $\mathcal{T}/\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. Moreover, by [1, Theorem 6.4], $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$ is compactly generated in $\mathcal{T}$, and so by the Neeman-Ravenel-Thomason localization theorem, $\mathcal{T}_{\mathfrak{p}}$ is compactly generated (see Section 2.4).

Our first result shows that $L_{\mathcal{Z}(\mathfrak{p})}$, for any prime $\mathfrak{p}$, is a localization. It is, in fact, contained in [1, Proposition 6.1(2)]. Here we present a slightly different proof.

PROPOSITION 3.1

Let $\mathfrak{q} \subseteq \mathfrak{p}$ be prime ideals of R. Then $L_{\mathcal{Z}(\mathfrak{q})}L_{\mathcal{Z}(\mathfrak{p})}X \cong L_{\mathcal{Z}(\mathfrak{q})}X$ or, in our notation, $(X_{\mathfrak{p}})_{\mathfrak{q}} \cong X_{\mathfrak{q}}$.

Proof

Since $\mathfrak{q} \subseteq \mathfrak{p}$, $\mathcal{Z}(\mathfrak{q}) \cap \mathcal{Z}(\mathfrak{p}) = \mathcal{Z}(\mathfrak{p})$. So by [1, Proposition 1.6(1)], $\Gamma_{\mathcal{Z}(\mathfrak{q})}\Gamma_{\mathcal{Z}(\mathfrak{p})}X \cong \Gamma_{\mathcal{Z}(\mathfrak{p})}X$. This implies that $L_{\mathcal{Z}(\mathfrak{q})}\Gamma_{\mathcal{Z}(\mathfrak{p})}X = 0$. Now the result follows by applying the triangulated functor $L_{\mathcal{Z}(\mathfrak{q})}$ on exact triangle

$$\Gamma_{\mathcal{Z}(\mathfrak{p})}X \longrightarrow X \longrightarrow L_{\mathcal{Z}(\mathfrak{p})}X \rightsquigarrow . \qquad \square$$

DEFINITION 3.2

Let X be an object of $\mathcal{T}$. We define the *(big) support* of X, denoted $\mathrm{Supp}_R X$, to be the set

$$\mathrm{Supp}_R X = \big\{\mathfrak{p} \in \mathrm{Spec}(R) \,\big|\, X_{\mathfrak{p}} \neq 0\big\}.$$

This definition of support for an object is completely related to the usual support of the cohomology of objects. The following theorem establishes this fact.

THEOREM 3.3

Let X be an object of $\mathcal{T}$. Then

$$\mathrm{Supp}_R X = \bigcup_{C \in \mathcal{T}^c} \mathrm{Supp}_R \mathrm{H}^*_C(X).$$

In particular, for any object X of $\mathcal{T}$ we have $X = 0$ if and only if $\mathrm{Supp}_R X = \emptyset$.

Proof

Let $\mathfrak{p} \in \mathrm{Spec}(R)$ be such that $X_{\mathfrak{p}} \neq 0$. So $X \notin \mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. This means that there exists a compact object $C \in \mathcal{T}^c$ such that $\mathrm{supp}_R \mathrm{H}^*_C(X) \nsubseteq \mathcal{Z}(\mathfrak{p})$. This, in turn, implies that there exists a prime ideal $\mathfrak{q}$ contained in $\mathfrak{p}$ such that $\mathrm{H}^*_C(X)_\mathfrak{q} \neq 0$. Hence $\mathrm{H}^*_C(X)_\mathfrak{p} \neq 0$.

Conversely, let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $\mathrm{H}^*_C(X)_\mathfrak{p} \neq 0$ for some compact object C of $\mathcal{T}^c$. Assume to the contrary that $X_\mathfrak{p} = 0$. So $X \in \mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. Therefore, by definition, $\operatorname{supp}_R \mathrm{H}^*_C(X) \subseteq \mathcal{Z}(\mathfrak{p})$ for any compact object C. Hence $\mathrm{H}^*_C(X)_\mathfrak{p}$ should vanish for all $C \in \mathcal{T}^c$, which contradicts the assumption. □

Some properties of support are listed below.

PROPOSITION 3.4

Let $\mathcal{T}$ be a triangulated category.

(1) *For any object X of $\mathcal{T}$,* $\operatorname{Supp}_R X = \operatorname{Supp}_R \Sigma X$.
(2) *For any object X of $\mathcal{T}$,* $\operatorname{supp}_R X \subseteq \operatorname{Supp}_R X$.
(3) *For any exact triangle* $X \longrightarrow Y \longrightarrow Z \rightsquigarrow$ *in $\mathcal{T}$, we have*

$$\operatorname{Supp}_R Y \subseteq \operatorname{Supp}_R X \cup \operatorname{Supp}_R Z.$$

(4) *For any exact triangle* $\Gamma_\vartheta X \longrightarrow X \longrightarrow L_\vartheta X \rightsquigarrow$ *in $\mathcal{T}$, we have*

$$\operatorname{Supp}_R X = \operatorname{Supp}_R \Gamma_\vartheta X \cup \operatorname{Supp}_R L_\vartheta X,$$

where ϑ is a specialization closed subset of $\operatorname{Spec}(R)$.
(5) *For any objects X and Y of $\mathcal{T}$,* $\operatorname{Hom}_\mathcal{T}(X, Y) = 0$ *if* $\operatorname{Supp}_R X \cap \operatorname{Supp}_R Y = \emptyset$.

Proof

Parts (1)–(3) are easy to see.

(4) By part (3), we should show only that $\operatorname{Supp}_R \Gamma_\vartheta X \cup \operatorname{Supp}_R L_\vartheta X \subseteq \operatorname{Supp}_R X$. This follows if we show that $\operatorname{Supp}_R \Gamma_\vartheta X \subseteq \operatorname{Supp}_R X$, and this is clear because, in view of [1, Proposition 6.1(3)], we have $\operatorname{Supp}_R \Gamma_\vartheta X = \operatorname{Supp}_R X \cap \vartheta$.

(5) Set $\vartheta = \operatorname{Supp}_R X$ and $\omega = \operatorname{Supp}_R Y$. By definition, $\operatorname{Hom}_\mathcal{T}(X, L_\vartheta Y) = 0$. So it follows from the triangle $\Gamma_\vartheta Y \longrightarrow Y \longrightarrow L_\vartheta Y \rightsquigarrow$ that $\operatorname{Hom}_\mathcal{T}(X, Y) = \operatorname{Hom}_\mathcal{T}(X, \Gamma_\vartheta Y)$. But $\Gamma_\vartheta Y = 0$ because, by [1, Proposition 6.1(1)], $\Gamma_\vartheta Y = \Gamma_\vartheta \Gamma_\omega Y = \Gamma_{\vartheta \cap \omega} Y$, and by assumption, $\vartheta \cap \omega = \emptyset$. So the result follows. □

DEFINITION 3.5

Let $C \in \mathcal{T}^c$. We say that an object X of $\mathcal{T}$ is *cohomologically finite* with respect to C if $H^*_C(X)$ is finitely generated as a graded R-module. X is called cohomologically finite if it is cohomologically finite with respect to C for any $C \in \mathcal{T}^c$.

3.6

Let X be a cohomologically finite object of $\mathcal{T}$. Then it is easy to see that ΣX; $X//r$ for any homogeneous element r of R and $X_\mathfrak{p}$ as an object of $\mathcal{T}_\mathfrak{p}$ are cohomologically finite.

LEMMA 3.7

Let X be an object of $\mathcal{T}$, and let r be a homogeneous element of R. Then

$$\operatorname{Supp}_R X//r \subseteq \operatorname{Supp}_R X \cap V\big((r)\big).$$

The equality holds when X is cohomologically finite and r is of degree zero.

Proof

Let r be of degree d, and let C be a compact object of $\mathcal{T}$. Applying the functor $\operatorname{Hom}^*_{\mathcal{T}}(C,)$ on the triangle

$$X \xrightarrow{\ r\ } \Sigma^d X \longrightarrow X//r \rightsquigarrow$$

induces exact sequence

$$\cdots \longrightarrow \mathrm{H}^*_C(X) \xrightarrow{r} \mathrm{H}^*_C(\Sigma^d X) \longrightarrow \mathrm{H}^*_C(X//r) \longrightarrow \mathrm{H}^*_C(X)[1] \longrightarrow \cdots$$

of cohomology modules. We note that

$$\operatorname{Supp}_R \mathrm{H}^*_C(X//r) = \operatorname{Supp}_R \mathrm{H}^*_C(X)[d]/r\mathrm{H}^*_C(X) \cup \operatorname{Supp}_R(0 :_{\mathrm{H}^*_C(X)[1]} r).$$

Assume that $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}^*_C(X//r)$. So $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}^*_C(X)$, and it is straightforward to check that $r \in \mathfrak{p}$. This implies the result.

Now assume that X is cohomologically finite and r is of degree zero. Let $\mathfrak{p} \in \operatorname{Supp}_R X \cap V((r))$. There exists a compact object $C \in \mathcal{T}^c$ such that $\mathfrak{p} \in \operatorname{Supp}_R H^*_C(X)$. By localizing the above exact sequence at $\mathfrak{p}$, we get the exact sequence

$$\mathrm{H}^*_C(X)_\mathfrak{p} \xrightarrow{r/1} \mathrm{H}^*_C(X)_\mathfrak{p} \longrightarrow \mathrm{H}^*_C(X//r)_\mathfrak{p} \longrightarrow \mathrm{H}^*_C(X)_\mathfrak{p}[1].$$

If $\mathrm{H}^*_C(X//r)_\mathfrak{p} = 0$, then $\mathrm{H}^*_C(X)_\mathfrak{p} = r/1\mathrm{H}^*_C(X)_\mathfrak{p}$ and, since X is cohomologically finite, the Nakayama lemma implies that $\mathrm{H}^*_C(X)_\mathfrak{p} = 0$, which contradicts the fact that $\mathfrak{p} \in \operatorname{Supp}_R X$. Hence $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}^*_C(X//r)$, and so we have the equality. $\square$

The above lemma implies that for any prime ideal $\mathfrak{p}$ of R and any homogeneous element $r \in R \setminus \mathfrak{p}$, $\operatorname{Supp}_R X//r \subseteq \mathcal{Z}(\mathfrak{p})$. This, in particular, implies that $X//r \in \mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. So for any exact triangle

$$X \xrightarrow{\ r\ } \Sigma^d X \longrightarrow X//r \rightsquigarrow,$$

the induced morphism $L_{\mathcal{Z}(\mathfrak{p})}X \xrightarrow{r} L_{\mathcal{Z}(\mathfrak{p})}\Sigma^d X$ is an isomorphism in $\mathcal{T}_\mathfrak{p}$. Therefore any homomorphism $\phi : R \longrightarrow Z(\mathcal{T})$ of graded rings induces a homomorphism $\phi_\mathfrak{p} : R_\mathfrak{p} \longrightarrow Z(\mathcal{T}_\mathfrak{p})$ such that the diagram

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d] & Z(\mathcal{T}) \arrow[d] \\
R_\mathfrak{p} \arrow[r, "\phi_\mathfrak{p}"] & Z(\mathcal{T}_\mathfrak{p})
\end{tikzcd}$$

with natural vertical maps, is commutative.

Ignoring our notation, which always denote $\phi(r)$ by r itself, we may write $\phi_{\mathfrak{p}}(r/s) = \phi(r)/1\phi(s)^{-1}/1$, where r and s are homogeneous elements of R with $s \notin \mathfrak{p}$.

PROPOSITION 3.8

Let C be a compact object of $\mathcal{T}$. Then for any object X of C and any prime ideal $\mathfrak{p}$ of R, there exists an isomorphism

$$\mathrm{H}^*_C(X)_{\mathfrak{p}} \cong \mathrm{H}^*_{C_{\mathfrak{p}}}(X_{\mathfrak{p}})$$

of $R_{\mathfrak{p}}$-modules.

Proof

By [1, Theorem 4.7], there exists an isomorphism $\mathrm{H}^*_C(X)_{\mathfrak{p}} \cong \mathrm{H}^*_C(X_{\mathfrak{p}})$ of R-modules. On the other hand, since the localization functor $L_{\mathcal{Z}(\mathfrak{p})}$ is a left adjoint of the inclusion functor, the adjoint duality gives us the isomorphism $\mathrm{H}^*_C(X_{\mathfrak{p}}) \cong \mathrm{H}^*_{C_{\mathfrak{p}}}(X_{\mathfrak{p}})$ of R-modules. So to complete the proof, we should check that the composite isomorphism $\mathrm{H}^*_C(X)_{\mathfrak{p}} \cong \mathrm{H}^*_{C_{\mathfrak{p}}}(X_{\mathfrak{p}})$ is an isomorphism of $R_{\mathfrak{p}}$-modules. The easy argument is left to the reader. □

4. Local cohomology functor and boundedness

Our results in this section show that the local cohomology functor $\Gamma_{\mathcal{V}}$ preserves a certain kind of boundedness in the case where R is a ring of finite Krull dimension. This, in case we are working in a derived category of a module category, simply means that the local cohomology of a bounded complex supported on a variety is again a bounded complex. In this section we assume that $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a Noetherian $\mathbb{N}_0$-graded ring.

Let us begin with some notation and definitions. Let X be an object of $\mathcal{T}$, and let C be a compact object. Since $\mathrm{H}^*_C(X)$ is a graded R-module, we may consider two invariants

$$\inf_C X = \inf\bigl(\mathrm{H}^*_C(X)\bigr) = \inf\bigl\{n \in \mathbb{Z} \bigm| \mathrm{H}^n_C(X) \neq 0\bigr\},$$
$$\sup_C X = \sup\bigl(\mathrm{H}^*_C(X)\bigr) = \sup\bigl\{n \in \mathbb{Z} \bigm| \mathrm{H}^n_C(X) \neq 0\bigr\}.$$

We say that an object X of $\mathcal{T}$ is cohomologically bounded above (resp., bounded below) if, for any compact object C, there exists a positive integer $n(C)$ such that $\sup_C X \leq n(C)$ (resp., $\inf_C X \geq -n(C)$). X is called cohomologically bounded if it is both cohomologically bounded above and cohomologically bounded below. Let $\mathcal{T}^-$ (resp., $\mathcal{T}^+, \mathcal{T}^b$) denote the full subcategory of $\mathcal{T}$, consisting of all cohomologically bounded above (resp., bounded below, bounded) objects. Our results in this section, in fact, show that the local cohomology functor is a functor from $\mathcal{T}^b$ to itself.

DEFINITION 4.1

Let X be an object of $\mathcal{T}$. We define the *dimension* of X to be

$$\dim_R X = \sup\{\dim_R \mathrm{H}_C^*(X) \mid C \in \mathcal{T}^c\}.$$

Obviously, for any object X of $\mathcal{T}$, we have $\dim_R X = \dim_R \Sigma X$.

DEFINITION 4.2

(1) Let $X \neq 0$ be an object of $\mathcal{T}^-$, and let $\mathfrak{a}$ be a graded ideal of R. We define the *cohomological dimension* of X with respect to $\mathfrak{a}$, denoted $\mathrm{cd}(\mathfrak{a}, X)$, to be

$$\mathrm{cd}(\mathfrak{a}, X) = \sup\{\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - \sup_C X \mid C \text{ is a compact object of } \mathcal{T}\}.$$

(2) Let $X \neq 0$ be an object of $\mathcal{T}^+$, and let $\mathfrak{a}$ be a graded ideal of R. We define the *cohomological grade* of X with respect to $\mathfrak{a}$, denoted $\mathrm{cg}(\mathfrak{a}, X)$, to be

$$\mathrm{cg}(\mathfrak{a}, X) = \inf\{\inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - \inf_C X \mid C \text{ is a compact object of } \mathcal{T}\}.$$

Note that, for any nonzero object X of $\mathcal{T}^-$ (resp., $\mathcal{T}^+$), $\mathrm{cd}(\mathfrak{a}, \Sigma X) = \mathrm{cd}(\mathfrak{a}, X)$ (resp., $\mathrm{cg}(\mathfrak{a}, \Sigma X) = \mathrm{cg}(\mathfrak{a}, X)$).

4.3

Let $\{\vartheta_i\}_{i \in \mathcal{I}}$ be a nonempty family of pairwise disjoint specially closed subsets of $\mathrm{Spec}(R)$. Set $\vartheta = \bigcup_{i \in \mathcal{I}} \vartheta_i$. Then one can apply [1, Theorem 7.1.] to $\Gamma_\vartheta X$ to get

$$\Gamma_\vartheta X \cong \coprod_{i \in \mathcal{I}} \Gamma_{\vartheta_i} X.$$

Throughout we use this fact.

As a corollary of this fact, we have the following.

COROLLARY 4.4

Let $\mathfrak{a}$ be a graded ideal of R, and let $X \neq 0$ be an object of $\mathcal{T}$. If $\dim_R X = 0$, then $\Gamma_{\mathcal{V}(\mathfrak{a})} X$ is a summand of X. In particular, if $X \in \mathcal{T}^-$, then $\mathrm{cd}(\mathfrak{a}, X) \leq 0$.

Proof

As $\dim_R X = 0$, the support of X contains only maximal graded ideals of R. Let $\vartheta = \mathrm{Supp}_R X$. So, by Section 4.3, $\Gamma_\vartheta X \cong \coprod_{\mathfrak{m} \in \vartheta} \Gamma_{\mathcal{V}(\mathfrak{m})} X \cong X$. Therefore

$$\Gamma_{\mathcal{V}(\mathfrak{a})} X \cong \Gamma_{\mathcal{V}(\mathfrak{a})} \Gamma_\vartheta X \cong \Gamma_{\mathcal{V}(\mathfrak{a}) \cap \vartheta} X \cong \coprod_{\mathfrak{m} \in \mathcal{V}(\mathfrak{a}) \cap \vartheta} \Gamma_{\mathcal{V}(\mathfrak{m})} X.$$

Hence the result follows. □

LEMMA 4.5

(1) *Let $r \in R$ be a homogeneous element of degree d, and let C be a compact object of $\mathcal{T}$. Then for any $X \in \mathcal{T}$,*

$$\sup_C X /\!/ r \leq \sup_C X.$$

If, moreover, we know that $d > 0$, then $\sup_C X /\!/ r = \sup_C X - 1$.

(2) *Let $\mathfrak{a}$ be a graded ideal of R, let $r \in \mathfrak{a}$ be a homogeneous element of $\mathfrak{a}$, and let C be a compact object of $\mathcal{T}$. Then for any $X \in \mathcal{T}$,*

$$\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X//r \geq \sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - 1.$$

Proof

(1) Both assertions follow immediately from the long exact sequence

$$\mathrm{H}_C^{i-1}(X//r) \longrightarrow \mathrm{H}_C^i(X) \xrightarrow{r} \mathrm{H}_C^{i+d}(X) \longrightarrow \mathrm{H}_C^i(X//r) \longrightarrow \mathrm{H}_C^{i+1}(X)$$

of R_0-modules, induced from the exact triangle

$$X \xrightarrow{r} \Sigma^d X \longrightarrow X//r \rightsquigarrow .$$

(2) By part (1) we should consider only the case $\deg(r) = 0$. Consider the exact triangle

$$\Gamma_{\mathcal{V}(\mathfrak{a})} X \xrightarrow{r} \Gamma_{\mathcal{V}(\mathfrak{a})} X \longrightarrow \Gamma_{\mathcal{V}(\mathfrak{a})} X//r \rightsquigarrow .$$

This induces the exact sequence

$$\mathrm{H}_C^{i-1}(\Gamma_{\mathcal{V}(\mathfrak{a})} X//r) \longrightarrow \mathrm{H}_C^i(\Gamma_{\mathcal{V}(\mathfrak{a})} X) \xrightarrow{r} \mathrm{H}_C^i(\Gamma_{\mathcal{V}(\mathfrak{a})} X)$$

of cohomology modules. Let $\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})}(X) = t$. If $\mathrm{H}_C^{t-1}(\Gamma_{\mathcal{V}(\mathfrak{a})} X//r) = 0$, then the morphism $\mathrm{H}_C^t(\Gamma_{\mathcal{V}(\mathfrak{a})} X) \xrightarrow{r} \mathrm{H}_C^t(\Gamma_{\mathcal{V}(\mathfrak{a})} X)$ has to be injective, which is impossible as any element of $\mathrm{H}_C^t(\Gamma_{\mathcal{V}(\mathfrak{a})} X)$ is annihilated by a power of r. So in this case also $\mathrm{H}_C^{t-1}(\Gamma_{\mathcal{V}(\mathfrak{a})} X//r) \neq 0$. The proof is hence complete.

□

Let $\mathfrak{a}$ be a graded ideal of R. The arithmetic rank of $\mathfrak{a}$, denoted $\operatorname{ara}(\mathfrak{a})$, is defined by

$$\operatorname{ara}(\mathfrak{a}) = \min\bigl\{n \in \mathbb{N} : \exists \text{ homogeneous elements } b_1, \ldots, b_n \in R \text{ with } \sqrt{(b_1, \ldots, b_n)} = \sqrt{\mathfrak{a}}\bigr\}.$$

Note that $\operatorname{ara}(0R) = 0$.

THEOREM 4.6

Let $\mathfrak{a}$ be a graded ideal of R, and let $X \neq 0$ be an object of $\mathcal{T}^-$. Then $\operatorname{cd}(\mathfrak{a}, X) \leq \operatorname{ara}(\mathfrak{a})$.

Proof

Let $\operatorname{ara}(\mathfrak{a}) = n$. We proceed by induction on n. Case $n = 0$ is trivial. Assume that $n = 1$ and $\mathfrak{a} = (r)$, with $\deg(r) = d$. Clearly, $\Gamma_{\mathcal{V}(\mathfrak{a})} X//r = X//r$. So we have the triangle

$$\Gamma_{\mathcal{V}(\mathfrak{a})} X \xrightarrow{r} \Sigma^d \Gamma_{\mathcal{V}(\mathfrak{a})} X \longrightarrow X//r \rightsquigarrow .$$

Consider the induced long exact sequence of R_0-modules

$$\mathrm{H}^i_C(X//r) \longrightarrow \mathrm{H}^{i+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) \xrightarrow{r} \mathrm{H}^{d+i+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X).$$

Let $t = \sup_C X$. Since, by Lemma 4.5, $\mathrm{H}^i_C(X//r) = 0$ for all $i \geq t+1$, the homomorphism $\mathrm{H}^{i+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) \xrightarrow{r} \mathrm{H}^{d+i+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$ is injective for all $i \geq t+1$. This implies that $\mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) = 0$ for all $i > t+1$, because any element of $\mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$ is annihilated by some power of r. The result hence follows in this case.

Suppose inductively that $n > 1$ and the result has been proved for all ideals with arithmetic rank less than n. Assume that $\mathfrak{a} = (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n$ are homogeneous elements of R, and consider the ideal $\mathfrak{b} = (r_2, \ldots, r_n)$. Now the result follows from the Mayer-Vietoris triangle

$$\Gamma_{\mathcal{V}(\mathfrak{b})\cap\mathcal{V}(r_1)}X \longrightarrow \Gamma_{\mathcal{V}(\mathfrak{b})}X \coprod \Gamma_{\mathcal{V}(r_1)}X \longrightarrow \Gamma_{\mathcal{V}(\mathfrak{b})\cup\mathcal{V}(r_1)}X \rightsquigarrow$$

in view of the facts that $\mathcal{V}(\mathfrak{b}) \cup \mathcal{V}(r_1) = \mathcal{V}(\mathfrak{b}r_1)$ and $\mathrm{ara}(\mathfrak{b}r_1) \leq \mathrm{ara}(\mathfrak{b})$. □

THEOREM 4.7

Let $\mathfrak{a}$ *be a graded ideal of* R*, and let* $X \neq 0$ *be an object of* $\mathcal{T}^-$*. Then* $\mathrm{cd}(\mathfrak{a}, X) \leq \dim R$*.*

Proof

Clearly, we may assume that $\dim R < \infty$. If $\dim R = 0$, the result is clear because then $\dim_R X = 0$ for any object X of $\mathcal{T}$, and so the result follows from Corollary 4.4. So assume that $\dim R = d > 0$. Assume that there exists $\mathfrak{p} \in \min R$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Rearrange the ideals in $\min R$ in a way that $\mathfrak{a} \subseteq \mathfrak{p}_i$ for $i = 1, \ldots, n$ and $\mathfrak{a} \nsubseteq \mathfrak{p}_i$ for $i = n+1, \ldots, t$. Choose $r \in \bigcap_{i=n+1}^{t} \mathfrak{p}_i \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. Set $\mathfrak{b} = \mathfrak{a} \cap Rr$ and $\mathfrak{c} = \mathfrak{a} + Rr$. Clearly $\mathfrak{b} \subseteq \bigcap_{\mathfrak{p}\in\min R} \mathfrak{p}$ and $\mathfrak{c} \nsubseteq \bigcup_{\mathfrak{p}\in\min R} \mathfrak{p}$. But $\Gamma_{\mathcal{V}(\mathfrak{b})}X = X$, so it follows from the Mayer-Vietoris triangle that we should consider the case only for $\Gamma_{\mathcal{V}(\mathfrak{c})}X$. Therefore, we may assume that $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{p}\in\min R} \mathfrak{p}$. This we do.

Let $r_1 \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p}\in\min R} \mathfrak{p}$ be a homogeneous element. So $\dim R/r_1R < \dim R$. If $\mathfrak{a} \subseteq \bigcap_{\mathfrak{p}\in\min(R/r_1R)} \mathfrak{p}$, then since $r_1 \in \mathfrak{a}$, we get $\sqrt{\mathfrak{a}} = \sqrt{r_1R}$, and so $\mathrm{ara}(\mathfrak{a}) = 1$. Hence the result follows from Theorem 4.6. Otherwise, we may find $r_2 \in \mathfrak{a}$ such that $\dim R/(r_1, r_2) < \dim R/r_1R$. Continuing in this way, after d steps, we may deduce that $\dim R/(r_1, \ldots, r_d) = 0$. So $\dim_R X/(r_1, \ldots, r_d)X = 0$. Hence, by Corollary 4.4 and Lemma 4.5, for any compact object C of $\mathcal{T}$, we have

$$\begin{aligned}\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1, \ldots, r_d)X &\leq \sup_C X/(r_1, \ldots, r_d)X \\ &\leq \sup_C X = s.\end{aligned}$$

This means that $\mathrm{H}^{s+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1, \ldots, r_d)X) = 0$. Now the long exact sequence of cohomology modules arising from the exact triangle

$$\begin{aligned}\Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1, \ldots, r_{d-1})X &\xrightarrow{r_d} \Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1, \ldots, r_{d-1})X \\ &\longrightarrow \Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1, \ldots, r_d)X \rightsquigarrow\end{aligned}$$

implies that $\mathrm{H}_C^{s+2}(\Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1,\ldots,r_{d-1})X)=0$. This follows because $r_d\in\mathfrak{a}$ and any element of $\mathrm{H}_C^*(\Gamma_{\mathcal{V}(\mathfrak{a})}X/(r_1,\ldots,r_{d-1})X)$ vanishes by a power of r_d. Following back from this argument gives us $\mathrm{H}_C^{s+d+1}(\Gamma_{\mathcal{V}(\mathfrak{a})}X)=0$. That is, $\sup_C\Gamma_{\mathcal{V}(\mathfrak{a})}X\leq \sup_C X+\dim R$. The proof is complete because C was arbitrary. □

LEMMA 4.8

Let $X\in\mathcal{T}$, let $C\in\mathcal{T}^c$ be a compact object, and let $r\in R$ be a homogeneous element of degree d. Then

$$\inf_C X/\!/r\geq\inf_C X-1\quad \text{if } d=0$$

and

$$\inf_C X/\!/r=\inf_C X-d\quad \text{if } d>0.$$

Proof

This can be proved easily using the long exact sequence of cohomology modules arising from the exact triangle

$$X\xrightarrow{r}\Sigma^d X\longrightarrow X/\!/r\rightsquigarrow. \qquad \square$$

PROPOSITION 4.9

Let $\mathfrak{a}$ be a graded ideal of R, and let $X\neq 0$ be an object of $\mathcal{T}^+$. Then $\mathrm{cg}(\mathfrak{a},X)\geq 0$.

Proof

The result trivially holds, if $X/\!/\mathfrak{a}=0$. So we may assume that $X/\!/\mathfrak{a}\neq 0$. Let C be a compact object of $\mathcal{T}$. We use induction on the number of generators of $\mathfrak{a}$ to show that $\inf_C\Gamma_{\mathcal{V}(\mathfrak{a})}X\geq\inf_C X$. To begin, assume that $\mathfrak{a}=(r)$ and $\deg(r)=d$. The triangle

$$X\xrightarrow{r}\Sigma^d X\longrightarrow X/\!/r\rightsquigarrow$$

induces the triangle

$$\Gamma_{\mathcal{V}(\mathfrak{a})}X\xrightarrow{r}\Sigma^d\Gamma_{\mathcal{V}(\mathfrak{a})}X\longrightarrow\Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r\rightsquigarrow.$$

If $d>0$, then in view of Lemma 4.8, we have the following equalities:

$$\inf_C X-d=\inf_C X/\!/r=\inf_C\Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r=\inf_C\Gamma_{\mathcal{V}(\mathfrak{a})}X-d.$$

Therefore $\inf_C\Gamma_{\mathcal{V}(\mathfrak{a})}X=\inf_C X$. Now, assume that $d=0$. The latter exact triangle induces the following exact sequence of R_0-modules:

$$\mathrm{H}_C^i(\Gamma_{\mathcal{V}(\mathfrak{a})}X)\xrightarrow{r}\mathrm{H}_C^i(\Gamma_{\mathcal{V}(\mathfrak{a})}X)\longrightarrow\mathrm{H}_C^i(X/\!/r).$$

Assume that $\inf_C\Gamma_{\mathcal{V}(\mathfrak{a})}X=t$. So this exact sequence implies that $\mathrm{H}_C^{t-1}(X/\!/r)\neq 0$. Otherwise, we get the monomorphism $\mathrm{H}_C^t(\Gamma_{\mathcal{V}(\mathfrak{a})}X)\xrightarrow{r}\mathrm{H}_C^t(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$, and since any element of $\mathrm{H}^t_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$ is annihilated by a power of r, we deduce that $\mathrm{H}^t_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) = 0$, which is a contradiction. Furthermore, $\mathrm{H}^i_C(X//r) = 0$ for all $i < t - 1$. Thus $\inf_C X//r = \inf_C \Gamma_{\mathcal{V}(\mathfrak{a})}X - 1$. Lastly, in view of Lemma 4.8, we get $\inf_C X \leq \inf_C \Gamma_{\mathcal{V}(\mathfrak{a})}X$. Thus the result follows in this case. Now, assume inductively that the result has been proved for all values smaller than n. Let $\mathfrak{a} = (r_1, \ldots, r_n)$. Set $\mathfrak{b} = (r_2, \ldots, r_n)$. In view of the induction step, we have the following inequalities:

$$\inf_C X \leq \inf_C \Gamma_{\mathcal{V}(\mathfrak{b})}X \leq \inf_C \Gamma_{\mathcal{V}(r_1)}\Gamma_{\mathcal{V}(\mathfrak{b})}X = \inf_C \Gamma_{\mathcal{V}(\mathfrak{a})}X.$$

Therefore the result follows. □

4.10. Artinianness of the local cohomology functors

An object X of $\mathcal{T}$ is said to be Artinian with respect to the compact object $C \in \mathcal{T}^c$ if $H^*_C(X)$ is a graded Artinian R-module. X is called Artinian if it is Artinian with respect to any compact object C of $\mathcal{T}$. Let $(R, \mathfrak{m})$ be a graded local ring, and let X be a cohomologically finite object of $\mathcal{T}$. Then using the same techniques as we used above, and following the same arguments as in [2, Chapter 7], one can show that $\Gamma_{\mathcal{V}(\mathfrak{m})}X$ is Artinian. Let us just outline the proof. We proceed by induction on $\dim_R X = n$ to show that $\Gamma_{\mathcal{V}(\mathfrak{m})}X$ is Artinian. If $n = 0$, then $\mathrm{Supp}_R X = \{\mathfrak{m}\}$, and so $\Gamma_{\mathcal{V}(\mathfrak{m})}X = X$. Now, since the graded R-module $\mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X) = \mathrm{H}^*_C(X)$ is cohomologically finite, it has finite length, and so we are done in this case. Now, assume that $n > 0$ and the result has been proved for all integers smaller than n. As $n > 0$, $\mathfrak{m} \nsubseteq \bigcup_{\mathfrak{p}\in\min(R)} \mathfrak{p}$, and hence there exists a homogeneous element $r \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p}\in\min(R)} \mathfrak{p}$. Let $\deg(r) = d$, and consider the exact triangle

$$X \xrightarrow{r} \Sigma^d X \longrightarrow X//r \rightsquigarrow$$

in $\mathcal{T}$. If $\dim_R X//r < n$, then we apply the functor $\Gamma_{\mathcal{V}(\mathfrak{m})}$ to the above exact triangle and then apply the functor $\mathrm{H}^*_C(\ )$ to the new exact triangle to get the following exact sequence of graded R-modules:

$$\mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X//r)[1] \longrightarrow \mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X) \xrightarrow{r} \mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X)[d] \longrightarrow \mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X//r).$$

So by using the induction hypothesis, we deduce that $\mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X//r)$ is Artinian. Thus the graded module $(0 :_{\mathrm{H}^*_C(\Gamma_{\mathcal{V}(\mathfrak{m})}X//r)} r)$ is Artinian. Now, since $\Gamma_{\mathcal{V}(\mathfrak{m})}X//r$ is r-torsion, Melkersson's lemma (cf. [2, Theorem 7.1.2]) implies the result. If otherwise, $\dim_R X//r = n$, we may find $r_2 \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p}\in\min(R/r_1R)} \mathfrak{p}$, and so on. Finally, we get $r_1, \ldots, r_t$ in R such that $\dim_R X//(r_1, \ldots, r_t) < n$. Therefore the procedure can be continued as in the previous case.

5. Triangulated categories generated by a compact object

In this section we assume that $\mathcal{T}$ is generated by one compact object C. As in the Section 4, $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a Noetherian $\mathbb{N}_0$-graded ring. One of the main results of this section, Theorem 5.6, provides us with a formula connecting the dimension and the end of local cohomology objects. This, in particular, reduces to a nice result in the local cohomology theory over a commutative Noetherian ring.

THEOREM 5.1

Let $\mathfrak{a}$ be an ideal of R, and let $X \neq 0$ be a cohomologically finite object of $\mathcal{T}^-$. Then $\operatorname{cd}(\mathfrak{a}, X) \leq \dim_R X$.

Proof

Clearly, we may assume that $\dim_R X < \infty$. If $\dim_R X = 0$, the result follows from Corollary 4.4. So assume that $\dim_R X > 0$ and the result has been proved for all objects of dimension less than $\dim_R X$. Similarly to the proof of Theorem 4.7, we may assume that $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{p} \in \min \mathrm{H}^*_C(X)} \mathfrak{p}$.

We should show that $\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X \leq \dim_R X + \sup_C X$. By the induction assumption, we have $\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X//r \leq \dim_R X//r + \sup_C X//r$, where $r \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \min \mathrm{H}^*_C(X)} \mathfrak{p}$. But by Lemma 4.5, we have the inequalities $\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - 1 \leq \sup_C \Gamma_{\mathcal{V}(\mathfrak{a})} X//r$ and $\sup_C X//r \leq \sup_C X$. Hence the result follows because $\dim_R X//r = \dim_R X - 1$. □

LEMMA 5.2

Let R be a commutative graded ring with local base ring $(R_0, \mathfrak{m}_0)$, and let X be a cohomologically finite object of $\mathcal{T}$. Let $r \in \mathfrak{m}_0$. Then

$$\sup_C X//r = \sup_C X.$$

Proof

For each $i \in \mathbb{Z}$, consider the induced exact sequence

$$\mathrm{H}^i_C(X) \xrightarrow{r} \mathrm{H}^i_C(X) \longrightarrow \mathrm{H}^i_C(X//r) \longrightarrow \mathrm{H}^{i+1}_C(X)$$

of R_0-modules. Since X is cohomologically finite, $H^i_C(X)$ is finitely generated for each i. So if $\mathrm{H}^i_C(X//r) = 0$, the Nakayama lemma implies that $\mathrm{H}^i_C(X) = 0$. Therefore $\sup_C X \leq \sup_C X//r$. The reverse inequality follows from Lemma 4.5. □

LEMMA 5.3

Let $\mathfrak{a}$ be a graded ideal of R, and let $r \in \mathfrak{a}$ be a homogeneous element of degree zero. Then for any $X \in \mathcal{T}$ and $C \in \mathcal{T}^c$,

$$\inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X//r = \inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - 1.$$

Proof

By Lemma 4.8, we have $\inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X//r \geq \inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X - 1$. For the converse assume that $\inf_C \Gamma_{\mathcal{V}(\mathfrak{a})} X//r = t$. Then $\mathrm{H}^{t-1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})} X//r) = 0$, and so the homomorphism

$$\mathrm{H}^t_C(\Gamma_{\mathcal{V}(\mathfrak{a})} X) \xrightarrow{r} \mathrm{H}^t_C(\Gamma_{\mathcal{V}(\mathfrak{a})} X)$$

is injective. But any element of $\mathrm{H}^t_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$ is annihilated by some powers of $\mathfrak{a}$, and so $\mathrm{H}^t_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) = 0$. Furthermore, $\mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) = 0$ for all $i < t$. Hence we have the result. □

DEFINITION 5.4

Let $\mathfrak{a}$ be a graded ideal of R, and let X be a cohomologically finite object of $\mathcal{T}$. We define the *grade* of X relative to $\mathfrak{a}$, denoted $\mathrm{grade}(\mathfrak{a}, X)$, by

$$\mathrm{grade}(\mathfrak{a}, X) = \mathrm{grade}\big(\mathfrak{a}, \mathrm{H}^*_C(X)\big).$$

In the rest of this section we assume that R is a trivially graded (concentrated in degree zero) commutative Noetherian ring.

THEOREM 5.5

Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal of R. Then for any cohomologically finite object $0 \neq X \in \mathcal{T}^+$, we do have

$$\mathrm{grade}(\mathfrak{a}, X) \leq \mathrm{cg}(\mathfrak{a}, X).$$

Proof

We use induction on $\mathrm{grade}(\mathfrak{a}, X) = g$. When $g = 0$, the result follows from Proposition 4.9. Now suppose that $g > 0$ and that the result has been proved for any cohomologically finite object Y with $\mathrm{grade}(\mathfrak{a}, Y) < g$. Since $g > 0$, $\mathfrak{a} \nsubseteq Z_R(H^*_C(X))$, and so there exists a homogeneous element $r \in \mathfrak{a} \setminus Z_R(H^*_C(X))$. This induces the exact triangle

$$X \xrightarrow{\ r\ } X \longrightarrow X//r \rightsquigarrow$$

in $\mathcal{T}$ which, in turn, induces an exact sequence

$$0 \longrightarrow \mathrm{H}^*_C(X) \xrightarrow{r} \mathrm{H}^*_C(X) \longrightarrow \mathrm{H}^*_C(X//r) \longrightarrow 0$$

of graded modules. So

$$\begin{aligned}\mathrm{grade}(\mathfrak{a}, X//r) &= \mathrm{grade}\big(\mathfrak{a}, H^*_C(X//r)\big) = \mathrm{grade}\big(\mathfrak{a}, H^*_C(X)/rH^*_C(X)\big)\\ &= \mathrm{grade}(\mathfrak{a}, X) - 1 = g - 1.\end{aligned}$$

Hence the induction hypothesis for $X//r$ implies that

$$\inf\nolimits_C X//r + \mathrm{grade}(\mathfrak{a}, X//r) \leq \inf\nolimits_C \Gamma_{\mathcal{V}(\mathfrak{a})}X//r.$$

By Lemma 5.3, $\inf_C \Gamma_{\mathcal{V}(\mathfrak{a})}X//r = \inf_C \Gamma_{\mathcal{V}(\mathfrak{a})}X - 1$. On the other hand, assume that $\inf_C X = t$. But if $\mathrm{H}^t_C(X//r) = 0$, from the above exact sequence of graded R-modules we get the exact sequence $\mathrm{H}^t_C(X) \xrightarrow{r} \mathrm{H}^t_C(X) \longrightarrow 0$, which, in view of the Nakayama lemma, is impossible. So $\inf_C X//r = \inf_C X$. Therefore we get the desired formula. □

THEOREM 5.6

Let $\mathfrak{a}$ be an ideal of R, and let $X \in \mathcal{T}$ be cohomologically finite. Then for any *integer i we have*

$$\dim_R \mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) + i \le \dim_R X + \sup_C X.$$

Proof
Set $\dim_R X = t$ and $\sup_C X = s$. Since $\mathrm{Supp}_R \mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) \subseteq \mathrm{Supp}_R X$, the result is clear for $i \le s$. Moreover, by Theorem 5.1, we have $\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})}X \le t+s$. So for $i > t+s$, $\mathrm{H}^i_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) = 0$, and the result holds in this case. On the other hand, the result follows if $t = 0$ because in this case we deduce from Corollary 4.4 that $\Gamma_{\mathcal{V}(\mathfrak{a})}X$ is a direct summand of X, and hence $\sup_C \Gamma_{\mathcal{V}(\mathfrak{a})}X \le \sup_C X$.

So assume that $s < i \le s+t$ and $t > 0$. Let $t = 1$. Let $\mathfrak{p}$ be a minimal element of $\mathrm{Supp}_R X$. So $\mathrm{Supp}_R X_{\mathfrak{p}} = \mathcal{V}(\mathfrak{p})$, and hence for any ideal $\mathfrak{a} \subseteq \mathfrak{p}$ with $\mathrm{Ann}_R \mathrm{H}^*_C(X) \subseteq \mathfrak{a}$, we have $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(\mathfrak{p})$. Therefore $\Gamma_{\mathcal{V}(\mathfrak{a})}(X_{\mathfrak{p}}) = X_{\mathfrak{p}}$. But by Proposition 3.8, $\sup_{C_{\mathfrak{p}}} X_{\mathfrak{p}} \le \sup_C X$, and so

$$\mathrm{H}^{s+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}} \cong \mathrm{H}^{s+1}_{C_{\mathfrak{p}}}(\Gamma_{\mathcal{V}(\mathfrak{a})}X_{\mathfrak{p}}) = \mathrm{H}^{s+1}_{C_{\mathfrak{p}}}(X_{\mathfrak{p}}) = 0.$$

Therefore $\mathfrak{p} \notin \mathrm{Supp}_R \mathrm{H}^{s+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$, and hence $\dim_R \mathrm{H}^{s+1}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) \le 0$. So we have the result in this case.

Now suppose inductively that $t > 1$. We should show that for $0 < i \le t$, $\dim_R \mathrm{H}^{s+i}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X) \le t - i$. To this end, consider $\mathfrak{p} \in \mathrm{Supp}_R \mathrm{H}^*_C(X)$ with $\mathrm{ht}_{\mathrm{H}^*_C(X)}\,\mathfrak{p} < i$. We claim that $\mathfrak{p} \notin \mathrm{Supp}_R \mathrm{H}^{s+i}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X)$. If $\mathfrak{a} \nsubseteq \mathfrak{p}$, the result is clear. So assume that $\mathfrak{a} \subseteq \mathfrak{p}$. Using the same technique as in the proof of Theorem 4.7, we may assume that $\mathfrak{a} \nsubseteq \bigcup_{\mathfrak{q} \in \min X} \mathfrak{q}$. So there exists $r \in \mathfrak{a} \setminus \bigcup_{\mathfrak{q} \in \min X} \mathfrak{q}$. Consider the exact triangle

$$\Gamma_{\mathcal{V}(\mathfrak{a})}X \xrightarrow{\;r\;} \Gamma_{\mathcal{V}(\mathfrak{a})}X \longrightarrow \Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r \rightsquigarrow,$$

and localize it at $\mathfrak{p}$ to get the triangle

$$(\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}} \xrightarrow{\;r\;} (\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}} \longrightarrow (\Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r)_{\mathfrak{p}} \rightsquigarrow.$$

By the induction assumption, for $0 < i \le t-1$, $\dim_R \mathrm{H}^{s+i}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r) \le t-1-i$. So since $\mathrm{ht}_{\mathrm{H}^*_C(X/\!/r)}\,\mathfrak{p} < i-1$, $\mathfrak{p} \notin \mathrm{Supp}_R \mathrm{H}^{s+i}_C(\Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r)$. So the first term of the induced exact sequence

$$\mathrm{H}^{s+i-1}_C\big((\Gamma_{\mathcal{V}(\mathfrak{a})}X/\!/r)_{\mathfrak{p}}\big) \longrightarrow \mathrm{H}^{s+i}_C\big((\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}}\big) \xrightarrow{\;r\;} \mathrm{H}^{s+i}_C\big((\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}}\big)$$

of cohomology modules vanishes. But since $r \in \mathfrak{a} \subseteq \mathfrak{p}$ and any element of $\mathrm{H}^{s+i}_C((\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}})$ vanishes by a power of $\mathfrak{a}$, we deduce that $\mathrm{H}^{s+i}_C((\Gamma_{\mathcal{V}(\mathfrak{a})}X)_{\mathfrak{p}}) = 0$. This completes the induction step and hence the proof. □

Local cohomology in $\mathbb{D}(R)$

Let R be a commutative Noetherian ring, and let $\mathcal{T} = \mathbb{D}(R)$ be the derived category of complexes of R-modules. It is known that $\mathcal{T}$ is a compactly generated triangulated category with compact object R, considered as a complex concentrated in degree zero.

It is shown in [1, Theorem 9.1] that for any specialization closed subset ϑ of $\operatorname{Spec}(R)$, the local cohomology functor Γ_ϑ is equivalent to the right derived functor $\mathbf{R}F_\vartheta : \mathcal{T} \longrightarrow \mathcal{T}$ of F_ϑ, where for any R-module M, $F_\vartheta M$ is the kernel of the morphism $M \longrightarrow \prod_{\mathfrak{q} \notin \vartheta} M_\mathfrak{q}$ (for details on local cohomology in this case, see [4], [6]). Using this, the above theorem interprets to the following result related to the usual local cohomology functors. These two results have been known for a while (see, e.g., [3]).

COROLLARY 5.7

For any cohomologically finite complex X in $\mathbb{D}(R)$ and any integer i, there is an inequality

$$\dim_R \mathrm{H}^i\big(\mathbf{R}F_{\mathcal{V}(\mathfrak{a})}(X)\big)\big) + i \leq \dim_R X + \sup_R X.$$

In the special case when we consider the R-module M as a complex concentrated in degree zero, we have the following.

COROLLARY 5.8

Let M be a finitely generated R-module. Then for each $i \geq 0$, there is an inequality

$$\dim_R \mathrm{H}^i_\mathfrak{a}(M) + i \leq \dim_R M.$$

Acknowledgments. We would like to thank Srikanth Iyengar and Henning Krause for reading the first draft of the manuscript and for their useful hints. We also thank the referees for their comments on the paper that improved our exposition. The first and second authors thank the Center of Excellence for Mathematics (University of Isfahan).

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