Multiplicity bounds in graded rings

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To the memory of Professor Masayoshi Nagata and to his many wonderful contributions to mathematics

Abstract The F-threshold $c^J(\mathfrak{a})$ of an ideal $\mathfrak{a}$ with respect to an ideal J is a positive characteristic invariant obtained by comparing the powers of $\mathfrak{a}$ with the Frobenius powers of J. We study a conjecture formulated in an earlier article that we authored with M. Mustaţă, which bounds $c^J(\mathfrak{a})$ in terms of the multiplicities $e(\mathfrak{a})$ and $e(J)$ when $\mathfrak{a}$ and J are zero-dimensional ideals and J is generated by a system of parameters. We prove the conjecture when $\mathfrak{a}$ and J are generated by homogeneous systems of parameters in a Noetherian graded k-algebra. We also prove a similar inequality involving, instead of the F-threshold, the jumping number for the generalized parameter test submodules.

0. Introduction

Let R be a Noetherian ring of prime characteristic p. For every ideal $\mathfrak{a}$ in R and for every ideal J whose radical contains $\mathfrak{a}$, one can define asymptotic invariants that measure the containment of the powers of $\mathfrak{a}$ in the Frobenius powers of J. These invariants, dubbed F-thresholds, were introduced in the case of a regular local ring in [MTW] and in full generality in the article [HMTW]. In this article we work in the general setting.

A conjecture was made in [HMTW] which connects F-thresholds with the multiplicities of the ideals $\mathfrak{a}$ and J (see Conjecture 2.1 below). A second question was stated in the same article which does not explicitly refer to F-thresholds but which implies Conjecture 2.1. This second question is easy to state.

QUESTION 0.1

Suppose that $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals in a d-dimensional Noetherian local or Noetherian graded k-algebra $(R, \mathfrak{m})$, where k is a field of arbitrary characteristic.

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Further, suppose that J is generated by a system of parameters (homogeneous in the graded case). If $\mathfrak{a}^{N+1} \subseteq J$ for some integer $N \geq 0$, then does the inequality

$$e(\mathfrak{a}) \geq \left(\frac{d}{d+N}\right)^d e(J)$$

hold? Here $e(K)$ denotes the multiplicity of the ideal K.

In [HMTW, Theorem 5.8] this question was answered in the affirmative if R is a Cohen-Macaulay graded ring and $\mathfrak{a}$ is generated by a homogeneous system of parameters. In this article we generalize this result to the case of arbitrary graded rings.

THEOREM 0.2 (THEOREM 2.8)
The answer to Question 0.1 is yes if R is a graded ring and $\mathfrak{a}$ is generated by a homogeneous system of parameters for R.

In fact, we prove that if R is in addition a domain of positive characteristic, then the power N in the statement of Question 0.1 can be changed to be the least integer $N \geq 0$ such that $\mathfrak{a}^{N+1} \subseteq J^{+\mathrm{gr}}$, where $J^{+\mathrm{gr}}$ is a certain graded plus closure of the ideal J (see Discussion 2.11). This result not only removes the Cohen-Macaulay assumption on R but even strengthens the previous result in the case when R is a Cohen-Macaulay graded domain of positive characteristic. The proof of Theorem 0.2 uses reduction to characteristic $p > 0$ and takes advantage of the fact that the graded plus closure of a graded Noetherian domain in positive characteristic is a big Cohen-Macaulay algebra (see [HH1]).

Another ingredient of this article is a comparison of F-thresholds and F-jumping numbers, jumping numbers for the generalized parameter test submodules of Schwede and Takagi [ST] (see Definition 4.3 for their definition). In [MTW, Proposition 2.7] and [HMTW, Remark 2.5], it was shown that those two invariants coincide with each other in cases when the ring is regular but not in general. In this article, we show that if the ideal J is generated by a full system of parameters and if the ring is F-rational away from $V(\mathfrak{a})$, then the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ coincides with the F-threshold $c^J(\mathfrak{a})$. Also, when $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters in a Noetherian graded domain R over a field of positive characteristic, we prove an inequality similar to that of the original conjecture in [HMTW] (Conjecture 2.1), involving the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ instead of the F-threshold $c^J(\mathfrak{a})$.

The article is structured as follows. In Section 1 we recall some notions needed throughout the rest of the article and introduce basic facts about F-thresholds. We prove their existence in a new case in Theorem 1.6. In Section 2 we prove our main result, Theorem 2.8, in positive characteristic, and in Section 3 we prove the case of characteristic zero. In Section 4 we compare F-thresholds and F-jumping numbers. We give a lower bound on the F-jumping number in terms of multiplicities in Corollary 4.7.

1. Preliminaries and basic results

In this section we review some definitions and notation that are used throughout the article and state and prove basic results on F-thresholds. All rings are Noetherian commutative rings with unity unless explicitly stated otherwise. A particularly important exception is the graded plus closure of a graded Noetherian domain, which essentially is never Noetherian. For a ring R, we denote by R° the set of elements of R that are not contained in any minimal prime ideal. Elements $x_1, \dots, x_r$ in R are called *parameters* if they generate an ideal of height r.

For a real number u, we denote by $\lfloor u \rfloor$ the largest integer $\le u$ and by $\lceil u \rceil$ the smallest integer $\ge u$.

DEFINITION 1.1

Let R be a ring of prime characteristic p.

(i) We always let $q = p^e$ denote a power of p. If I is an ideal of R, then $I^{[q]}$ is the ideal generated by the set of all i^q for $i \in I$. The *Frobenius closure* I^F of I is defined as the ideal of R consisting of all elements $x \in R$ such that $x^q \in I^{[q]}$ for some $q = p^e$.

(ii) R is *F-finite* if the Frobenius map $R \to R$ sending r to r^p is a finite map.

(iii) R is *F-pure* if the Frobenius map $R \to R$ is pure.

Let R be a Noetherian ring of dimension d and of characteristic $p > 0$. Let $\mathfrak{a}$ be a fixed proper ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. To each ideal J of R such that $\mathfrak{a} \subseteq \sqrt{J}$ we associate the F-threshold $c^J(\mathfrak{a})$ as follows. For every $q = p^e$, let

$$\nu_{\mathfrak{a}}^J(q) := \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq J^{[q]}\}.$$

These numbers can be thought of as characteristic p analogues of Samuel's asymptotic function (for example, see [SH, Corollary 6.9.1]). Since $\mathfrak{a} \subseteq \sqrt{J}$, these are nonnegative integers. (If $\mathfrak{a} \subseteq J^{[q]}$, then we put $\nu_{\mathfrak{a}}^J(q) = 0$.) We put

$$c_+^J(\mathfrak{a}) = \limsup_{q \to \infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}, \qquad c_-^J(\mathfrak{a}) = \liminf_{q \to \infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}.$$

When $c_+^J(\mathfrak{a}) = c_-^J(\mathfrak{a})$, we call this limit the *F-threshold* of the pair $(R, \mathfrak{a})$ (or simply of $\mathfrak{a}$) with respect to J, and we denote it by $c^J(\mathfrak{a})$. We refer to [BMS1], [BMS2], [HM], [HMTW], and [MTW] for further information.

REMARK 1.2 (CF. [MTW, REMARK 1.2])

One has

$$0 \le c_-^J(\mathfrak{a}) \le c_+^J(\mathfrak{a}) < \infty.$$

REMARK 1.3 ([HMTW, PROPOSITION 2.2])

Let $\mathfrak{a}$, J be ideals as above.

(1) If $I \supseteq J$, then $c_{\pm}^{I}(\mathfrak{a}) \le c_{\pm}^{J}(\mathfrak{a})$.

(2) If $\mathfrak{b} \subseteq \mathfrak{a}$, then $c_{\pm}^{J}(\mathfrak{b}) \le c_{\pm}^{J}(\mathfrak{a})$. Moreover, if $\mathfrak{a} \subseteq \overline{\mathfrak{b}}$, then $c_{\pm}^{J}(\mathfrak{b}) = c_{\pm}^{J}(\mathfrak{a})$. Here $\overline{\mathfrak{b}}$ denotes the integral closure of $\mathfrak{b}$.

(3) We have $c_{\pm}^{I}(\mathfrak{a}^r) = (1/r)c_{\pm}^{I}(\mathfrak{a})$ for every integer $r \ge 1$.

(4) We have $c_{\pm}^{J^{[q]}}(\mathfrak{a}) = qc_{\pm}^{J}(\mathfrak{a})$ for every $q = p^e$.

(5) We have $c_{+}^{J}(\mathfrak{a}) \le c$ (resp., $c_{-}^{J}(\mathfrak{a}) \ge c$) if and only if for every power q_0 of p, we have $\mathfrak{a}^{\lceil cq \rceil + q/q_0} \subseteq J^{[q]}$ (resp., $\mathfrak{a}^{\lceil cq \rceil - q/q_0} \not\subseteq J^{[q]}$) for all $q = p^e \gg q_0$.

The F-threshold $c^J(\mathfrak{a})$ exists in many cases.

LEMMA 1.4 ([HMTW, LEMMA 2.3])

Let $\mathfrak{a}$, J be as above.

(1) *If $J^{[q]} = (J^{[q]})^F$ for all large $q = p^e$, then the F-threshold $c^J(\mathfrak{a})$ exists; that is, $c_{+}^{J}(\mathfrak{a}) = c_{-}^{J}(\mathfrak{a})$. In particular, if R is F-pure, then $c^J(\mathfrak{a})$ exists.*

(2) *If $\mathfrak{a}$ is principal, then $c^J(\mathfrak{a})$ exists.*

REMARK 1.5

Let $\mathfrak{a}$, J be as above. Set $R_{\mathrm{red}} := R/\sqrt{(0)}$, and let $\overline{\mathfrak{a}}, \overline{J}$ be the images of $\mathfrak{a}$ and J in R_{red}, respectively. Let μ be the least number of generators for the ideal $\mathfrak{a}$, and let q' be a power of p such that $\sqrt{(0)}^{[q']} = 0$. Then for all $q = p^e$, $\nu_{\overline{\mathfrak{a}}}^{\overline{J}}(qq')/qq' \le \nu_{\mathfrak{a}}^{J}(qq')/qq' \le \nu_{\overline{\mathfrak{a}}}^{\overline{J}}(q)/q + \mu/q$. In particular, if the F-threshold $c^{\overline{J}}(\overline{\mathfrak{a}})$ exists, then the F-threshold $c^J(\mathfrak{a})$ also exists.

Our first new result is that the F-threshold exists more generally when the ring is F-pure away from $V(\mathfrak{a})$.

THEOREM 1.6

Let $(R, \mathfrak{m})$ be a local F-finite Noetherian ring of characteristic p, and let $\mathfrak{a}$ and J be ideals such that the radical of J contains $\mathfrak{a}$. Assume that R_P is F-pure for all primes P which do not contain $\mathfrak{a}$. Then the F-threshold $c^J(\mathfrak{a})$ exists, that is, $c_{+}^{J}(\mathfrak{a}) = c_{-}^{J}(\mathfrak{a})$.

Proof

We use the following notation. ${}^eR$ is R thought of as an R-module via the eth iterate of the Frobenius map F. Thus the R-module structure on ${}^eR$ is given by $r \cdot s = r^q s$ for $r \in R$, $s \in {}^eR$, and $q = p^e$.

By Remark 1.5, we may assume that R is reduced. The map from R_P to ${}^1(R_P)$ is split for every prime P not containing $\mathfrak{a}$. Since the Frobenius map commutes with localization and R is F-finite, there exists an R-homomorphism $f_P : {}^1R \to R$ such that $f_P(1) = u_P \notin P$. By [HH1, Lemma 6.21], for each $e \ge 1$ there exists an R-linear map ${}^eR \to R$ taking 1 to u_P^2. (In [HH1, Lemma 6.21] it is assumed that the element $u_P \in R^o$, but that is not used in the proof of this lemma.)

Let I be the ideal generated by the set of u_P^2 for P ranging over all prime ideals not containing $\mathfrak{a}$. We claim that $I(J^{[q]})^F \subseteq J^{[q]}$ for all $q = p^e$. Suppose that $r \in (J^{[q]})^F$. Then there exists a power $q' = p^{e'}$ of p such that $r^{q'} \in J^{[qq']}$, and so $r \in J^{[q]}({}^{e'}R)$. For each prime P not containing $\mathfrak{a}$, there is an R-linear map $g_P : {}^{e'}R \to R$ such that $g_P(1) = u_P^2 \notin P$. Then $ru_P^2 = g_P(r \cdot 1) \in g_P(J^{[q]}({}^{e'}R)) \subseteq J^{[q]}$, showing that $Ir \subseteq J^{[q]}$ as claimed.

Since $u_P \notin P$, I is not contained in any prime P which does not contain $\mathfrak{a}$. Hence $\mathfrak{a} \subseteq \sqrt{I}$, and there exists an integer k such that $\mathfrak{a}^k \subseteq I$. We claim that for all powers of p, q, and Q,

$$\frac{v_{\mathfrak{a}}^J(qQ)+1}{qQ} \geq \frac{v_{\mathfrak{a}}^J(q)}{q} - \frac{k}{q}.$$

Since the set of values $\{(v_{\mathfrak{a}}^J(q))/q\}$ is bounded above, this implies that the limit exists. To prove this claim, fix Q, and write $\lceil (v_{\mathfrak{a}}^J(qQ)+1)/Q \rceil = a$. Then

$$(\mathfrak{a}^a)^{[Q]} \subseteq \mathfrak{a}^{aQ} \subseteq \mathfrak{a}^{v_{\mathfrak{a}}^J(qQ)+1} \subseteq J^{[qQ]},$$

where the last containment follows from the definition of $v_{\mathfrak{a}}^J(qQ)$. Hence $\mathfrak{a}^a \subseteq (J^{[q]})^F$, and the work above shows that $\mathfrak{a}^{a+k} \subseteq I(J^{[q]})^F \subseteq J^{[q]}$. It follows that $v_{\mathfrak{a}}^J(q) \leq a + k - 1$. Since $a \leq \big((v_{\mathfrak{a}}^J(qQ)+1)/Q\big) + 1$, dividing by q gives the required inequality and finishes the proof. $\square$

2. Connections between F-thresholds and multiplicity

The following conjecture was proposed in [HMTW, Conjecture 5.1].

CONJECTURE 2.1

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic $p > 0$. If $J \subseteq \mathfrak{m}$ is an ideal generated by a full system of parameters, and if $\mathfrak{a} \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal, then

$$e(\mathfrak{a}) \geq \Big(\frac{d}{c_-^J(\mathfrak{a})}\Big)^d e(J).$$

Given an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in a regular local ring $(R, \mathfrak{m})$, essentially of finite type over a field of characteristic zero, de Fernex, Ein, and Mustaţă proved in [dFEM] an inequality involving the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ and the multiplicity $e(\mathfrak{a})$. Later, Takagi and Watanabe gave in [TW] a characteristic p-analogue of this result, replacing the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ by the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$. The conjecture above generalizes these inequalities.

We list some known facts.

REMARK 2.2

(1) The condition in Conjecture 2.1 that J is generated by a system of parameters is critical; the inequality can fail if we drop this condition (see [HMTW, Remark 5.2(c)]).

(2) If $(R,\mathfrak{m})$ is a one-dimensional analytically irreducible local domain of characteristic $p>0$, and if $\mathfrak{a}, J$ are $\mathfrak{m}$-primary ideals in R, then

$$c^J(\mathfrak{a}) = \frac{e(J)}{e(\mathfrak{a})}.$$

In particular, Conjecture 2.1 holds in R (see [HMTW, Proposition 5.5]).

(3) If $(R,\mathfrak{m})$ is a regular local ring of characteristic $p>0$ and $J=(x_1^{a_1},\dots, x_d^{a_d})$ with $x_1,\dots,x_d$ a full regular system of parameters for R, and with $a_1,\dots,a_d$ positive integers, then Conjecture 2.1 holds (see [HMTW, Theorem 5.6]).

(4) Let $R=\bigoplus_{n\geq 0} R_n$ be a d-dimensional graded Cohen-Macaulay ring with R_0 a field of characteristic $p>0$. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R, then

$$e(\mathfrak{a}) \geq \left(\frac{d}{c_-^J(\mathfrak{a})}\right)^d e(J)$$

(see [HMTW, Corollary 5.9]).

The next proposition shows that Conjecture 2.1 actually is equivalent to the special case of itself in which $c_-^J(\mathfrak{a}) \leq d$.

PROPOSITION 2.3

Let $(R,\mathfrak{m})$ be a d-dimensional formally equidimensional Noetherian local ring of characteristic $p>0$. Then the following are equivalent:

(1) *$e(\mathfrak{a}) \geq e(J)$ for all ideals $J\subseteq R$ generated by full systems of parameters and for all $\mathfrak{m}$-primary ideals $\mathfrak{a}\subseteq R$ with $c_-^J(\mathfrak{a})\leq d$;*

(2) *$e(\mathfrak{a}) \geq \big(d/(c_-^J(\mathfrak{a}))\big)^d e(J)$ for all ideals $J\subseteq R$ generated by full systems of parameters and for all $\mathfrak{m}$-primary ideals $\mathfrak{a}\subseteq R$.*

Proof

First assume (1). Given any positive $\epsilon>0$, choose n and $q=p^e$ so that $d-\epsilon \leq \frac{q}{n} c_-^J(\mathfrak{a}) \leq d$. Since $\frac{q}{n}c_-^J(\mathfrak{a}) = c_-^{J^{[q]}}(\mathfrak{a}^n)$ by Remark 1.3(3), (4), we get

$$c_-^{J^{[q]}}(\mathfrak{a}^n) \leq d,$$

and then by (1), we obtain $n^d e(\mathfrak{a}) = e(\mathfrak{a}^n) \geq e(J^{[q]}) = q^d e(J)$. Thus we have shown that for any $\epsilon>0$, $e(\mathfrak{a})/e(J) \geq ((d-\epsilon)/c_-^J(\mathfrak{a}))^d$, proving the inequality in (2).

On the other hand, assuming (2) immediately gives (1).* $\square$

*In [HMTW, Theorem 3.3.1], it is incorrectly claimed that $c_+^J(I)\leq d$ implies that I is integral over J (which in particular forces $e(I)\geq e(J)$). This is false. The mistake occurs in the following section of the proof: "$\dots c_+^J(I)\leq d$ implies that for all $q_0=p^{e_0}$ and for all large $q=p^e$, we have $I^{q(d+(1/q_0))}\subseteq J^{[q]}$. Hence $I^q J^{q(d-1+(1/q_0))}\subseteq J^{[q]},\dots$" This latter statement does not follow unless $J\subseteq I$.

REMARK 2.4

One can think of the condition (1) in Proposition 2.3 as a converse to the main point of the tight closure Briançon-Skoda theorem ([HH1, Theorem 5.4]). Namely, suppose that we are in the special case in which $J \subseteq \mathfrak{a}$. If $e(\mathfrak{a}) \geq e(J)$, then $\mathfrak{a}$ is integral over J by a theorem of Rees. In this case there is a constant l such that for all $q = p^e$, $\mathfrak{a}^{qd+l} \subseteq J^{[q]}$. Hence $c_-^J(\mathfrak{a}) \leq d$. What (1) is claiming in this case is that the converse holds. If $c_-^J(\mathfrak{a}) \leq d$, then $e(\mathfrak{a}) \geq e(J)$, and therefore $\mathfrak{a}$ is integral over J (assuming as above that $J \subseteq \mathfrak{a}$). This is interesting as it shows that integrality over parameter ideals can be detected at the level of Frobenius powers.

REMARK 2.5

In [HMTW], another problem was raised which does not depend upon the characteristic and which implies the conjecture above. We state it here as a conjecture.

CONJECTURE 2.6

Suppose that $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals in a d-dimensional Noetherian local or Noetherian graded k-algebra $(R, \mathfrak{m})$, where k is a field of arbitrary *characteristic. Further, assume that J is generated by a full system of parameters (homogeneous in the graded case). If $\mathfrak{a}^{N+1} \subseteq J$ for some integer $N \geq 0$, then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{d+N}\right)^d e(J).$$

REMARK 2.7

Conjecture 2.6 reduces to the domain case as follows. Let $P_1, \ldots, P_l$ be the minimal primes of R such that the dimension of R/P_i is equal to the dimension of R. Write $\mathfrak{a}_i = (\mathfrak{a} + P_i)/P_i$, $J_i = (J + P_i)/P_i$. Suppose that the conjecture holds in each R/P_i. The ideal J_i is generated by parameters in R/P_i since the dimension of R/P_i is d. Moreover, if $\mathfrak{a}^{N+1} \subseteq J$, then $\mathfrak{a}_i^{N+1} \subseteq J_i$ for each i. Hence $e(\mathfrak{a}_i) \geq (d/(d+N))^d e(J_i)$. By [SH, Theorem 11.2.4], we then have

$$e(\mathfrak{a}) = \sum_i e(\mathfrak{a}_i) l_R(R_{P_i}) \geq \sum_i \left(\frac{d}{d+N}\right)^d e(J_i) l_R(R_{P_i}) = \left(\frac{d}{d+N}\right)^d e(J).$$

THEOREM 2.8

Let $R = \bigoplus_{n \geq 0} R_n$ be a d-dimensional Noetherian graded ring with R_0 a field of arbitrary characteristic. Suppose that $\mathfrak{a}$ (resp., J) is an ideal generated by a full homogeneous system of parameters of degrees $a_1 \leq \cdots \leq a_d$ (resp., $b_1 \leq \cdots \leq b_d$) for R. If $\mathfrak{a}^{N+1} \subseteq J$ for some integer $N \geq 0$, then

$$e(\mathfrak{a}) \geq \left(\frac{d}{d+N}\right)^d e(J).$$

If the equality holds in the above, then $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ are proportional, that is, $a_1/b_1 = \cdots = a_d/b_d$.

As in the proof of [HMTW, Corollary 5.9] we can immediately obtain that the first conjecture holds in this case as well.

COROLLARY 2.9

Let $R = \bigoplus_{n\geq 0} R_n$ be a d-dimensional Noetherian graded ring with R_0 a field of characteristic $p > 0$. Suppose that $\mathfrak{a}$ (resp., J) is an ideal generated by a full homogeneous system of parameters of degrees $a_1 \leq \cdots \leq a_d$ (resp., $b_1 \leq \cdots \leq b_d$) for R. Then

$$e(\mathfrak{a}) \geq \Big(\frac{d}{c_-^J(\mathfrak{a})}\Big)^d e(J).$$

If the equality holds in the above, then $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ are proportional, that is, $a_1/b_1 = \cdots = a_d/b_d$.

Proof

The proof of the former assertion is exactly as in the proof of [HMTW, Corollary 5.9], but we repeat it here for the convenience of the reader since it is quite short.

Note that each $J^{[q]}$ is again generated by a full homogeneous system of parameters. It follows from the theorem and from the definition of $\nu_{\mathfrak{a}}^J(q)$ that for every $q = p^e$ we have

$$e(\mathfrak{a}) \geq \Big(\frac{d}{d+\nu_{\mathfrak{a}}^J(q)}\Big)^d e(J^{[q]}) = \Big(\frac{qd}{d+\nu_{\mathfrak{a}}^J(q)}\Big)^d e(J).$$

On the right-hand side we can take a subsequence converging to $\big(d/(c_-^J(\mathfrak{a}))\big)^d e(J)$; hence, we get the inequality in the corollary.

For the latter assertion, we postpone the proof to Remark 2.16. $\square$

The proof of the main theorem is in two parts. First we prove it in characteristic p in this section, and then we do the characteristic zero case in Section 3, by reducing to characteristic p. Before beginning the proof of the main theorem in positive characteristic, there are several preliminary results we need to recall as well as some new results on multiplicity which we need.

An important point for us is that the multiplicity of an ideal generated by a full homogeneous system of parameters is determined only by the degrees of the parameters up to a constant which does not depend on the parameters themselves. This is well known in case the ring is Cohen-Macaulay, and a generalization to the non-Cohen-Macaulay case can be found in [To, Lemma 1.5]. We give here a more general statement with a short proof.

PROPOSITION 2.10

Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring of dimension d over an Artinian local ring $R_0 = (A, \mathfrak{m})$, and let $\mathfrak{M} = \mathfrak{m}R + R_+$ be the unique homogeneous maximal ideal of R. Let J be an ideal generated by a homogeneous system of parameters

$f_1, \ldots, f_d$ for R. Let

$$P(R,t) = \sum_{n \geq 0} l_A(R_n) t^n$$

be the Poincaré series of R. Then the multiplicity $e(J)$ of J is given by

$$e(J) = \deg f_1 \cdots \deg f_d \lim_{t \to 1} (1-t)^d P(R,t).$$

Proof

We compute the multiplicity $e(J)$ by using Koszul homology of J.* By a theorem of Auslander and Buchsbaum [AuB, Theorem 4.1] one has $e(J) = \chi(J)$, where $\chi(J)$ denotes the Euler characteristic of the Koszul complex $K_\bullet((f_1, \ldots, f_d), R)$.

Now let $K_i = K_i((f_1, \ldots, f_d), R)$ (resp., $H_i = H_i((f_1, \ldots, f_d), R)$) be the component of degree i (resp., ith homology module) of the Koszul complex considered as graded R-modules, and put $\deg f_i = b_i$ for each $1 \leq i \leq d$. Then the assertion follows from the theorem of Auslander and Buchsbaum and the following equalities:

$$\begin{aligned}\chi(J) = \lim_{t \to 1} \sum_{i \geq 0} (-1)^i P(H_i, t) &= \lim_{t \to 1} \sum_{i \geq 0} (-1)^i P(K_i, t) \\ &= \lim_{t \to 1} \prod_{i=1}^{d} (1 - t^{b_i}) P(R,t) \\ &= b_1 \cdots b_d \lim_{t \to 1} (1-t)^d P(R,t).\end{aligned}$$

□

Note that if R is standard graded over a field, then the value $\lim_{t \to 1}(1-t)^d P(R,t)$ can be computed from this theorem by choosing a homogeneous system of parameters of degree 1. This means all $b_i = 1$, and one sees that $\lim_{t \to 1}(1-t)^d \times P(R,t) = e(\mathfrak{M})$, the multiplicity of the irrelevant ideal.

DISCUSSION 2.11

We need some of the results of the article [HH2], which we discuss here. Let R be an $\mathbb{N}$-graded Noetherian domain over a field $R_0 = k$ of positive characteristic p. Let Ω be an algebraic closure of the fraction field of R, and let R^+ be the integral closure of R in Ω. We refer to an element $\theta \in \Omega \setminus \{0\}$ as a homogeneous element of Ω if θ is a root of a nonzero polynomial $F(X) \in R[X]$ such that X can be assigned a degree in $\mathbb{Q}$ making F homogeneous. By [HH2, Lemma 4.1], this condition is equivalent to saying that the grading on R (uniquely) extends to a grading on $R[\theta]$ indexed by $\mathbb{Q}$. In this way (see [HH2, Lemma 4.1] for the detail), the homogeneous elements in Ω span a domain graded by $\mathbb{Q}$, extending the grading

*We thank Kazuhiko Kurano for advising us to use the Koszul complex, which makes the proof much simpler.

on R. The homogeneous elements of R^+ with degrees in $\mathbb{N}$ span a subring of R^+ graded by $\mathbb{N}$. This ring is denoted by $R^{+\mathrm{gr}}$. The inclusion of $R \subset R^{+\mathrm{gr}}$ is a graded inclusion of degree zero. For an ideal $I \subseteq R$, we let $I^{+\mathrm{gr}} = IR^{+\mathrm{gr}} \cap R$. Note that by the grading of $R^{+\mathrm{gr}}$, $J^{+\mathrm{gr}} \neq R^{+\mathrm{gr}}$ for every proper homogeneous ideal J of R. Part of the main theorem of [HH2], Theorem 5.15, gives the following statement.

THEOREM 2.12

Let R be a Noetherian $\mathbb{N}$-graded domain over a field $R_0 = k$ of characteristic $p > 0$. Every homogeneous system of parameters for R is a regular sequence in $R^{+\mathrm{gr}}$.

Again, let R be a Noetherian $\mathbb{N}$-graded domain over a field $R_0 = k$ of characteristic $p > 0$, and let $x_1, \dots, x_d$ be homogeneous elements of R. Set $A = k[X_1, \dots, X_d]$. Consider the natural map $f : A \to R$ taking X_i to x_i. Let I be an ideal of R generated by monomials in $x_1, \dots, x_d$, and let $L \subseteq A$ be the corresponding ideal of monomials in A. Let $T_\bullet$ denote the Taylor resolution of A/L (see [Ei]).

The main fact we need is the following.

LEMMA 2.13

In the situation above, if $x_1, \dots, x_d$ are a homogeneous system of parameters for R, and if $S = R^{+\mathrm{gr}}$, then $T_\bullet \otimes_A S$ is exact. In fact, S is flat over A.

Proof

Since $T_\bullet$ is a free A-resolution of A/L, where L is generated by monomials, the first statement follows from the second statement. The flatness of S over A is proved as in the discussion in [HH2, 6.7]. See also [Hu, Theorem 9.1] and [Ho, proof of 2.3]. $\square$

DISCUSSION 2.14

One of the consequences of S being flat over A is that we can compute the quotient of monomial ideals in the x_i in S as if the monomials belong to A. Specifically, let I and I' be two ideals in R generated by monomials in $x_1, \dots, x_d$, which are a homogeneous system of parameters for R. Let L and L' be the corresponding ideals in A generated by the same monomials in $X_1, \dots, X_d$. Then $IS :_S I' = (L :_A L')S$. This follows at once from the flatness of S over A. Note that we *cannot* replace S by R in this equality unless R is Cohen-Macaulay.

We can now prove the following.

THEOREM 2.15

Let $R = \bigoplus_{n \geq 0} R_n$ be a d-dimensional Noetherian graded ring with R_0 a field of characteristic $p > 0$. Suppose that $\mathfrak{a}$ and J are ideals each generated by a full homogeneous system of parameters for R, and put a to be the least degree of *the generators of $\mathfrak{a}$. Let $c \in R^\circ$ be a homogeneous element of degree n such that $c(R/P_i) \not\subseteq (JR/P_i)^{+\mathrm{gr}}$ for all i, where the P_i range over the minimal primes of R with R/P_i having the same dimension as R. If $N \geq 0$ is an integer such that $c\mathfrak{a}^{N+1}(R/P_i) \subseteq (JR/P_i)^{+\mathrm{gr}}$ for all i, then*

$$e(\mathfrak{a}) \geq \Big(\frac{ad}{a(d+N)+n}\Big)^d e(J).$$

In particular, if $\mathfrak{a}^{N+1}(R/P_i) \subseteq (JR/P_i)^{+\mathrm{gr}}$ for all i, then

$$e(\mathfrak{a}) \geq \Big(\frac{d}{d+N}\Big)^d e(J).$$

Proof
By Remark 2.7, we may assume without loss of generality that R is a domain. Throughout the proof, we set $S = R^{+\mathrm{gr}}$.

We chiefly follow the same proof that was given in [HMTW, Theorem 5.8], making suitable modifications to take advantage of the fact that S is a big Cohen-Macaulay algebra for R; however, we have a simplification at the end of the proof.

Suppose that $\mathfrak{a}$ is generated by a full homogeneous system of parameters $x_1, \ldots, x_d$ of degrees $a_1 \leq \cdots \leq a_d$, and suppose that J is generated by another homogeneous system of parameters $f_1, \ldots, f_d$ of degrees $b_1 \leq \cdots \leq b_d$.

To prove the theorem it suffices to show that*

$$\Big(N + d + \frac{n}{a_1}\Big)^d a_1 \ldots a_d \geq d^d b_1 \ldots b_d.$$

That this inequality is enough to prove the theorem follows from Proposition 2.10 since $\frac{e(\mathfrak{a})}{a_1 \cdots a_d} = \frac{e(J)}{b_1 \cdots b_d}$ (this common ratio is $\lim_{t \to 1}(1-t)^d P(R,t)$ where $P(R,t)$ is the Poincaré series of R).

Define positive integers $t_1, \ldots, t_d$ inductively as follows: t_1 is the smallest integer t such that $cx_1^t \in J^{+\mathrm{gr}} (= JS \cap R)$. If $2 \leq i \leq d$, then t_i is the smallest integer t such that $cx_1^{t_1-1} \cdots x_{i-1}^{t_{i-1}-1} x_i^t \in J^{+\mathrm{gr}}$.

We show the following inequality for every $i = 1, \ldots, d$:

$$t_1 a_1 + \cdots + t_i a_i + n \geq b_1 + \cdots + b_i. \tag{1}$$

Let I_i be the ideal of R generated by $x_1^{t_1}, x_1^{t_1-1}x_2^{t_2}, \ldots, x_1^{t_1-1} \cdots x_{i-1}^{t_{i-1}-1} x_i^{t_i}$ for each $1 \leq i \leq d$. We let L_i be the corresponding ideal of monomials in A, where $A = k[X_1, \ldots, X_d]$ is as in the discussion above. Note that the definition of the integers t_j implies that $cI_iS \subseteq JS$. We use the Taylor resolution $T_\bullet$ for A/L_i. After tensoring with S, this complex is exact by Lemma 2.13. The multiplication map by c from S/I_iS into S/JS induces a comparison map of degree n between $T_\bullet \otimes_A S$ and the Koszul complex on the generators $f_1, \ldots, f_d$ of J. This Koszul complex is acyclic since the f_i form a regular sequence in S by

*In the proof of [HMTW, Theorem 5.8], it is claimed that in the Cohen-Macaulay case the multiplicity of $x_1, \ldots, x_d$ (resp., $f_1, \ldots, f_d$) is $a_1 \ldots a_d$ (resp., $b_1 \ldots b_d$). This is not correct; however, in that article we used only $e(x_1, \ldots, x_d)/a_1 \ldots a_d = e(f_1, \ldots, f_d)/b_1 \ldots b_d$. That this is true is easy to see in the Cohen-Macaulay case and follows more generally by Proposition 2.10.

Theorem 2.12. Note that the ith step in the Taylor complex $T_\bullet$ for the monomials $X_1^{t_1}, X_1^{t_1-1}X_2^{t_2}, \ldots, X_1^{t_1-1}\cdots X_{i-1}^{t_{i-1}-1}X_i^{t_i}$ in A is a free module of rank one with a generator corresponding to the monomial

$$\mathrm{lcm}(X_1^{t_1}, X_1^{t_1-1}X_2^{t_2}, \ldots, X_1^{t_1-1}\cdots X_{i-1}^{t_{i-1}}X_i^{t_i}) = X_1^{t_1}\cdots X_{i-1}^{t_{i-1}}X_i^{t_i}$$

(see [Ei, Exercise 17.11]). It follows that the map of degree n between the ith steps in the resolutions of S/I_iS and S/JS is of the form

$$S(-t_1a_1 - \cdots - t_ia_i) \to \bigoplus_{1\le v_1<\cdots<v_i\le d} S(-b_{v_1} - \cdots - b_{v_i}).$$

In particular, unless this map is zero, we have

$$t_1a_1 + \cdots + t_ia_i + n \ge \min_{1\le v_1<\cdots<v_i\le d}\{b_{v_1} + \cdots + b_{v_i}\} = b_1 + \cdots + b_i.$$

We now show that this map cannot be zero. If it is zero, then also the induced map

$$\mathrm{Tor}_i^S(S/I_iS, S/\mathfrak{b}_i) \to \mathrm{Tor}_i^S(S/JS, S/\mathfrak{b}_i) \tag{2}$$

is zero, where $\mathfrak{b}_i$ is the ideal in S generated by $x_1, \ldots, x_i$. On the other hand, using the Koszul resolution on $x_1, \ldots, x_i$ to compute the above Tor modules (these elements are a regular sequence in S), we see that the map (2) can be identified with the multiplication map by c,

$$(I_iS\colon {}_S\mathfrak{b}_i)/I_iS \xrightarrow{\times c} (JS\colon {}_S\mathfrak{b}_i)/JS.$$

Since $x_1^{t_1-1}\cdots x_i^{t_i-1} \in (I_iS\colon \mathfrak{b}_i)$, if the map in (2) is zero, then it follows that $cx_1^{t_1-1}\cdots x_i^{t_i-1}$ lies in $J^{+\mathrm{gr}}$, a contradiction. This proves (1).

To finish the proof, we use the following claim, which is a slight modification of the claim in the proof of [HMTW, Theorem 5.8].

CLAIM

Let $\alpha_i, \beta_i, \gamma_i$ be real numbers for $1 \le i \le d$, and let ω be a real number. If $1 = \gamma_1 \le \gamma_2 \le \cdots \le \gamma_d$ and if $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i + \omega \ge \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for all $i = 1, \ldots, d$, then $\alpha_1 + \cdots + \alpha_d + \omega \ge \beta_1 + \cdots + \beta_d$.

Proof

The proof is essentially the same as that of the claim in the proof of [HMTW, Theorem 5.8]. $\square$

As in [HMTW],* we now set $\alpha_i = t_i$, $\beta_i = b_i/a_i$ and $\gamma_i = a_i/a_1$ for $1 \le i \le d$. We put $\omega = n/a_1$. Since $a_1 \le \cdots \le a_d$, we deduce $1 = \gamma_1 \le \cdots \le \gamma_d$. Moreover, using (1), we obtain that $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i + \omega \ge \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for $1 \le i \le d$. Using

*The end of the proof is simpler here than in [HMTW], due to the fact that we are able to remove an argument by using linkage. This improvement comes from a suggestion by Hailong Dao, whom we thank.

the above claim, we conclude that

$$t_1+\cdots+t_d+\frac{n}{a_1}=\alpha_1+\cdots+\alpha_d+\omega\geq\beta_1+\cdots+\beta_d=\frac{b_1}{a_1}+\cdots+\frac{b_d}{a_d}.$$

The inductive definition of the t_i shows that the monomial $cx_1^{t_1-1}\cdots x_d^{t_d-1}\notin JS$. Since $c\mathfrak{a}^{N+1}\subseteq JS$, we see that $N+d\geq t_1+\cdots+t_d$. Comparing the arithmetic mean with the geometric mean of the b_i/a_i yields the conclusion that

$$\Big(N+d+\frac{n}{a_1}\Big)^d a_1\ldots a_d\geq d^d b_1\ldots b_d,$$

which finishes the proof of Theorem 2.15. □

REMARK 2.16

Let the notation be as in Theorem 2.8. Since we compare the arithmetic mean with the geometric mean of the b_i/a_i in the proof of Theorem 2.15, if equality holds in the inequality of Theorem 2.8, then $(a_1,\ldots,a_d)$ and $(b_1,\ldots,b_d)$ have to be proportional, that is, $b_1/a_1=\cdots=b_d/a_d$. We also remark that if equality holds in the inequality of Corollary 2.9, then $(a_1,\ldots,a_d)$ and $(b_1,\ldots,b_d)$ also have to be proportional. Suppose to the contrary that $(a_1,\ldots,a_d)$ and $(b_1,\ldots,b_d)$ are not proportional. In this case, the rational number

$$\epsilon:=\frac{((b_1/a_1)+\cdots+(b_d/a_d))^d a_1\cdots a_d}{d^d b_1\cdots b_d}-1$$

is strictly positive. For all $q=p^e$, one has

$$\Big(\frac{b_1q}{a_1}+\cdots+\frac{b_dq}{a_d}\Big)^d a_1\cdots a_d=(1+\epsilon)d^d(b_1q)\cdots(b_dq).$$

By the argument in the proof of Theorem 2.15, this implies that

$$e(\mathfrak{a})\geq(1+\epsilon)\Big(\frac{d}{\nu_{\mathfrak{a}}^J(q)+d}\Big)^d e(J^{[q]})=(1+\epsilon)\Big(\frac{qd}{\nu_{\mathfrak{a}}^J(q)+d}\Big)^d e(J)$$

for all $q=p^e$. Since ϵ is independent of q, we conclude that

$$e(\mathfrak{a})\geq(1+\epsilon)\Big(\frac{d}{c_-^J(\mathfrak{a})}\Big)^d e(J)\gneq\Big(\frac{d}{c_-^J(\mathfrak{a})}\Big)^d e(J).$$

REMARK 2.17

If equality holds in the inequality of Corollary 2.9, we think that $\mathfrak{a}$ and J are "equivalent" in the sense of the following conjecture, which is true if $d=2$ or, more generally, if the set $\{a_1,\ldots,a_d\}$ consists of at most two elements.

CONJECTURE 2.18

Let R, $\mathfrak{a}$, and J be as in Corollary 2.9. Suppose that

$$e(\mathfrak{a})=\Big(\frac{d}{c_-^J(\mathfrak{a})}\Big)^d e(J).$$

If we put $b_1/a_1=\cdots=b_d/a_d=t/s$, where s and t are positive integers, then $\mathfrak{a}^t$ and J^s have the same integral closure.

3. Main theorem in characteristic zero

In this section we outline the proof of Theorem 2.8 in characteristic zero. The reduction to characteristic p is fairly standard. We prove the following.

THEOREM 3.1

Let $R=\bigoplus_{n\geq 0} R_n$ be a d-dimensional Noetherian graded ring with $R_0=k$ a field of characteristic zero. Suppose that $\mathfrak{a}$ and J are ideals each generated by a full homogeneous system of parameters for R. If $\mathfrak{a}^{N+1}\subseteq J$ for some integer $N\geq 0$, then

$$e(\mathfrak{a})\geq\Big(\frac{d}{d+N}\Big)^d e(J).$$

Proof

Let $\mathfrak{a}$ be generated by the homogeneous parameters $x_1,\dots,x_d$, and let J be generated by the homogeneous parameters $f_1,\dots,f_d$. Without loss of generality we may assume that the degree of x_i is a_i with $a_1\leq a_2\cdots\leq a_d$ and, similarly, that the degree of f_i is b_i with $b_1\leq b_2\leq\cdots\leq b_d$. As in the proof of Theorem 2.15, it suffices to prove that

$$a_1\cdots a_d\geq\Big(\frac{d}{d+N}\Big)^d b_1\cdots b_d.$$

We begin by writing $R=k[t_1,\dots,t_n]\cong B/I$, where $B=k[T_1,\dots,T_n]$ is a graded polynomial ring with each T_i having a positive degree, such that I is homogeneous and the isomorphism of R with B/I is degree preserving, taking each T_i to the homogeneous algebra generator t_i of R. We lift each x_i to a homogeneous polynomial $h_i\in B$ and each f_i to a homogeneous polynomial $g_i\in B$. We also fix homogeneous generators $F_1,\dots,F_l$ of I. Furthermore, since the maximal homogeneous ideal of R has a power contained in the ideal $(x_1,\dots,x_d)$, there are equations in B which express this. A typical one, for example, would be of the form

$$T_j^M=\sum_{1\leq i\leq d} p_{ji}h_i+\sum_{1\leq i\leq l} q_{ji}F_i$$

for some fixed large power M.

Since the maximal homogeneous ideal of R also has a power contained in the ideal $(f_1,\dots,f_d)$, there is another set of equations expressing the fact that the T_j are nilpotent modulo the ideal I plus the ideal generated by $g_1,\dots,g_d$. There are also equations which express the fact that each monomial of total degree $N+1$ in the x_i is in J; these are expressed by equations which give that every monomial $h_1^{m_1}\cdots h_d^{m_d}$ in the h_i of total degree $N+1=m_1+\cdots+m_d$ is equal to an element in the ideal $I+(g_1,\dots,g_d)B$.

Now let A be a finitely generated $\mathbb{Z}$-subalgebra of k which has the coefficients of all polynomials in all the equations and defining ideals above. We let $B_A=A[T_1,\dots,T_n]$. We can assume that I_A is generated by $F_1,\dots,F_l\in B_A$. We let x_{iA} be the image of h_i in $R_A=B_A/I_A$, and we let f_{iA} be the image of g_i in the same ring. By the lemma on generic flatness, we can invert one element of A to make R_A free over A. (Here we replace A by the localization of A at that one element. It is still finitely generated over $\mathbb{Z}$.) Note that $R = R_A \otimes_A k$.

Choose any maximal ideal $\mathfrak{n}$ of A, and let $\kappa = A/\mathfrak{n}$, a finite field. We use κ as a subscript to denote images after tensoring over A with κ. The dimension d of R is equal to the dimension of R_κ, and R_κ is a positively graded Noetherian ring over the field κ. For the proofs and a discussion of the dimension of the fiber R_κ, see [Hu, Appendix, Section 2] or [HH3, Section 2.3].

The images of the x_{iA} and the f_{iA} in R_κ form homogeneous systems of parameters generating ideals $\mathfrak{a}_\kappa$ and J_κ, respectively, and furthermore, $\mathfrak{a}_\kappa^{N+1} \subseteq J_\kappa$. Moreover, the degrees of these elements are the same as the degrees of the corresponding elements in characteristic zero. By the main theorem in characteristic p (Theorem 2.15),

$$a_1 \cdots a_d \geq \left(\frac{d}{d+N}\right)^d b_1 \cdots b_d.$$

□

4. A comparison of F-thresholds and F-jumping numbers

In this section, we compare F-thresholds and jumping numbers for the generalized parameter test submodules, which were introduced by Schwede and Takagi [ST]. Throughout this section, we use the following notation.

NOTATION 4.1

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian excellent reduced ring of equal characteristic satisfying one of the following conditions:

(a) R is a complete local ring with the maximal ideal $\mathfrak{m}$;
(b) $R = \bigoplus_{n \geq 0} R_n$ is a graded ring with R_0 a field and $\mathfrak{m} = \bigoplus_{n \geq 1} R_n$.

Then R admits a (graded) *canonical module* ω_R: in case (a), ω_R is the finitely generated R-module $\mathrm{Hom}_R(H^d_\mathfrak{m}(R), E_R(R/\mathfrak{m}))$, where $E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$. In case (b), ω_R is the finitely generated graded R-module $\underline{\mathrm{Hom}}_{R_0}(H^d_\mathfrak{m}(R), R_0) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{R_0}([H^d_\mathfrak{m}(R)]_{-n}, R_0)$.

Also, in dealing with graded rings, we assume that all the ideals and systems of parameters considered are homogeneous.

First we recall the definition of a generalization of tight closure introduced by Hara and Yoshida [HY].

DEFINITION 4.2

Assume that R is a ring of characteristic $p > 0$. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number.

(i) For any ideal $I \subseteq R$, the $\mathfrak{a}^t$-*tight closure* I^* of I is defined to be the ideal of R consisting of all elements $x \in R$ for which there exists $c \in R^\circ$ such that

$$c\mathfrak{a}^{\lceil tq \rceil} x^q \subseteq I^{[q]}$$

for all large $q = p^e$. When $\mathfrak{a} = R$, we denote this ideal simply by I^* (see [HY, Definition 6.1]).

(ii) The $\mathfrak{a}^t$-*tight closure* $0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$ of the zero submodule in $H^d_{\mathfrak{m}}(R)$ is defined to be the submodule of $H^d_{\mathfrak{m}}(R)$ consisting of all elements $\xi \in H^d_{\mathfrak{m}}(R)$ for which there exists $c \in R^\circ$ such that

$$c\mathfrak{a}^{\lceil tq \rceil} \xi^q = 0$$

in $H^d_{\mathfrak{m}}(R)$ for all large $q = p^e$, where $\xi^q := F^e(\xi) \in H^d_{\mathfrak{m}}(R)$ denotes the image of ξ via the induced e-times iterated Frobenius map $F^e : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$. When $\mathfrak{a} = R$, we denote this submodule simply by $0^*_{H^d_{\mathfrak{m}}(R)}$ (see [HY, Definition 6.1]).

(iii) An element $c \in R^\circ$ is called a *parameter* $\mathfrak{a}^t$*-test element* if, for every ideal I generated by a system of parameters for R, one has $c\mathfrak{a}^{\lceil tq \rceil} x^q \subseteq I^{[q]}$ for all $q = p^e$ whenever $x \in I^{*\mathfrak{a}^t}$. When $\mathfrak{a} = R$, we call such an element simply a parameter test element (see [ST, Definition 6.6]).

(iv) R is said to be F*-rational* if $I^* = I$ for every ideal $I \subseteq R$ generated by a system of parameters for R. This is equivalent to saying that R is Cohen-Macaulay and $0^*_{H^d_{\mathfrak{m}}(R)} = 0$ (see [FW]).

Now we are ready to state the definitions of generalized parameter test submodules and their jumping numbers.

DEFINITION 4.3

Assume that R is a ring of characteristic $p > 0$, and let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$.

(i) For every real number $t \geq 0$, the *generalized parameter test submodule* $\tau(\omega_R, \mathfrak{a}^t)$ associated to the pair $(R, \mathfrak{a}^t)$ is defined to be

$$\tau(\omega_R, \mathfrak{a}^t) = \mathrm{Ann}_{\omega_R}(0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}) \subseteq \omega_R$$

(see [ST, Remark 6.4]).

(ii) For every ideal $J \subseteq R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, the F*-jumping number* $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ of $\mathfrak{a}$ with respect to $J\omega_R$ is defined to be

$$\mathrm{fjn}^J(\omega_R, \mathfrak{a}) = \inf\{t \geq 0 \mid \tau(\omega_R, \mathfrak{a}^t) \subseteq J\omega_R\}$$

(see [ST, Definition 7.9]).

REMARK 4.4

(1) For every real number $t \geq 0$, there exists $\varepsilon > 0$ such that $\tau(\omega_R, \mathfrak{a}^t) = \tau(\omega_R, \mathfrak{a}^{t+\varepsilon})$. In particular,

$$\mathrm{fjn}^J(\omega_R, \mathfrak{a}) = \min\{t \geq 0 \mid \tau(\omega_R, \mathfrak{a}^t) \subseteq J\omega_R\}$$

(see [ST, Lemma 7.10]).

(2) In the case when R is a regular ring, the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ coincides with the F-threshold $c^J(\mathfrak{a})$ for all ideals $\mathfrak{a}$ and J such that $\mathfrak{a} \subseteq \sqrt{J}$ (see [MTW, Proposition 2.7]).

(3) In general, the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ disagrees with the F-threshold $c^J(\mathfrak{a})$ (see [HMTW, Remark 2.5]).

If J is generated by a full system of parameters and if the ring is F-rational away from $V(\mathfrak{a})$, then the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ coincides with the F-threshold $c^J(\mathfrak{a})$.

THEOREM 4.5

Suppose that R is an integral domain of characteristic $p > 0$. Let $\mathfrak{a}$ be a nonzero ideal of R and J be an ideal generated by a full system of parameters for R. Assume in addition that R_P is F-rational for all prime ideals P not containing $\mathfrak{a}$. Then

$$\mathrm{fjn}^J(\omega_R, \mathfrak{a}) = c_+^J(\mathfrak{a}).$$

Proof

Suppose that J is generated by a full system of parameters $x_1, \dots, x_d$ for R. Let $\xi = [1/x_1 \cdots x_d] \in H^d_{\mathfrak{m}}(R)$. It is clear that $\xi \in (0 :_{H^d_{\mathfrak{m}}(R)} J\omega_R)$.

Suppose that $\tau(\omega_R, \mathfrak{a}^t) \subseteq J\omega_R$. Since $\xi \in (0 :_{H^d_{\mathfrak{m}}(R)} J\omega_R) \subseteq 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$, there exists $c \in R^\circ$ such that $c\mathfrak{a}^{\lceil tq \rceil}\xi^q = 0$ in $H^d_{\mathfrak{m}}(R)$ for all large $q = p^e$. This implies that there exists $s \in \mathbb{N}$ such that $c\mathfrak{a}^{\lceil tq \rceil}(x_1 \cdots x_d)^s \subseteq (x_1^{q+s}, \dots, x_d^{q+s})$. We then use the colon-capturing property of tight closure (see [HH1, Theorem 7.15(a)]) to have $c\mathfrak{a}^{\lceil tq \rceil} \subseteq (x_1^q, \dots, x_d^q)^*$. Hence, for any parameter test element $c' \in R^\circ$, one has $cc'\mathfrak{a}^{\lceil tq \rceil} \subseteq (x_1^q, \dots, x_d^q) = J^{[q]}$ for all large $q = p^e$, so that $1 \in J^{*\mathfrak{a}^t}$. By the definition of parameter $\mathfrak{a}^t$-test elements, $c''\mathfrak{a}^{\lceil tq \rceil} \subseteq J^{[q]}$ for all parameter $\mathfrak{a}^t$-test elements $c'' \in R^\circ$ and for all $q = p^e$. Since R_P is F-rational for all prime ideals P not containing $\mathfrak{a}$, the localized ring R_a is F-rational for every nonzero element $a \in \mathfrak{a}$. It then follows from [ST, Lemma 6.8] that there exists an integer $n \geq 1$ such that every nonzero element of $\mathfrak{a}^n$ is a parameter $\mathfrak{a}^t$-test element. Therefore $\mathfrak{a}^{\lceil tq \rceil + n} \subseteq J^{[q]}$; that is, $\nu^J_{\mathfrak{a}}(q) \leq \lceil tq \rceil + n - 1$ for all $q = p^e$. By dividing by q and taking the limit as q goes to the infinity, we have $t \geq c_+^J(\mathfrak{a})$. Thus, $\mathrm{fjn}^J(\omega_R, \mathfrak{a}) \geq c_+^J(\mathfrak{a})$.

To prove the converse inequality, suppose that $t > c_+^J(\mathfrak{a})$. Fix any $\eta \in (0 :_{H^d_{\mathfrak{m}}(R)} J\omega_R)$. Here note that $(0 :_{H^d_{\mathfrak{m}}(R)} J\omega_R) = (0 :_{H^d_{\mathfrak{m}}(R)} J)$, because $\omega_R \times H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(\omega_R)$ is the duality pairing. Hence $J^{[q]}\eta^q = 0$ for all $q = p^e$. Since $\mathfrak{a}^{\lceil tq \rceil} \subseteq J^{[q]}$ for all large $q = p^e$ by the definition of $c_+^J(\mathfrak{a})$, one has $\mathfrak{a}^{\lceil tq \rceil}\eta^q = 0$ for all large $q = p^e$; that is, $\eta \in 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$. Thus, $(0 :_{H^d_{\mathfrak{m}}(R)} J\omega_R) \subseteq 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$ or, equivalently, $J\omega_R \supseteq \tau(\omega_R, \mathfrak{a}^t)$. Taking the infimum of such t's, we conclude that $\mathrm{fjn}^J(\omega_R, \mathfrak{a}) \leq c_+^J(\mathfrak{a})$. $\square$

QUESTION 4.6

Does the equality in Theorem 4.5 still hold true if R is not F-rational away from $V(\mathfrak{a})$? We have seen in the proof of Theorem 4.5 that the inequality $\mathrm{fjn}^J(\omega_R, \mathfrak{a}) \leq c_+^J(\mathfrak{a})$ holds even in this case.

We can replace the F-threshold $c^J(\mathfrak{a})$ with the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ in the inequality of Corollary 2.9. The following is a corollary of Theorem 2.15.

COROLLARY 4.7

Let $R = \bigoplus_{n \geq 0} R_n$ be a d-dimensional Noetherian equidimensional reduced graded ring with R_0 a field of characteristic $p > 0$. Suppose that $\mathfrak{a}$ (resp., J) is an ideal generated by a full homogeneous system of parameters of degrees $a_1 \leq \cdots \leq a_d$ (resp., $b_1 \leq \cdots \leq b_d$) for R. Then

$$e(\mathfrak{a}) \geq \Bigl(\frac{d}{\mathrm{fjn}^J(\omega_R, \mathfrak{a})}\Bigr)^d e(J).$$

If the equality holds in the above inequality, then $(a_1, \dots, a_d)$ and $(b_1, \dots, b_d)$ are proportional; that is, $a_1/b_1 = \cdots = a_d/b_d$.

Proof

Suppose that $\tau(\omega_R, \mathfrak{a}^t) \subseteq J\omega_R$. By the same argument as the proof of Theorem 4.5, there exists a homogeneous element $c \in R^\circ$ of degree n such that $c\mathfrak{a}^{\lceil tq \rceil} \subseteq J^{[q]}$ for all $q = p^e$. Here note that each $J^{[q]}$ is again generated by a full homogeneous system of parameters for R. It follows from Theorem 2.15 that for all large $q = p^e$, we have

$$e(\mathfrak{a}) \geq \Bigl(\frac{a_1 d}{a_1(d + \lceil tq \rceil - 1) + n}\Bigr)^d e(J^{[q]}) = \Bigl(\frac{a_1 dq}{a_1(d + \lceil tq \rceil - 1) + n}\Bigr)^d e(J).$$

The right-hand side converges to $(d/t)^d e(J)$ as q goes to the infinity. Taking the infimum of such t's, we obtain the inequality in the corollary.

The latter assertion follows from an argument similar to Remark 2.16. $\square$

In general, $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ disagrees with the F-jumping number

$$\mathrm{fjn}^J(\mathfrak{a}) = \mathrm{fjn}^J(R, \mathfrak{a}) := \inf\bigl\{t \geq 0 \mid \tau(R, \mathfrak{a}^t) \subseteq J\bigr\}$$

of $\mathfrak{a}$ with respect to J. (See [HY] for the definition of the generalized test ideal $\tau(R, \mathfrak{a}^t)$).

EXAMPLE 4.8

Let $S = k[x, y]$ be the two-dimensional polynomial ring over an F-finite field k, and let $R = S^{(3)}$ be the third Veronese subring of S. Let $J = (x^3, y^3)$ be a parameter ideal of R, and let $\mathfrak{m}_S = (x, y)$ (resp., $\mathfrak{m}_R = (x^3, x^2y, xy^2, y^3)$) be a maximal ideal of S (resp., R). Note that

$$\tau(R, J^t) = \tau(R, \mathfrak{m}_R^t) = \tau(S, \mathfrak{m}_S^{3t}) \cap R = \mathfrak{m}_S^{\lfloor 3t \rfloor - 1} \cap R$$

for $t \geq 1/3$, where the second equality follows from [HT, Theorem 3.3]. Since $\mathfrak{m}_R^2 \subseteq J$, we have $\mathrm{fjn}^J(J) = 5/3 < 2 = c^J(J) = \mathrm{fjn}^J(\omega_R, \mathfrak{a})$. This example also shows that we cannot replace $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$ with $\mathrm{fjn}^J(\mathfrak{a})$ in Corollary 4.7.

We can define a geometric analogue of the F-jumping number $\mathrm{fjn}^J(\omega_R, \mathfrak{a})$.

DEFINITION 4.9

Assume that R is a normal domain essentially of finite type over a field of characteristic zero, and let $\mathfrak{a}$ be a nonzero ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$.

(i) Let $\pi : \widetilde{X} \to X := \operatorname{Spec} R$ be a log resolution of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$. For every real number $t \geq 0$, the *multiplier submodule* $\mathcal{J}(\omega_R, \mathfrak{a}^t)$ associated to the pair $(R, \mathfrak{a}^t)$ is defined to be

$$\mathcal{J}(\omega_R, \mathfrak{a}^t) = \pi_*\big(\omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(\lceil -tF \rceil)\big) \subseteq \omega_R$$

(see [Bl, Definition 2]).

(ii) For every ideal $J \subseteq R$ such that $\mathfrak{a} \subseteq \sqrt{J}$, the *jumping number* $\lambda^J(\omega_R, \mathfrak{a})$ of $\mathfrak{a}$ with respect to $J\omega_R$ is defined to be

$$\lambda^J(\omega_R, \mathfrak{a}) = \inf\big\{t \geq 0 \,\big|\, \mathcal{J}(\omega_R, \mathfrak{a}^t) \subseteq J\omega_R\big\}.$$

By virtue of [ST, Remark 6.12], we obtain the following inequality between jumping numbers and F-jumping numbers.

PROPOSITION 4.10

Let $R, \mathfrak{a}, J$ be as in Definition 4.9, and let $(R_p, \mathfrak{a}_p, J_p)$ be a reduction of $(R, \mathfrak{a}, J)$ to sufficiently large characteristic $p \gg 0$. Then

$$\mathrm{fjn}^{J_p}(\omega_{R_p}, \mathfrak{a}_p) \leq \lambda^J(\omega_R, \mathfrak{a}).$$

The following can be viewed as a strengthening of [dFEM, Theorem 0.1] in the graded case.

COROLLARY 4.11

Let $R = \bigoplus_{n \geq 0} R_n$ be a d-dimensional Noetherian normal graded domain with R_0 a field of characteristic zero. Suppose that $\mathfrak{a}$ (resp., J) is an ideal generated by a full homogeneous system of parameters of degrees $a_1 \leq \cdots \leq a_d$ (resp., $b_1 \leq \cdots \leq b_d$) for R. Then

$$e(\mathfrak{a}) \geq \Big(\frac{d}{\lambda^J(\omega_R, \mathfrak{a})}\Big)^d e(J).$$

If the equality holds in the above, then $(a_1, \ldots, a_d)$ and $(b_1, \ldots, b_d)$ are proportional; that is, $a_1/b_1 = \cdots = a_d/b_d$.

Proof

We use an argument similar to the proof of Theorem 3.1. The assertion is then immediate from Corollary 4.7 and Proposition 4.10. □

REMARK 4.12

Let $R = \bigoplus_{n \geq 0} R_n$ be a d-dimensional Noetherian normal graded domain with R_0 a field of positive characteristic (resp., a field of characteristic zero). Suppose that $\mathfrak{a}$ and J are ideals each generated by a full homogeneous system of parameters for R. By a Skoda-type theorem for $\tau(\omega_R, \mathfrak{a}^t)$ (resp., $\mathcal{J}(\omega_R, \mathfrak{a}^t)$; see [ST, Lemma 7.10(3)]), we have

$$\tau(\omega_R, \mathfrak{a}^t) = \mathfrak{a}\tau(\omega_R, \mathfrak{a}^{t-1}) \quad (\text{resp., } \mathcal{J}(\omega_R, \mathfrak{a}^t) = \mathfrak{a}\mathcal{J}(\omega_R, \mathfrak{a}^{t-1}))$$

for all real numbers $t \geq d$. Hence, if $\mathfrak{a}^{N+1} \subseteq J$ for some integer $N \geq 0$, then

$$\tau(\omega_R, \mathfrak{a}^{d+N}) = \mathfrak{a}^{N+1}\tau(\omega_R, \mathfrak{a}^{d-1}) \subseteq J\omega_R,$$
$$(\text{resp., } \mathcal{J}(\omega_R, \mathfrak{a}^{d+N}) = \mathfrak{a}^{N+1}\mathcal{J}(\omega_R, \mathfrak{a}^{d-1}) \subseteq J\omega_R)$$

so that $\operatorname{fjn}^J(\omega_R, \mathfrak{a}) \leq d + N$ (resp., $\lambda^J(\omega_R, \mathfrak{a}) \leq d + N$). Thus, we can think of Corollaries 4.7 and 4.11 as a strengthening of Theorem 2.8 when the ring is a normal domain.

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