Surjectivity of the global-to-local map defining a Selmer group

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In memory of Masayoshi Nagata

Abstract This article studies a map from a global Galois cohomology group to a direct sum of quotients of local Galois cohomology groups. The kernel of such a *global-to-local* map is often called a *Selmer group*. The objective of this article is to study the cokernel of such a map. We do so in a very general context. In particular, we find various sets of assumptions which imply that a global-to-local map is surjective.

1. Introduction

The term *Selmer group* was first used in the 1960s to refer to a certain group that proved to be useful in studying the arithmetic properties of an elliptic curve defined over a number field. The classical definition is easily extended to abelian varieties defined over number fields. We will recall that definition later. Over the years, it was found that one could define such objects in a much more general context. Such definitions occur in the formulation of the Bloch-Kato conjecture in [BK] as well as in generalizations of a conjecture of Iwasawa in [Gr1] and [Gr2]. Roughly speaking, a Selmer group is a subgroup of a global Galois cohomology group defined by imposing local restrictions of some kind on the cocycle classes. These local conditions take a rather specific form in the examples cited above. However, in this article, a Selmer group is defined simply as the kernel of a very general type of map which we call *global-to-local*.

Our objective is to study the cokernel of such a global-to-local map in a very general setting. Suppose that K is a finite extension of $\mathbf{Q}$ and that Σ is a finite set of primes of K. Let K_Σ denote the maximal algebraic extension of K unramified outside of Σ. We always assume that Σ contains all archimedean primes and all primes lying over some fixed rational prime p. The Selmer groups that we consider are associated to a continuous representation

$$\rho : \operatorname{Gal}(K_\Sigma/K) \longrightarrow \mathrm{GL}_n(R),$$

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where R is a complete Noetherian local ring. Let $\mathfrak{M}$ denote the maximal ideal of R. We assume that the residue field $R/\mathfrak{M}$ is finite and has characteristic p. Hence, R is compact in its $\mathfrak{M}$-adic topology. Let $\mathcal{T}$ be the underlying free R-module on which $\mathrm{Gal}(K_\Sigma/K)$ acts via ρ. We define $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$, where $\widehat{R} = \mathrm{Hom}(R, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin dual of R with a trivial action of $\mathrm{Gal}(K_\Sigma/K)$. That Galois group acts on $\mathcal{D}$ through its action on the first factor $\mathcal{T}$. Thus, $\mathcal{D}$ is a discrete abelian group which is isomorphic to $\widehat{R}^n$ as an R-module and has a continuous R-linear action of $\mathrm{Gal}(K_\Sigma/K)$.

The Galois cohomology group $H^1(K_\Sigma/K, \mathcal{D})$ can be considered as a discrete R-module too. It is a cofinitely generated R-module in the sense that its Pontryagin dual is finitely generated as an R-module (see [Gr3, Proposition 3.2]). For each prime v of K, let K_v denote the completion of K at v. Suppose that one specifies an R-submodule $L(K_v, \mathcal{D})$ of $H^1(K_v, \mathcal{D})$ for each $v \in \Sigma$. We denote such a specification simply by $\mathcal{L}$. Let

$$P(K,\mathcal{D}) = \prod_{v\in\Sigma} H^1(K_v,\mathcal{D}) \qquad \text{and} \qquad L(K,\mathcal{D}) = \prod_{v\in\Sigma} L(K_v,\mathcal{D}).$$

Now $L(K, \mathcal{D})$ is an R-submodule of $P(K, \mathcal{D})$, and the corresponding quotient module is

$$Q_{\mathcal{L}}(K,\mathcal{D}) = \prod_{v\in\Sigma} Q_{\mathcal{L}}(K_v,\mathcal{D}), \quad \text{where } Q_{\mathcal{L}}(K_v,\mathcal{D}) = H^1(K_v,\mathcal{D})/L(K_v,\mathcal{D}).$$

The natural global-to-local restriction maps for $H^1(\cdot, \mathcal{D})$ induce a map

$$\phi_{\mathcal{L}} : H^1(K_\Sigma/K,\mathcal{D}) \longrightarrow Q_{\mathcal{L}}(K,\mathcal{D}). \tag{1}$$

The *Selmer group* for $\mathcal{D}$ over K corresponding to the specification $\mathcal{L}$ is defined to be $\ker(\phi_{\mathcal{L}})$ and will be denoted by $S_{\mathcal{L}}(K, \mathcal{D})$. We refer to $\phi_{\mathcal{L}}$ as the global-to-local map defining $S_{\mathcal{L}}(K, \mathcal{D})$.

In the definition given above, one fixes an embedding of $\overline{K}$ into $\overline{K}_v$ for every prime v of K. Here $\overline{K}$ denotes an algebraic closure of K, and $\overline{K}_v$ denotes an algebraic closure of K_v. Thus, one has an embedding of K_Σ into $\overline{K}_v$. The restriction maps $G_{K_v} \to \mathrm{Gal}(K_\Sigma/K)$ for $v \in \Sigma$ then induce the restriction maps for the cohomology groups occurring in the definition of $\phi_{\mathcal{L}}$. However, the Selmer group does not depend on the choice of embeddings.

It is clear that $S_{\mathcal{L}}(K, \mathcal{D})$ is an R-submodule of $H^1(K_\Sigma/K, \mathcal{D})$ and so is also a discrete, cofinitely generated R-module. For a fixed set Σ, the smallest possible Selmer group occurs when we take $L(K_v, \mathcal{D}) = 0$ for all $v \in \Sigma$. Following the notation in [Gr3], we denote that Selmer group by $\text{Ш}^1(K, \Sigma, \mathcal{D})$. In general, for any $i \geq 0$, we define

$$\text{Ш}^i(K,\Sigma,\mathcal{D}) = \ker\Big(H^i(K_\Sigma/K,\mathcal{D}) \longrightarrow \prod_{v\in\Sigma} H^1(K_v,\mathcal{D})\Big).$$

Obviously, we have $\text{Ш}^1(K, \Sigma, \mathcal{D}) \subseteq S_{\mathcal{L}}(K, \mathcal{D})$ for any choice of the specification $\mathcal{L}$.

We do not want to assume necessarily that R is a domain. But we will assume that R contains a subring Λ of the following type: Λ is isomorphic to one of the formal power series rings $\mathbf{Z}_p[[x_1, \ldots, x_m]]$ or $\mathbf{F}_p[[x_1, \ldots, x_{m+1}]]$, where

$m \geq 0$. Furthermore, we assume that R is finitely generated and torsion free as a Λ-module. Such a subring Λ is known to exist if R is a domain. This is a special case of a classical theorem of Cohen (see [Coh, Theorem 16]). The Krull dimension of R is the same as that for Λ and is equal to $m+1$. Both R and Λ have the same characteristic. In general, even if R is not a domain, the assumptions about R imply that $\widehat{R}$ is a divisible Λ-module. Consequently, $\mathcal{D}$ is a divisible Λ-module. In most of the results of this article, it is this last property of $\mathcal{D}$ that is important.

The results that we prove in this article assert that $\phi_{\mathcal{L}}$ is surjective under various sets of hypotheses. In a subsequent article [Gr4], and under additional hypotheses, we apply such results to prove that $S_{\mathcal{L}}(K, \mathcal{D})$ has the following property as a Λ-module: *There exists a nonzero element $\theta \in \Lambda$ such that $\alpha S_{\mathcal{L}}(K, \mathcal{D}) = S_{\mathcal{L}}(K, \mathcal{D})$ for all nonzero $\alpha \in \Lambda$ which are relatively prime to θ.* We then say that $S_{\mathcal{L}}(K, \mathcal{D})$ is an *almost-divisible* Λ-module. Equivalently, the assertion that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible as a Λ-module means that the Pontryagin dual of $S_{\mathcal{L}}(K, \mathcal{D})$ has no nonzero, pseudonull Λ-submodules. The hypotheses that we need for this to be so are more stringent than the ones needed to prove the surjectivity of $\phi_{\mathcal{L}}$. This is partly because we apply the results concerning surjectivity not just to $\mathcal{D}$ but also to the (Λ/Π)-module $\mathcal{D}[\Pi]$, where Π varies over $\mathrm{Spec}_{ht=1}(\Lambda)$, the set of prime ideals of Λ of height 1. It is useful for that reason to keep the assumptions about $\mathcal{D}$ and $\mathcal{L}$ to a minimum.

One of the hypotheses that we will need in [Gr4] is purely ring-theoretic in nature. It is a condition which guarantees that $\mathcal{D}[\Pi]$ is divisible as a (Λ/Π)-module for the prime ideals Π in $\mathrm{Spec}_{ht=1}(\Lambda)$ and has already played a role in our previous article [Gr3]. The hypothesis is that R is reflexive as a Λ-module. We then say that R is a *reflexive ring*. In the case where R is also assumed to be a domain, one can equivalently require that R be the intersection of all its localizations at the prime ideals in $\mathrm{Spec}_{ht=1}(R)$. (One can find an explanation of this equivalence in [Gr3, Section 2, Part D].) That condition also occurs as part of the definition of a Krull domain. For example, it is stated as condition (2.b) on page 116 of Nagata's book [Nag2]. In the literature, one sometimes finds the term *weakly Krull domain* for a domain R satisfying that condition together with a certain finiteness condition (automatically satisfied if R is Noetherian). The class of reflexive domains is rather large. For example, if R is integrally closed or Cohen-Macaulay, then it turns out that R is reflexive. There are important examples (from Hida theory) where R is not necessarily a domain but is still a free (and hence reflexive) module over a suitable subring Λ.

The map $\phi_{\mathcal{L}}$ can certainly fail to be surjective. We can regard $Q_{\mathcal{L}}(K, \mathcal{D})$ as a discrete Λ-module. We have already mentioned that the Pontryagin dual of $H^1(K_\Sigma/K, \mathcal{D})$ is a finitely generated Λ-module. The same is true for the local cohomology groups $H^1(K_v, \mathcal{D})$ and hence for $Q_{\mathcal{L}}(K, \mathcal{D})$. For any discrete, cofinitely generated Λ-module $\mathcal{A}$, we define $\mathrm{corank}_\Lambda(\mathcal{A})$ to be $\mathrm{rank}_\Lambda(\widehat{\mathcal{A}})$, where $\widehat{\mathcal{A}}$ denotes the Pontryagin dual of $\mathcal{A}$. Let $s_{\mathcal{L}}(K,\mathcal{D}), h_1(K,\mathcal{D}), q_{\mathcal{L}}(K,\mathcal{D})$, and $c_{\mathcal{L}}(K,\mathcal{D})$ denote the Λ-coranks of $S_{\mathcal{L}}(K,\mathcal{D}), H^1(K_\Sigma/K,\mathcal{D}), Q_{\mathcal{L}}(K,\mathcal{D})$, and $\mathrm{coker}(\phi_{\mathcal{L}})$, respectively. Although the definitions of these objects all involve the set Σ, we omit it from the notation. In fact, Σ is fixed throughout, except in Sections 3.3 and 4.5. The following equation relating these coranks is an immediate consequence of the definitions:

$$s_{\mathcal{L}}(K,\mathcal{D}) = h_1(K,\mathcal{D}) - q_{\mathcal{L}}(K,\mathcal{D}) + c_{\mathcal{L}}(K,\mathcal{D}). \tag{2}$$

In particular, if $h_1(K,\mathcal{D}) < q_{\mathcal{L}}(K,\mathcal{D})$, then $c_{\mathcal{L}}(K,\mathcal{D}) > 0$ and $\phi_{\mathcal{L}}$ is far from surjective. However, $\phi_{\mathcal{L}}$ can fail to be surjective even if $c_{\mathcal{L}}(K,\mathcal{D}) = 0$. A classical theorem of Cassels provides one important illustration of this behavior, which we discuss briefly now and also in Section 4 with more details.

Suppose that A is an abelian variety defined over K. Let $g = \dim(A)$. We denote the dual abelian variety by B. The classical Selmer group $\mathrm{Sel}_A(K)$ for A over K is a torsion group. For any prime p, its p-primary subgroup $\mathrm{Sel}_A(K)_p$ is a subgroup of $H^1(K,\mathcal{D})$, where $\mathcal{D} = A[p^\infty]$, the group of p-power torsion points on A. As we explain in Section 4.5, it turns out that $\mathrm{Sel}_A(K)_p$ is isomorphic to $S_{\mathcal{L}}(K,\mathcal{D})$, where we take Σ to be a finite set of primes containing the primes over p and ∞ and the primes of K where A has bad reduction, and the specification $\mathcal{L}$ is defined in the following way. For every prime $v \in \Sigma$, let

$$L(K_v,\mathcal{D}) = \mathrm{im}(\kappa_v), \quad \text{where } \kappa_v : A(K_v) \otimes (\mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow H^1(K_v, A[p^\infty]) \tag{3}$$

is the p-power Kummer map for A over K_v. Now $A(K_v) \otimes (\mathbf{Q}_p/\mathbf{Z}_p) = 0$ if $v \nmid p$. Thus, if $v \in \Sigma$ and $v \nmid p$, then $L(K_v,\mathcal{D}) = 0$. There is also a relatively simple description of $L(K_v,\mathcal{D})$ for a prime v lying over p in the case where A has good, ordinary reduction at v. This can be found in [CG, Proposition 4.5] and does not play a role here. It is the inflation map $H^1(K_\Sigma/K,\mathcal{D}) \to H^1(K,\mathcal{D})$ which identifies $S_{\mathcal{L}}(K,\mathcal{D})$ with $\mathrm{Sel}_A(K)_p$.

In terms of our general notation, we are taking $R = \Lambda = \mathbf{Z}_p$ and $\mathcal{T} = T_p(A)$, the p-adic Tate module for A. Thus, $\mathcal{T}$ is a free $\mathbf{Z}_p$-module of rank $n = 2g$, and $\mathrm{Gal}(K_\Sigma/K)$ acts $\mathbf{Z}_p$-linearly on $\mathcal{T}$. The definition of $S_{\mathcal{L}}(K,\mathcal{D})$ is the kernel of the global-to-local map $\phi_{\mathcal{L}}$. The theorem of Cassels mentioned above is equivalent to the following assertion about the cokernel of $\phi_{\mathcal{L}}$. If $\mathrm{Sel}_A(K)_p$ is finite, then $\mathrm{coker}(\phi_{\mathcal{L}})$ is isomorphic to the Pontryagin dual of $H^0(K, B[p^\infty])$, the p-primary subgroup of $B(K)$. Thus, if $\mathrm{Sel}_A(K)_p$ is finite, then $\mathrm{coker}(\phi_{\mathcal{L}})$ is finite. If we assume in addition that $B(K)$ has no elements of order p, then $\phi_{\mathcal{L}}$ is surjective.

Returning to the general setting, [Gr3, Proposition 4.3] gives an explicit lower bound $b_1(K,\mathcal{D})$ for $h_1(K,\mathcal{D})$, where $b_1(K,\mathcal{D})$ is defined in terms of the Λ-ranks or coranks of various global and local H^0's. This lower bound is derived directly from the Poitou-Tate duality theorems. One has

$$h_1(K,\mathcal{D}) = b_1(K,\mathcal{D}) + \mathrm{corank}_\Lambda\big(\text{Ш}^2(K,\Sigma,\mathcal{D})\big),$$

where $Ш^2(K,\Sigma,\mathcal{D})$ is as defined earlier. One therefore has an inequality

$$s_{\mathcal{L}}(K,\mathcal{D}) \geq b_1(K,\mathcal{D}) - q_{\mathcal{L}}(K,\mathcal{D}). \tag{4}$$

The main results of this article are based on the assumption that equality holds in (4). By (2), equality means that $h_1(K,\mathcal{D}) = b_1(K,\mathcal{D})$ and $c_{\mathcal{L}}(K,\mathcal{D}) = 0$. We do not need to recall the precise definition of $b_1(K,\mathcal{D})$ here because the assumption of equality in (4) is equivalent to the assumption that both of the Λ-modules $Ш^2(K,\Sigma,\mathcal{D})$ and $\mathrm{coker}(\phi_{\mathcal{L}})$ have corank zero. That assumption is part of the hypothesis in many of our results.

In general, additional assumptions may be needed to conclude that $\phi_{\mathcal{L}}$ is surjective. For example, returning to the theorem of Cassels, where $\Lambda = \mathbf{Z}_p$ and $\mathcal{D} = A[p^\infty]$ for an abelian variety A of dimension g, it turns out that $b_1(K,\mathcal{D})$ and $q_{\mathcal{L}}(K,\mathcal{D})$ are both equal to $[K:\mathbf{Q}]g$, and so equality holds in (4) if and only if $\mathrm{Sel}_A(K)_p$ is finite. In that case, the surjectivity of $\phi_{\mathcal{L}}$ requires the additional assumption that $H^0(K, B[p^\infty]) = 0$. There are some situations where no extra assumption is needed. Proposition 5.3.1 is an example.

It will become evident that this article relies very much on results proved in our previous article [Gr3]. The results that we prove here together with results in [Gr3] in turn play an important role in [Gr4] and [Gr5]. Our objective in all of these articles is to study basic questions that have arisen naturally in Iwasawa theory over the years. Our approach is to study these questions from a very general point of view.

It is a privilege to dedicate this article to Masayoshi Nagata. We want to mention one specific theorem of Nagata which has already played a role in [Gr2] and promises to be useful in the future. Suppose that R is a domain and that $\mathfrak{R}$ is the integral closure of R in its field of fractions. Theorem 7 in [Nag1] asserts that if R is a complete Noetherian local ring, then $\mathfrak{R}$ is finitely generated as an R-module. Combining this with [Coh, Theorem 7], it follows that $\mathfrak{R}$ is also a complete Noetherian local ring. We will have reason to cite these theorems again in Section 3.4.

The result just described was needed in [Gr2] to formulate a generalization of the so-called *main conjectures* of Iwasawa and of Mazur. It provided a way to associate a *characteristic divisor* to a Selmer group. This result of Nagata may also provide a way of gaining additional insight into the kinds of divisors that can arise from the Selmer groups introduced in [Gr2]. Very little is known about this. If one has a representation ρ of $\mathrm{Gal}(K_\Sigma/K)$ over R, as discussed above, then one can define a representation $\sigma : \mathrm{Gal}(K_\Sigma/K) \to \mathrm{GL}_n(\mathfrak{R})$ simply be extending scalars. A Selmer group associated to ρ will be an R-module. But there is then a natural way to associate a Selmer group to σ, and that will be an $\mathfrak{R}$-module. The relationship between those Selmer groups is not understood at present. We hope that studying this relationship is one step in learning more about the characteristic divisors of Selmer groups.

The organization of this article is as follows. Section 3 contains the main general results concerning the cokernel of $\phi_{\mathcal{L}}$ as well as sufficient conditions for surjectivity. Those results are based on Section 2, which discusses the structure of various relevant Λ-modules. Sections 4 and 5 discuss special situations where the results become more precise. The Tate module of an abelian variety is discussed in Section 4. One then has $R = \Lambda = \mathbf{Z}_p$, and the Krull dimension of R is 1. Section 5 concerns what we call a *twist deformation* associated to an infinite Galois extension K_∞/K such that $\mathrm{Gal}(K_\infty/K) \cong \mathbf{Z}_p^m$ for some $m \geq 1$. In that case, $R = \Lambda$ is the completed group algebra for $\mathrm{Gal}(K_\infty/K)$ over $\mathbf{Z}_p$ and the Krull dimension of R is $m+1$. The results discussed in Section 5 will be useful in [Gr5].

2. The structure of certain Λ-modules

Suppose that $\mathcal{D}$ is a discrete, cofinitely generated Λ-module and that $\mathrm{Gal}(K_\Sigma/K)$ acts continuously on $\mathcal{D}$. We assume that this action is Λ-linear. Let $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$, a compact, finitely generated Λ-module. The Λ-modules to be considered in this section include $H^1(K_\Sigma/K, \mathcal{T}^*)$ and its maximal torsion Λ-submodule $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$. Cohomology groups with values in $\mathcal{T}^*$ will always be the continuous cohomology groups, which are defined by requiring continuity of cocycles. We refer the reader to Chapter 2, [NSW, Section 3] for the basic properties. For any $i \geq 0$, define

$$Ш^i(K, \Sigma, \mathcal{T}^*) = \ker\Big(H^i(K_\Sigma/K, \mathcal{T}^*) \longrightarrow \bigoplus_{v \in \Sigma} H^i(K_v, \mathcal{T}^*)\Big).$$

Of course, $Ш^1(K, \Sigma, \mathcal{T}^*)$ is a Λ-submodule of $H^1(K_\Sigma/K, \mathcal{T}^*)$.

We will use the global and local duality theorems of Tate and Poitou, extended from the case of finite Galois modules to direct and inverse limits of finite Galois modules. Assume that we have fixed a choice of the specification $\mathcal{L}$ for $\mathcal{D}$ and Σ, that is, a choice of Λ-submodules $L(K_v, \mathcal{D})$ of $H^1(K_v, \mathcal{D})$ for all $v \in \Sigma$. By definition, we have a perfect pairing $\mathcal{D} \times \mathcal{T}^* \to \mu_{p^\infty}$. Thus, for each prime v of K, there is a nondegenerate pairing:

$$H^1(K_v, \mathcal{D}) \times H^1(K_v, \mathcal{T}^*) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p. \tag{5}$$

The pairing behaves well with respect to the Λ-module structure on the two groups. Denoting the pairing by $\langle \cdot, \cdot \rangle_v$, it has the property that $\langle \lambda\alpha, \beta \rangle_v = \langle \alpha, \lambda\beta \rangle_v$ for $\lambda \in \Lambda$, $\alpha \in H^1(K_v, \mathcal{D})$, and $\beta \in H^1(K_v, \mathcal{T}^*)$. We accordingly say that the pairing is a Λ-pairing.

To define a useful Selmer group for $\mathcal{T}^*$, we choose the following specification, which we denote by $\mathcal{L}^*$: For all $v \in \Sigma$, define $L(K_v, \mathcal{T}^*)$ to be the orthogonal complement of $L(K_v, \mathcal{D})$ under the pairing (5). Thus, $L(K_v, \mathcal{T}^*)$ and the quotient $Q_{\mathcal{L}^*}(K_v, \mathcal{T}^*) = H^1(K_v, \mathcal{T}^*)/L(K_v, \mathcal{T}^*)$ are compact Λ-modules and are isomorphic to the Pontryagin duals of the discrete Λ-modules $Q_{\mathcal{L}}(K_v, \mathcal{D}) = H^1(K_v, \mathcal{D})/L(K_v, \mathcal{D})$ and $L(K_v, \mathcal{D})$, respectively. Let $P(K, \mathcal{T}^*)$, $L(K, \mathcal{T}^*)$, and

$Q_{\mathcal{L}^*}(K,\mathcal{T}^*)$ be defined as the direct sums over all $v \in \Sigma$ of the Λ-modules $H^1(K_v,\mathcal{T}^*)$, $L(K_v,\mathcal{T}^*)$, and $Q_{\mathcal{L}^*}(K_v,\mathcal{T}^*)$, respectively. Thus, we have $Q_{\mathcal{L}^*}(K,\mathcal{T}^*) \cong P(K,\mathcal{T}^*)/L(K,\mathcal{T}^*)$. The Selmer group $S_{\mathcal{L}}(K,\mathcal{D})$ is the kernel of $\phi_{\mathcal{L}}$, as discussed in the introduction. We now define a Selmer group $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ to be the kernel of the following map (which is induced from the restriction maps $G_{K_v} \to \mathrm{Gal}(K_\Sigma/K)$ for $v \in \Sigma$):

$$\phi_{\mathcal{L}^*} : H^1(K_\Sigma/K,\mathcal{T}^*) \longrightarrow Q_{\mathcal{L}^*}(K,\mathcal{T}^*).$$

This map again is induced by the restriction maps $G_{K_v} \to \mathrm{Gal}(K_\Sigma/K)$ for $v \in \Sigma$.

All of the cohomology groups and the subgroups mentioned above are Λ-modules (either finitely or cofinitely generated), and the maps are Λ-module homomorphisms. In particular, $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ is a Λ-submodule of $H^1(K_\Sigma/K,\mathcal{T}^*)$.

Section 2.1 below concerns $Ш^1(K,\Sigma,\mathcal{T}^*)$. In Section 2.2, we study the maximal torsion Λ-submodule of $H^1(K_\Sigma/K,\mathcal{T}^*)$, especially when it vanishes. Section 2.3 concerns the maximal torsion Λ-submodule of $S_{\mathcal{L}}(K,\mathcal{T}^*)$. We use the following notation. For any compact Λ-module X, we let $X_{\Lambda\text{-tors}}$ denote the maximal Λ-torsion submodule of X. For a discrete Λ-module $\mathcal{A}$, we let $\mathcal{A}_{\Lambda\text{-div}}$ denote the maximal Λ-divisible submodule of $\mathcal{A}$. If $\theta \in \Lambda$, or if I is an ideal in Λ, then $X[\theta]$ denotes the kernel of multiplication by θ and $X[I]$ denotes the intersection of those kernels over all $\theta \in I$. The Λ-submodules $\mathcal{A}[\theta]$ and $\mathcal{A}[I]$ of $\mathcal{A}$ are defined similarly. We say that $\mathcal{A}$ is a cotorsion Λ-module if $\mathcal{A}[\theta] = 0$ for some nonzero $\theta \in \Lambda$. Assuming that $\mathcal{A}$ is cofinitely generated, $\mathcal{A}$ is cotorsion as a Λ-module if and only if $\mathrm{corank}_\Lambda(\mathcal{A}) = 0$.

2.1. The Λ-rank of $Ш^1(K,\Sigma,\mathcal{T}^*)$

The Poitou-Tate duality theorems include the following result. There is a perfect pairing

$$Ш^1(K,\Sigma,\mathcal{T}^*) \times Ш^2(K,\Sigma,\mathcal{D}) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p, \tag{6}$$

and therefore, the Λ-rank of $Ш^1(K,\Sigma,\mathcal{T}^*)$ is equal to the Λ-corank of its dual $Ш^2(K,\Sigma,\mathcal{D})$. In particular, $Ш^1(K,\Sigma,\mathcal{T}^*)$ is a torsion Λ-module if and only if $Ш^2(K,\Sigma,\mathcal{D})$ is cotorsion as a Λ-module. It is often useful to assume that these equivalent properties are satisfied. We formulate such a hypothesis in terms of $\mathcal{D}$.

LEO($\mathcal{D}$)

The Λ-module $Ш^2(K,\Sigma,\mathcal{D})$ is cotorsion.

LEO($\mathcal{D}$) is referred to as Hypothesis L in [Gr3, p. 361]. An equivalent statement is that the Λ-rank of $Ш^1(K,\Sigma,\mathcal{T}^*)$ is zero.

Under rather mild hypotheses, we now show that $Ш^1(K,\Sigma,\mathcal{T}^*)$ is a torsion-free Λ-module. Equivalently, such an assertion means that $Ш^2(K,\Sigma,\mathcal{D})$ is a divisible Λ-module. Consequently, LEO($\mathcal{D}$) then means that $Ш^2(K,\Sigma,\mathcal{D}) = 0$ and that $Ш^1(K,\Sigma,\mathcal{T}^*) = 0$ too.

PROPOSITION 2.1.1

Assume that $\mathcal{D}$ is a divisible Λ-module. Assume also that there is at least one prime $\eta \in \Sigma$ such that $H^0(K_\eta, \mathcal{T}^*) = 0$. Then $Ш^1(K, \Sigma, \mathcal{T}^*)$ is a torsion-free Λ-module and $Ш^2(K, \Sigma, \mathcal{D})$ is a divisible Λ-module.

Proof

The first assumption means that $\mathcal{T}^*$ is a torsion-free Λ-module. Thus, if θ is a nonzero element of Λ, then multiplication by θ gives an exact sequence

$$0 \longrightarrow \mathcal{T}^* \xrightarrow{\theta} \mathcal{T}^* \longrightarrow \mathcal{T}^*/\theta\mathcal{T}^* \longrightarrow 0$$

from which we obtain the following exact sequence of cohomology groups:

$$H^0(K, \mathcal{T}^*) \longrightarrow H^0(K, \mathcal{T}^*/\theta\mathcal{T}^*) \longrightarrow H^1(K_\Sigma/K, \mathcal{T}^*)[\theta] \longrightarrow 0. \tag{7}$$

We have a similar exact sequence for the cohomology groups over K_η. However, since we are assuming that $H^0(K_\eta, \mathcal{T}^*) = 0$, it follows that $H^0(K, \mathcal{T}^*) = 0$ too. Thus, for any nonzero element θ in Λ, the horizontal maps in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} H^0(K, \mathcal{T}^*/\theta\mathcal{T}^*) & \longrightarrow & H^1(K, \mathcal{T}^*)[\theta] \\ \big\downarrow & & \big\downarrow \\ H^0(K_\eta, \mathcal{T}^*/\theta\mathcal{T}^*) & \longrightarrow & H^1(K_\eta, \mathcal{T}^*)[\theta] \end{array}$$

The first vertical map is injective. Hence, so is the second. As a consequence, the map

$$H^1(K, \mathcal{T}^*)_{\Lambda\text{-tors}} \longrightarrow H^1(K_\eta, \mathcal{T}^*)_{\Lambda\text{-tors}} \tag{8}$$

is injective. By definition, $Ш^1(K, \Sigma, \mathcal{T}^*)_{\Lambda\text{-tors}}$ is contained in the kernel of the above map, and hence must vanish. This shows that $Ш^1(K, \Sigma, \mathcal{T}^*)$ is indeed a torsion-free Λ-module. $\square$

REMARK 2.1.2

Theorem 1 in [Gr3] includes a result which is analogous to the above proposition, although somewhat different. The hypotheses in that theorem are more stringent, but the conclusion is the stronger statement that $Ш^1(K, \Sigma, \mathcal{T}^*)$ is reflexive as a Λ-module. It is possible for $Ш^1(K, \Sigma, \mathcal{T}^*)$ to have positive Λ-rank. One finds several examples illustrating this possibility in [Gr3, Section 6, Part D].

It is also possible for $Ш^1(K, \Sigma, \mathcal{T}^*)_{\Lambda\text{-tors}}$ to be nontrivial. According to Proposition 2.1.1, this could happen only if $H^0(K_v, \mathcal{T}^*)$ has positive Λ-rank for all $v \in \Sigma$. As an example when $\Lambda = \mathbf{Z}_3$ and $p = 3$, one could take $\mathcal{T} = T_3(E)(1)$ and $\Sigma = \{\infty, 3, 7, 31\}$, where E is the elliptic curve 651E3 in Cremona's tables. One has $\mathcal{T}^* = T_3(E)(-1)$. The curve E has split multiplicative reduction at $v = 3, 7$, and 31. As a consequence, one finds that $H^0(\mathbf{Q}_v, \mathcal{T}^*) \cong \mathbf{Z}_3$ for all $v \in \Sigma$ and that $Ш^1(\mathbf{Q}, \Sigma, \mathcal{T}^*) \cong \mathbf{Z}/3\mathbf{Z}$. We hope to discuss such examples in the future.

There are situations where one does expect to have $\text{Ш}^2(K,\Sigma,\mathcal{D})=0$. This statement is equivalent to $\mathrm{LEO}(\mathcal{D})$ under the assumptions of Proposition 2.1.1. One very general conjecture in this direction is stated later, namely, Conjecture 5.2.1. If one makes the additional assumption that p is odd and that $H^0(K_v,\mathcal{T}^*)=0$ for all nonarchimedean $v\in\Sigma$, then one has $H^2(K_v,\mathcal{D})=0$ for all $v\in\Sigma$. One would then have $\text{Ш}^2(K,\Sigma,\mathcal{D})=H^2(K_\Sigma/K,\mathcal{D})$, and so $\mathrm{LEO}(\mathcal{D})$ would then mean that $H^2(K_\Sigma/K,\mathcal{D})=0$. Consider the special case where $\Lambda=\mathbf{Z}_p$, $\mathcal{D}=\mathbf{Q}_p/\mathbf{Z}_p$, and the Galois action on $\mathbf{Q}_p/\mathbf{Z}_p$ is trivial. One then has $H^2(K_v,\mathbf{Q}_p/\mathbf{Z}_p)=0$ for all v, even when $p=2$. In fact, $\mathrm{LEO}(\mathbf{Q}_p/\mathbf{Z}_p)$, or the equivalent statement that $H^2(K_\Sigma/K,\mathbf{Q}_p/\mathbf{Z}_p)=0$, is a reformulation of the famous Leopoldt conjecture for K and p. Thus, the more general formulations (such as Conjecture 5.2.1) are extensions of Leopoldt's conjecture in a sense and have often been referred to by the phrase *weak Leopoldt conjecture*. We refer the reader to [Per, appendice B] for a discussion of some important special cases.

REMARK 2.1.3

The prime η in Proposition 2.1.1 could be archimedean. Assume that $\mathcal{D}$ is a divisible Λ-module and hence that $\mathcal{T}^*$ is torsion free. Assume that $\mathcal{T}^*\neq 0$. Let $\mathcal{F}$ be the field of fractions of Λ. We may suppose that η is a real prime, and so G_{K_η} has order 2. Let σ_η be a generator. Of course, σ_η acts on the $\mathcal{F}$-vector space $\mathcal{V}^*=\mathcal{T}^*\otimes_\Lambda\mathcal{F}$. Then $H^0(K_\eta,\mathcal{T}^*)=0$ means that 1 is not an eigenvalue of σ_η. It follows that $H^0(K_\eta,\mathcal{T}^*)=0$ if and only if the characteristic of Λ is not 2 and σ_η acts on $\mathcal{V}^*$ as the scalar -1.

If Π is in $\mathrm{Spec}_{ht=1}(\Lambda)$, then $\mathrm{LEO}(\mathcal{D}[\Pi])$ should be interpreted to mean that $\text{Ш}^2(K,\Sigma,\mathcal{D}[\Pi])$ is cotorsion as a (Λ/Π)-module. The following proposition is sometimes useful because the Krull dimension of the underlying ring is reduced by 1. The proof uses the following general observation from [Gr3, Remark 2.1.3]. If $\mathcal{A}$ is a discrete, cofinitely generated Λ-module and $r=\mathrm{corank}_\Lambda(\mathcal{A})$, then $\mathrm{corank}_{\Lambda/\Pi}(\mathcal{A}[\Pi])\geq r$ for all prime ideals Π of Λ. Furthermore, equality holds for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$. The phrase *almost all* means *all but a finite number.*

PROPOSITION 2.1.4

Assume that Λ has Krull dimension ≥ 2. Then $\mathrm{LEO}(\mathcal{D})$ is satisfied if and only if $\mathrm{LEO}(\mathcal{D}[\Pi])$ is satisfied for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$. Furthermore, if $\mathcal{D}$ is Λ-divisible and $H^2(K_\Sigma/K,\mathcal{D}[\Pi])$ is a cotorsion (Λ/Π)-module for at least one Π in $\mathrm{Spec}_{ht=1}(\Lambda)$, then $H^2(K_\Sigma/K,\mathcal{D})$ is a cotorsion Λ-module, and hence $\mathrm{LEO}(\mathcal{D})$ is then satisfied.

Proof

Lemma 4.4.1 in [Gr3] states that $\text{Ш}^2(K,\Sigma,\mathcal{D}[\Pi])$ and $\text{Ш}^2(K,\Sigma,\mathcal{D})[\Pi]$ have the same (Λ/Π)-corank for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$. The general observation from [Gr3] cited above implies that the (Λ/Π)-corank of $\text{Ш}^2(K,\Sigma,\mathcal{D})[\Pi]$ is equal to the Λ-corank of $\text{Ш}^2(K,\Sigma,\mathcal{D})$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. The first part of the proposition follows immediately.

For the second part, let π be a generator of Π. Since $\mathcal{D}$ is divisible by π, the natural map from $H^2(K_\Sigma/K,\mathcal{D}[\Pi])$ to $H^2(K_\Sigma/K,\mathcal{D})[\Pi]$ is surjective. Combining that fact with the above observation from [Gr3] gives the inequalities

$$\begin{aligned}\mathrm{corank}_\Lambda\big(H^2(K_\Sigma/K,\mathcal{D})\big) &\le \mathrm{corank}_{\Lambda/\Pi}\big(H^2(K_\Sigma/K,\mathcal{D})[\Pi]\big)\\ &\le \mathrm{corank}_{\Lambda/\Pi}\big(H^2(K_\Sigma/K,\mathcal{D}[\Pi])\big).\end{aligned}$$

If the last corank is zero, then so is the first, and hence, $H^2(K_\Sigma/K,\mathcal{D})$ is indeed Λ-cotorsion. □

2.2. The torsion Λ-submodule of $H^1(K_\Sigma/K,\mathcal{T}^*)$

We first prove a result concerning the vanishing of the maximal torsion Λ-submodule of $H^1(K_\Sigma/K,\mathcal{T}^*)$. Let $\mathfrak{m}$ denote the maximal ideal of Λ. Note that $\Lambda/\mathfrak{m} \cong \mathbf{F}_p$. Thus, $\mathcal{D}[\mathfrak{m}]$ is a finite-dimensional representation space for $\mathrm{Gal}(K_\Sigma/K)$ over $\mathbf{F}_p$.

PROPOSITION 2.2.1

Assume that $\mathcal{D}$ is divisible as a Λ-module and that $\mathcal{D}[\mathfrak{m}]$ has no subquotient isomorphic to μ_p for the action of G_K. Then $H^1(K_\Sigma/K,\mathcal{T}^*)$ is torsion free as a Λ-module.

Proof

Using the exact sequence (7), it suffices to show that $H^0(K,\mathcal{T}^*/\theta\mathcal{T}^*)=0$ for all nonzero $\theta\in\Lambda$. Suppose that $j\ge 1$. Proposition 3.1 in [Gr3] implies that the composition factors in the G_K-module $\mathcal{D}[\mathfrak{m}^j]$ are the same as those in the G_K-module $\mathcal{D}[\mathfrak{m}]$, and hence, the second hypothesis implies that μ_p is not one of those composition factors. Consequently, none of the composition factors in $\mathcal{T}^*/\mathfrak{m}^j\mathcal{T}^*$ is isomorphic to the trivial Galois module $\mathbf{Z}/p\mathbf{Z}$. Now $\mathcal{T}^*/\theta\mathcal{T}^*$ is a projective limit of a sequence of finite G_K-modules A_n, each of which is a quotient of $\mathcal{T}^*/\mathfrak{m}^j\mathcal{T}^*$ for some value of j. If $H^0(K,\mathcal{T}^*/\theta\mathcal{T}^*)\ne 0$, then we have $H^0(K,A_n)\ne 0$ for some value of n. Thus, for such n, A_n has a submodule isomorphic to the trivial module $\mathbf{Z}/p\mathbf{Z}$. This cannot happen, and so we must indeed have $H^0(K,\mathcal{T}^*/\theta\mathcal{T}^*)=0$. □

The torsion Λ-submodule of $H^1(K_\Sigma/K,\mathcal{T}^*)$ can vanish even if $\mathcal{D}[\mathfrak{m}]$ has a subquotient isomorphic to μ_p. Propositions 2.2.5 and 2.2.7 below give some situations where that is so. They are based on the next proposition, which is itself a straightforward consequence of (7). For the first part, one just chooses θ to be a nonzero element in the annihilator of $H^1(K_\Sigma/K,\mathcal{T}^*)_{\Lambda\text{-tors}}$. For the second part, if $H^1(K_\Sigma/K,\mathcal{T}^*)_{\Lambda\text{-tors}}\ne 0$, then at least one irreducible factor π of θ has the stated property. Note that (7) is valid only under the assumption that $\mathcal{D}$ is a divisible Λ-module.

PROPOSITION 2.2.2

Suppose that $\mathcal{D}$ is divisible as a Λ-module and that $H^0(K, \mathcal{T}^*) = 0$. We have

$$H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}} \cong H^0(K, \mathcal{T}^*/\theta\mathcal{T}^*)$$

for some nonzero element θ in Λ. Furthermore, $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}} \neq 0$ if and only if there exists an irreducible element π in Λ such that $H^0(K, \mathcal{T}^*/\pi\mathcal{T}^*) \neq 0$.

REMARK 2.2.3

By definition, we have $\mathcal{T}^*/\pi\mathcal{T}^* \cong \mathrm{Hom}(\mathcal{D}[\pi], \mu_{p^\infty})$. Hence, $H^0(K, \mathcal{T}^*/\pi\mathcal{T}^*) \neq 0$ means that there exists a nontrivial G_K-homomorphism from $\mathcal{D}[\pi]$ to μ_{p^∞}.

We assume now that the discrete, cofinitely generated Λ-module $\mathcal{D}$ is actually cofree. This means that $\mathcal{T}^*$ is a free Λ-module of finite rank. This assumption is satisfied in a number of interesting cases. For example, it holds if $\mathcal{T}$ is a free R-module, as in the introduction, and R is a free Λ-module. If R is a domain, then R is free as a Λ-module if and only if R is a Cohen-Macaulay ring (see [BH, Proposition 2.2.11]). However, if R is reflexive and its Krull dimension is at least 3, then R may conceivably fail to be free as a Λ-module. Cofreeness of $\mathcal{D}$ has some useful implications, as we now discuss. The first is contained in the following result.

PROPOSITION 2.2.4

Suppose that $\mathcal{D}$ is cofree as a Λ-module and that $H^0(K, \mathcal{T}^*) = 0$. Then the Λ-module $H^1(K_\Sigma/K, \mathcal{T}^*)$ has no nonzero, pseudonull Λ-submodules.

The conclusion means that the associated prime ideals for the torsion Λ-module $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ are of height 1. That is, its support is pure of codimension 1.

Proof

Suppose to the contrary that there exists a nonzero pseudonull Λ-submodule Z of $H^1(K_\Sigma/K, \mathcal{T}^*)$. It is clear that Λ must have Krull dimension at least 2. According to [Gr3, Corollary 2.5.1], the annihilator of Z contains infinitely many prime ideals $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Choose any such Π. Since Λ is a unique factorization domain, Π must be principal. Let π be a generator. As in Proposition 2.2.2, Z is isomorphic to a Λ-submodule of $H^0(K, \mathcal{T}^*/\pi\mathcal{T}^*)$. Now Λ/Π has no nonzero, pseudonull Λ-submodules. Hence, the same is true for the free (Λ/Π)-module $\mathcal{T}^*/\pi\mathcal{T}^*$ and therefore also for the submodule $H^0(K, \mathcal{T}^*/\pi\mathcal{T}^*)$. $\square$

PROPOSITION 2.2.5

Suppose that $\mathcal{D}$ is cofree as a Λ-module and that $\mathcal{D}[\mathfrak{m}]$ has no quotient isomorphic to μ_p for the action of G_K. Then $H^1(K_\Sigma/K, \mathcal{T}^*)$ is torsion free as a Λ-module.

Proof
By definition, we have $\mathcal{T}^*/\mathfrak{m}\mathcal{T}^* \cong \mathrm{Hom}(\mathcal{D}[\mathfrak{m}], \mu_p)$. The assumption about μ_p means that $H^0(K, \mathcal{T}^*/\mathfrak{m}\mathcal{T}^*) = 0$. The stated result follows from Proposition 2.2.2 and the following lemma. One applies the lemma to first see that $H^0(K, \mathcal{T}^*) = 0$ and then to see that $H^0(K, \mathcal{T}^*/\pi\mathcal{T}^*) = 0$ for all irreducible elements π in Λ. $\square$

LEMMA 2.2.6
Suppose that $\mathcal{T}^*$ is free as a Λ-module. Suppose that Π_1 and Π_2 are prime ideals in Λ such that $\Pi_1 \subseteq \Pi_2$. If $H^0(K, \mathcal{T}^*/\Pi_2\mathcal{T}^*) = 0$, then $H^0(K, \mathcal{T}^*/\Pi_1\mathcal{T}^*) = 0$.

Proof
The action of $\mathrm{Gal}(K_\Sigma/K)$ on $\mathcal{T}^*$ factors through a quotient group which is topologically finitely generated. To see this, note that $\mathcal{T}^*$ is a free Λ-module since $\mathcal{D}$ is assumed to be cofree. After choosing a basis, the Galois action on $\mathcal{T}^*$ is given by a continuous homomorphism

$$\sigma : \mathrm{Gal}(K_\Sigma/K) \longrightarrow \mathrm{GL}_d(\Lambda),$$

where $d = \mathrm{rank}_\Lambda(\mathcal{T}^*)$. The Galois action on $\mathcal{T}^*/\mathfrak{m}\mathcal{T}^*$ is given by the reduction of σ modulo $\mathfrak{m}$, which factors through $\mathrm{Gal}(L/K)$ for some finite Galois extension L of K. One can verify that the kernel of the map $\mathrm{GL}_d(\Lambda) \to \mathrm{GL}_d(\mathbf{F}_p)$ is a pro-p group. Hence, σ factors through $\mathrm{Gal}(M/K)$, where M is the maximal pro-p extension of L contained in K_Σ. However, the Burnside basis theorem shows that $\mathrm{Gal}(M/L)$ is topologically finitely generated and hence so is $\mathrm{Gal}(M/K)$.

Thus, we can find a set $\{g_1, \dots, g_t\}$ in G_K such that, if X is any quotient of the G_K-module $\mathcal{T}^*$, then $H^0(K, X)$ coincides with the kernel of the map

$$\beta_X : X \longrightarrow X^t, \quad \text{defined by } \beta_X(x) = \big((g_1 - 1)x, \dots, (g_t - 1)x\big)$$

for all $x \in X$. The map $\beta_{\mathcal{T}^*}$ is given by a $(td \times d)$-matrix B with entries in Λ. The kernel of $\beta_{\mathcal{T}^*}$ has Λ-rank equal to $d - \mathrm{rank}(B)$. More generally, for any prime ideal Π of Λ, let B_Π denote the $(td \times d)$-matrix with entries in Λ/Π obtained by reducing B modulo Π. If $X = \mathcal{T}^*/\Pi\mathcal{T}^*$, then the kernel of β_X has (Λ/Π)-rank equal to $d - \mathrm{rank}(B_\Pi)$. The rank r of a matrix over a domain is the largest integer for which at least one $(r \times r)$-minor has a nonzero determinant. That description implies that

$$\mathrm{rank}(B_{\Pi_2}) \leq \mathrm{rank}(B_{\Pi_1}).$$

Now $\mathcal{T}^*/\Pi_1\mathcal{T}^*$ is free of rank n as a (Λ/Π_1)-module. If $H^0(K, \mathcal{T}^*/\Pi_1\mathcal{T}^*) \neq 0$, then B_{Π_1} has rank $\leq n - 1$. Hence, the same inequality is true for the rank of B_{Π_2}, and therefore we have $H^0(K, \mathcal{T}^*/\Pi_2\mathcal{T}^*) \neq 0$. $\square$

The following is a more refined result which is useful if $\mathcal{D}[\mathfrak{m}]$ does have a quotient isomorphic to μ_p. As in the introduction, we denote the Krull dimension of Λ by $m + 1$. We let $\mathrm{Spec}_{ht=m}(\Lambda)$ denote the set of prime ideals of Λ of height m. Note that if $\mathfrak{p}$ is in $\mathrm{Spec}_{ht=m}(\Lambda)$, then $\Lambda/\mathfrak{p}$ is a ring of Krull dimension 1 and hence is either a finite integral extension of $\mathbf{Z}_p$ if $\Lambda/\mathfrak{p}$ has characteristic zero, or a finite integral extension of a formal power series ring $\mathbf{F}_p[[x]]$ in one variable if $\Lambda/\mathfrak{p}$ has characteristic p. If $\mathcal{D}$ is cofree as a Λ-module, then $\mathcal{T}^*$ is free. Thus, for any prime ideal Π of Λ, $\mathcal{T}^*/\Pi\mathcal{T}^*$ is a free (Λ/Π)-module. Therefore, the (Λ/Π)-submodule $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*)$ either vanishes or has positive rank.

PROPOSITION 2.2.7

Suppose that $\mathcal{T}^*$ is free as a Λ-module. Assume that the Krull dimension of Λ is $m+1$, where $m \geq 1$. If $m = 1$, assume that $H^0(K, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*)$ vanishes for all $\mathfrak{p}$ in $\mathrm{Spec}_{ht=1}(\Lambda)$. If $m \geq 2$, assume that $H^0(K, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*)$ vanishes for all but finitely many $\mathfrak{p}$ in $\mathrm{Spec}_{ht=m}(\Lambda)$. Then $H^1(K_\Sigma/K, \mathcal{T}^*)$ is torsion free as a Λ-module.

Proof

The first assumption implies that $\mathcal{D}$ is Λ-cofree and hence certainly Λ-divisible. By Lemma 2.2.6, the other assumptions imply that $H^0(K, \mathcal{T}^*) = 0$. Therefore, according to Proposition 2.2.2, it suffices to show that $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*) = 0$ for all Π in $\mathrm{Spec}_{ht=1}(\Lambda)$. If $m = 1$, this vanishing statement is true by assumption. If $m \geq 2$, then every prime ideal Π of Λ of height 1 is contained in infinitely many prime ideals $\mathfrak{p}$ of height m, as follows from the lemma below. Therefore, in that case, the assumption implies that $H^0(K, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*) = 0$ for at least one such $\mathfrak{p}$, and Lemma 2.2.6 then implies the vanishing of $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*)$. □

LEMMA 2.2.8

Suppose that Λ has Krull dimension $m+1$, where $m \geq 2$. If Π is a prime ideal of height less than m, then there exist infinitely many prime ideals $\mathfrak{p} \in \mathrm{Spec}_{ht=m}(\Lambda)$ such that $\Pi \subset \mathfrak{p}$.

Proof

There exists a prime ideal containing Π of height $m-1$. Thus, we can assume that Π itself has height $m-1$. Consider Λ/Π, a complete Noetherian local domain of dimension 2. It is a finite integral extension of a subring Λ' which is a formal power series ring over $\mathbf{Z}_p$ or $\mathbf{F}_p$ of Krull dimension 2. Thus, Λ' has infinitely many prime ideals of height 1. It follows that the same is true for Λ/Π. The assertion in the lemma follows immediately. □

The next two propositions concern the case $m = 1$. The first concerns a global cohomology group. The second result is local, and its proof is virtually identical.

PROPOSITION 2.2.9

Suppose that Λ has Krull dimension 2, that $\mathcal{T}^*$ is free as a Λ-module, and that $H^0(K, \mathcal{T}^*) = 0$. Then $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}} \neq 0$ if and only if there exists at least one $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ with the following property: either $\mathcal{D}[\Pi]$ has a quotient *isomorphic to μ_{p^∞} as a G_K-module or $\mathcal{D}[\Pi]$ has infinitely many distinct quotients isomorphic to μ_p.*

Proof
According to Proposition 2.2.2 and Remark 2.2.3, $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}} \neq 0$ if and only if $\mathrm{Hom}_{G_K}(\mathcal{D}[\Pi], \mu_{p^\infty}) \neq 0$ for some $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. It is clear that $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*)$ is a torsion-free module over Λ/Π, a domain of Krull dimension 1. If Λ/Π has characteristic zero, then it follows that $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*)$ is a torsion-free $\mathbf{Z}_p$-module, and hence is either trivial or has positive $\mathbf{Z}_p$-rank. If Λ/Π has characteristic p, then it follows that $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*)$ is trivial or has infinite $\mathbf{F}_p$-dimension. Thus, $\mathrm{Hom}_{G_K}(\mathcal{D}[\Pi], \mu_{p^\infty}) \neq 0$ means that $\mathcal{D}[\Pi]$ has one of the two stated properties. $\square$

PROPOSITION 2.2.10
Suppose that Λ has Krull dimension 2, that $\mathcal{T}^*$ is free as a Λ-module, that $v \in \Sigma$, and that $H^0(K_v, \mathcal{T}^*) = 0$. Then $H^1(K_v, \mathcal{T}^*)_{\Lambda\text{-tors}} \neq 0$ if and only if there exists at least one $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ with the following property: either $\mathcal{D}[\Pi]$ has a quotient isomorphic to μ_{p^∞} as a G_{K_v}-module or $\mathcal{D}[\Pi]$ has infinitely many distinct quotients isomorphic to μ_p.

2.3. The vanishing of $S_{\mathcal{L}^*}(K, \mathcal{T}^*)_{\Lambda\text{-tors}}$

We first show that the vanishing of $S_{\mathcal{L}^*}(K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ and of $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ are equivalent under a certain assumption.

PROPOSITION 2.3.1
Assume that $L(K_v, \mathcal{D}) \subseteq H^1(K_v, \mathcal{D})_{\Lambda\text{-div}}$ for all $v \in \Sigma$. Then

$$S_{\mathcal{L}^*}(K, \mathcal{T}^*)_{\Lambda\text{-tors}} = H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}.$$

In particular, this equality is true if $L(K_v, \mathcal{D})$ is a divisible Λ-module for all $v \in \Sigma$.

Proof
The assumption means that $H^1(K_v, \mathcal{T}^*)_{\Lambda\text{-tors}} \subseteq L(K_v, \mathcal{T}^*)$ for all $v \in \Sigma$. Obviously, we have $S_{\mathcal{L}^*}(K, \mathcal{T}^*)_{\Lambda\text{-tors}} \subseteq H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$. The opposite inclusion follows by noting that the image of any element of $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ in $H^1(K_v, \mathcal{T}^*)$ must be in $H^1(K_v, \mathcal{T}^*)_{\Lambda\text{-tors}}$ and hence in $L(K_v, \mathcal{T}^*)$. $\square$

PROPOSITION 2.3.2
Assume that $\mathcal{D}$ is divisible as a Λ-module. Assume also that there exists a prime $\eta \in \Sigma$ with the following two properties:

(i) *$H^0(K_\eta, \mathcal{T}^*) = 0$, and*
(ii) *$Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is divisible as a Λ-module.*

Then $S_{\mathcal{L}^*}(K, \mathcal{T}^*)_{\Lambda\text{-tors}} = 0$.

Proof
Only the local condition at η occurring in the definition of $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ is needed. Consider the maps

$$H^1(K,\mathcal{T}^*)_{\Lambda\text{-tors}} \longrightarrow H^1(K_\eta,\mathcal{T}^*)_{\Lambda\text{-tors}} \longrightarrow H^1(K_\eta,\mathcal{T}^*)/L(K_\eta,\mathcal{T}^*).$$

Just as in the proof of Proposition 2.1.1, assumption (i) implies that the first map is injective. It is the map (8). Now $L(K_\eta,\mathcal{T}^*)$ is the Pontryagin dual of the divisible Λ-module $Q_{\mathcal{L}}(K_\eta,\mathcal{D})$ and is therefore a torsion-free Λ-submodule of $H^1(K_\eta,\mathcal{T}^*)$. It follows that the second map is also injective. By definition, any element of $S_{\mathcal{L}^*}(K,\mathcal{T}^*)_{\Lambda\text{-tors}}$ has trivial image under the composite of those maps and therefore must be trivial. □

REMARK 2.3.3
Assumption (ii) in Proposition 2.3.2 is obviously satisfied if $H^1(K_\eta,\mathcal{D})$ is a divisible Λ-module, but is a significantly less restrictive property in general. However, the two properties are actually equivalent if one makes the first assumption in Proposition 2.3.1 for $v=\eta$. To explain this, suppose that v is any prime of K. Then $H^1(K_v,\mathcal{D})/H^1(K_v,\mathcal{D})_{\Lambda\text{-div}}$ is a cotorsion Λ-module. It follows that the image of $H^1(K_v,\mathcal{D})_{\Lambda\text{-div}}$ in $Q_{\mathcal{L}}(K_v,\mathcal{D})$ is precisely $Q_{\mathcal{L}}(K_v,\mathcal{D})_{\Lambda\text{-div}}$. Therefore, $Q_{\mathcal{L}}(K_v,\mathcal{D})$ is a divisible Λ-module if and only if

$$L(K_v,\mathcal{D})H^1(K_v,\mathcal{D})_{\Lambda\text{-div}} = H^1(K_v,\mathcal{D}).$$

It follows that if $Q_{\mathcal{L}}(K_v,\mathcal{D})$ is a divisible Λ-module and if we assume the inclusion $L(K_v,\mathcal{D}) \subseteq H^1(K_v,\mathcal{D})_{\Lambda\text{-div}}$, then $H^1(K_v,\mathcal{D})$ is a divisible Λ-module. The converse is clearly true too.

3. The cokernel of $\phi_{\mathcal{L}}$

Section 3.1 describes $\operatorname{coker}(\phi_{\mathcal{L}})$ in terms of $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ and $\text{Ш}^1(K,\Sigma,\mathcal{T}^*)$. This is a direct consequence of the Poitou-Tate duality theorems and the basis for our results concerning $\operatorname{coker}(\phi_{\mathcal{L}})$. We apply this description together with results from Section 2 to obtain some rather general sufficient conditions for $\phi_{\mathcal{L}}$ to be surjective. In Section 3.3, under rather restrictive assumptions, we discuss what happens if Σ is allowed to vary.

3.1. Expressing $\operatorname{coker}(\phi_{\mathcal{L}})$ in terms of Selmer groups for $\mathcal{T}^*$

For a given specification $\mathcal{L}$, we have defined Λ-submodules $L(K,\mathcal{D})$ and $L(K,\mathcal{T}^*)$ of $P(K,\mathcal{D})$ and $P(K,\mathcal{T}^*)$, respectively. Furthermore, they are orthogonal complements of each other under the pairing

$$P(K,\mathcal{D}) \times P(K,\mathcal{T}^*) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \tag{9}$$

which is defined by the local pairings (5). It is a nondegenerate Λ-pairing. We define

$$\begin{aligned} G(K,\mathcal{D}) &= \operatorname{im}\big(H^1(K_\Sigma/K,\mathcal{D}) \to P(K,\mathcal{D})\big), \\ G(K,\mathcal{T}^*) &= \operatorname{im}\big(H^1(K_\Sigma/K,\mathcal{T}^*) \to P(K,\mathcal{T}^*)\big). \end{aligned} \tag{10}$$

For brevity, we denote $G(K,\mathcal{D}), P(K,\mathcal{D})$, and $L(K,\mathcal{D})$ by G, P, and L, respectively. Similarly, $G(K,\mathcal{T}^*), P(K,\mathcal{T}^*)$, and $L(K,\mathcal{T}^*)$ are denoted by G^*, P^*, and L^*. Thus, G and L are Λ-submodules of the discrete Λ-module P, while G^* and L^* are Λ-submodules of the compact Λ-module P^*. Under the pairing (9), the submodules G and G^* are orthogonal complements of each other, as are L and L^*.

By definition, the cokernel of $\phi_{\mathcal{L}}$ is isomorphic to P/GL. The pairing (9) shows that its Pontryagin dual is isomorphic to $G^* \cap L^*$. It is clear from the definition that $G^* \cap L^*$ is the image of $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ under the second map in (10). Denoting the kernel of that map by $Ш^1(K,\Sigma,\mathcal{T}^*)$, we obtain the following result concerning the cokernel of $\phi_{\mathcal{L}}$.

PROPOSITION 3.1.1

With the above notation and assumptions, we have the following Λ-module isomorphism for the Pontryagin dual of $\mathrm{coker}(\phi_{\mathcal{L}})$*:*

$$\widehat{\mathrm{coker}(\phi_{\mathcal{L}})} \cong S_{\mathcal{L}^*}(K,\mathcal{T}^*)/Ш^1(K,\Sigma,\mathcal{T}^*).$$

In particular, if $S_{\mathcal{L}^*}(K,\mathcal{T}^*) = 0$, then $\phi_{\mathcal{L}}$ is surjective.

The argument gives an isomorphism of $\mathbf{Z}_p$-modules if one just assumes that $\mathcal{D}$ is a discrete, p-primary abelian group with a continuous action of $\mathrm{Gal}(K_\Sigma/K)$.

REMARK 3.1.2

It follows that $\mathrm{coker}(\phi_{\mathcal{L}})$ is a cotorsion Λ-module if and only if $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ and $Ш^1(K,\Sigma,\mathcal{T}^*)$ have the same ranks as Λ-modules. If $\mathrm{LEO}(\mathcal{D})$ is satisfied, then $\mathrm{coker}(\phi_{\mathcal{L}})$ is cotorsion as a Λ-module if and only if $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ is a torsion Λ-module.

REMARK 3.1.3

Propositions 2.2.4 and 3.1.1 have the following consequence concerning $\mathrm{coker}(\phi_{\mathcal{L}})$. Suppose that $\mathcal{D}$ is Λ-cofree, that $H^0(K,\mathcal{T}^*) = 0$, and that $Ш^1(K,\Sigma,\mathcal{T}^*) = 0$. Then $\widehat{\mathrm{coker}(\phi_{\mathcal{L}})}_{\Lambda\text{-tors}}$ is isomorphic to a submodule of $H^1(K_\Sigma/K,\mathcal{T}^*)_{\Lambda\text{-tors}}$. Therefore, $\widehat{\mathrm{coker}(\phi_{\mathcal{L}})}$ has no nonzero, pseudonull Λ-submodules. That is, $\mathrm{coker}(\phi_{\mathcal{L}})$ is an almost-divisible Λ-module.

3.2. Surjectivity of $\phi_{\mathcal{L}}$

We can now give sufficient conditions for the surjectivity of $\phi_{\mathcal{L}}$. However, we first point out that Proposition 3.1.1 itself gives such a sufficient condition. If one assumes that $\mathcal{D}$ is a cofinitely generated Λ-module, that $\mathrm{LEO}(\mathcal{D})$ is satisfied, that $\mathrm{coker}(\phi_{\mathcal{L}})$ is a cotorsion Λ-module, and that $H^1(K_\Sigma/K,\mathcal{T}^*)$ is torsion free as a Λ-module, then it clearly follows that $\mathrm{coker}(\phi_{\mathcal{L}}) = 0$. Nevertheless, the following results often turn out to be useful.

PROPOSITION 3.2.1

Assume that $\mathcal{D}$ is divisible as a Λ-module, that $\mathrm{LEO}(\mathcal{D})$ *is satisfied, and that*

$\mathrm{coker}(\phi_{\mathcal{L}})$ *is a cotorsion* Λ*-module. Then* $\phi_{\mathcal{L}}$ *is surjective if at least one of the following assumptions is satisfied.*

(a) $\mathcal{D}[\mathfrak{m}]$ *has no subquotient isomorphic to* μ_p *for the action of* G_K*.*

(b) $\mathcal{D}$ *is a cofree* Λ*-module, and* $\mathcal{D}[\mathfrak{m}]$ *has no quotient isomorphic to* μ_p *for the action of* G_K*.*

(c) *There is a prime* $\eta \in \Sigma$ *satisfying properties (i) and (ii) in Proposition 2.3.2.*

Proof

As discussed in Section 2.1, LEO($\mathcal{D}$) implies that $\text{Ш}^1(K, \Sigma, \mathcal{T}^*)$ is a torsion Λ-module. By Proposition 3.1.1 and the assumption about the cokernel of $\phi_{\mathcal{L}}$, it follows that $S_{\mathcal{L}^*}(K, \mathcal{T}^*)$ is a torsion Λ-module. One can use Proposition 2.2.1 if assumption (a) is satisfied to conclude that $S_{\mathcal{L}^*}(K, \mathcal{T}^*) = 0$. If (b) is satisfied, then Proposition 2.2.5 gives that conclusion. On the other hand, if assumption (c) is satisfied, then Proposition 2.3.2 implies that $S_{\mathcal{L}^*}(K, \mathcal{T}^*)$ vanishes. In all three cases, Proposition 3.1.1 implies that $\mathrm{coker}(\phi_{\mathcal{L}}) = 0$. □

REMARK 3.2.2

Assumption (a) in Proposition 3.2.1 is satisfied in many interesting situations. As an example, suppose that ρ is a Galois representation of degree n over R as in the introduction, that $n \geq 2$, and that the residual representation $\widetilde{\rho}$ giving the action of G_K on $\mathcal{T}/\mathfrak{M}\mathcal{T}$ is irreducible over the finite field $R/\mathfrak{M}$. Regarding $\widetilde{\rho}$ as a representation space for G_K over $\Lambda/\mathfrak{m} = \mathbf{F}_p$, all of the irreducible constituents are conjugate over $\mathbf{F}_p$ and of dimension divisible by n. Hence, the Galois module μ_p cannot be a subquotient. Now $\widetilde{\rho}$ also gives the action of G_K on $\mathcal{D}[\mathfrak{M}]$. Thus, no subquotient of $\mathcal{D}[\mathfrak{M}]$ is isomorphic to μ_p. According to [Gr3, Proposition 3.8], the irreducible constituents of the $(\Lambda/\mathfrak{m})$-representation spaces $\mathcal{D}[\mathfrak{m}]$ and $\mathcal{D}[\mathfrak{M}]$ for G_K are the same (although with possibly different multiplicities). It therefore follows that no subquotient of $\mathcal{D}[\mathfrak{m}]$ is isomorphic to μ_p.

Concerning assumption (b), one useful remark is that $\mathcal{D}[\mathfrak{m}]$ has a quotient isomorphic to μ_p if and only if $\mathcal{D}[\mathfrak{M}]$ has such a quotient. To see this, note first that the intersection of the kernels of all G_K-equivariant homomorphisms from $\mathcal{D}[\mathfrak{m}]$ to μ_p is an R-submodule of $\mathcal{D}[\mathfrak{m}]$. Thus, $\mathrm{Hom}_{G_K}(\mathcal{D}[\mathfrak{m}], \mu_p) \neq 0$ if and only if $\mathrm{Hom}_{G_K}(\mathcal{D}[\mathfrak{m}]/\mathfrak{M}\mathcal{D}[\mathfrak{m}], \mu_p) \neq 0$. Now one can regard both $\mathcal{D}[\mathfrak{M}]$ and $\mathcal{D}[\mathfrak{m}]/\mathfrak{M}\mathcal{D}[\mathfrak{m}]$ as representation spaces for G_K over $R/\mathfrak{M}$. The first is isomorphic to $\widetilde{\rho}$. As we explain below, the second is isomorphic to $\widetilde{\rho}^t$, where t is the dimension of $\widehat{R}[\mathfrak{m}]/\mathfrak{M}\widehat{R}[\mathfrak{m}]$ as an $R/\mathfrak{M}$-vector space. Equivalently, we have $t = \dim_{R/\mathfrak{M}}((R/\mathfrak{m}R)[\mathfrak{M}])$. Regarding $\mathcal{D}[\mathfrak{M}]$ and $\mathcal{D}[\mathfrak{m}]/\mathfrak{M}\mathcal{D}[\mathfrak{m}]$ as representation spaces for G_K over $\Lambda/\mathfrak{m} = \mathbf{F}_p$, the second is isomorphic to a direct sum of t copies of the first, and so the above remark then follows.

Now, note that if $a \in R$, then multiplication by a gives an R-endomorphism of $\widehat{R}$ and the induced action on $\widehat{R}[\mathfrak{M}]$ is simply multiplication by the reduction of a modulo $\mathfrak{M}$. The induced action of a on $\widehat{R}[\mathfrak{m}]/\mathfrak{M}\widehat{R}[\mathfrak{m}]$ is also multiplication by the reduction of a modulo $\mathfrak{M}$ on that t-dimensional vector space over $R\mathfrak{M}$.

Now if $g \in G_K$, then $\rho(g)$ is an $(n \times n)$-matrix A_g over R. The action of $\rho(g)$ on $\mathcal{D} = \widehat{R}^n$ is multiplication by A_g. The action of $\rho(g)$ on $\mathfrak{D}[\mathfrak{M}] = \widehat{R}^n$ is given by the reduction of A_g modulo $\mathfrak{M}$. The action of $\rho(g)$ on $\mathcal{D}[\mathfrak{m}] = \widehat{R}[\mathfrak{m}]^n$ is given by the reduction of that matrix modulo $\mathfrak{m}R$. The action of $\rho(g)$ on $\mathcal{D}[\mathfrak{m}]/\mathfrak{M}\mathcal{D}[\mathfrak{m}]$ is given by t copies of the reduction of A_g modulo $\mathfrak{M}$. Thus, we do have $\mathcal{D}[\mathfrak{m}]/\mathfrak{M}\mathcal{D}[\mathfrak{m}]$ isomorphic to $\widetilde{\rho}^t$.

COROLLARY 3.2.3

Assume that $\mathcal{D}$ is divisible as a Λ-module, that $\mathrm{LEO}(\mathcal{D})$ *is satisfied, and that* $\mathrm{coker}(\phi_{\mathcal{L}})$ *is a cotorsion Λ-module. Suppose that $\Sigma_0 \subset \Sigma$ and that there exists a nonarchimedean prime $\eta \in \Sigma_0$ such that $H^0(K_\eta, \mathcal{T}^*) = 0$. Then the map*

$$\phi_{\mathcal{L},\Sigma_0} : H^1(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v \in \Sigma - \Sigma_0} Q_{\mathcal{L}}(K_v, \mathcal{D})$$

is surjective.

Proof

Denoting $\phi_{\mathcal{L}}$ by ϕ and $\phi_{\mathcal{L},\Sigma_0}$ by ϕ', it is clear that $\mathrm{coker}(\phi')$ is a quotient of $\mathrm{coker}(\phi)$ and hence is a cotorsion Λ-module. That is the assumption we actually need in this proof. If one defines a local specification $\mathcal{L}'$ by letting

$$L'(K_v, \mathcal{D}) = H^1(K_v, \mathcal{D}) \quad \text{for } v \in \Sigma_0, \qquad L'(K_v, \mathcal{D}) = L(K_v, \mathcal{D}) \quad \text{for } v \in \Sigma - \Sigma_0,$$

then ϕ' is just the map $\phi_{\mathcal{L}'}$. Note that $Q_{\mathcal{L}'}(K_\eta, \mathcal{D}) = 0$. The assumptions in Proposition 3.2.1(c) are satisfied for the specification $\mathcal{L}'$. It therefore follows that ϕ' is indeed surjective. □

REMARK 3.2.4

The kernel of $\phi_{\mathcal{L},\Sigma_0} = \phi_{\mathcal{L}'}$ is $S_{\mathcal{L}'}(K, \mathcal{D})$, which one can think of as a *nonprimitive* Selmer group $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})$. It is defined just as $S_{\mathcal{L}}(K, \mathcal{D})$, but one omits the local conditions for the specification $\mathcal{L}$ corresponding to the primes $v \in \Sigma_0$. Of course, we have the obvious inclusion $S_{\mathcal{L}}(K, \mathcal{D}) \subseteq S^{\Sigma_0}(K, \mathcal{D})$, and the corresponding quotient $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})/S_{\mathcal{L}}(K, \mathcal{D})$ is isomorphic to a Λ-submodule of $\prod_{v \in \Sigma_0} Q_{\mathcal{L}}(K_v, \mathcal{D})$. If $\phi_{\mathcal{L}}$ is itself surjective, then one has an isomorphism

$$S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})/S_{\mathcal{L}}(K, \mathcal{D}) \cong \prod_{v \in \Sigma_0} Q_{\mathcal{L}}(K_v, \mathcal{D}).$$

This provides a useful way to study the structure of $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})/S_{\mathcal{L}}(K, \mathcal{D})$.

The following results follow immediately from Corollary 3.2.3. One just takes $\Sigma_0 = \{\eta\}$.

COROLLARY 3.2.5

Under the assumptions of Corollary 3.2.3, the natural map from $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ to $\mathrm{coker}(\phi_{\mathcal{L}})$ *is surjective.*

COROLLARY 3.2.6

Assume that $\mathcal{D}$ is divisible as a Λ-module, that $\mathrm{LEO}(\mathcal{D})$ *is satisfied, and that η is a nonarchimedean prime in Σ such that $H^0(K_\eta, \mathcal{T}^*) = 0$. Then the map*

$$H^1(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v \in \Sigma - \{\eta\}} H^1(K_v, \mathcal{D})/H^1(K_v, \mathcal{D})_{\Lambda\text{-div}}$$

is surjective. The kernel of that map contains $H^1(K_\Sigma/K, \mathcal{D})_{\Lambda\text{-div}}$.

This last corollary is an improved version of [Gr3, Proposition 6.11]. It follows that

$$H^1(K_\Sigma/K, \mathcal{D})/H^1(K_\Sigma/K, \mathcal{D})_{\Lambda\text{-div}} \tag{11}$$

has a certain quotient Λ-module involving only local cohomology groups. Proposition 2.2.10 describes when $H^1(K_v, \mathcal{T}^*)_{\Lambda\text{-tors}}$ is nontrivial. One can often determine that Λ-module precisely. By (5), one then obtains equivalent statements about its Pontryagin dual $H^1(K_v, \mathcal{D})/H^1(K_v, \mathcal{D})_{\Lambda\text{-div}}$. One then obtains sufficient conditions for (11) to be nontrivial and some information about its structure as a Λ-module.

REMARK 3.2.7

Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are specifications for $\mathcal{D}$ and Σ. For $i \in \{1, 2\}$, let $L_i(K_v, \mathcal{D})$ be the Λ-submodule of $H^1(K_v, \mathcal{D})$ for the specification $\mathcal{L}_i$. We will write $\mathcal{L}_1 \subseteq \mathcal{L}_2$ if we have $L_1(K_v, \mathcal{D}) \subseteq L_2(K_v, \mathcal{D})$ for all $v \in \Sigma$. It is then obvious that $\mathrm{coker}(\phi_{\mathcal{L}_2})$ is a quotient of $\mathrm{coker}(\phi_{\mathcal{L}_1})$ as a Λ-module. Thus, if $\mathrm{coker}(\phi_{\mathcal{L}_1})$ is Λ-cotorsion, then so is $\mathrm{coker}(\phi_{\mathcal{L}_2})$. The converse is clearly true if the quotient $L_2(K_v, \mathcal{D})/L_1(K_v, \mathcal{D})$ is Λ-cotorsion for all $v \in \Sigma$. In particular, if $\mathcal{L}$ is a given specification for $\mathcal{D}$ and Σ, we can define a new specification $\mathcal{L}_{\mathrm{div}}$ by replacing $L(K_v, \mathcal{D})$ by $L(K_v, \mathcal{D})_{\Lambda\text{-div}}$ for all $v \in \Sigma$. With this notation, $\mathrm{coker}(\phi_{\mathcal{L}})$ is Λ-cotorsion if and only if $\mathrm{coker}(\phi_{\mathcal{L}_{\mathrm{div}}})$ is Λ-cotorsion. Also, if $\mathrm{coker}(\phi_{\mathcal{L}_{\mathrm{div}}}) = 0$, then $\mathrm{coker}(\phi_{\mathcal{L}}) = 0$ too. The converse of that statement is not true in general.

3.3. Varying Σ

We now discuss the dependence of the kernel and cokernel of $\phi_{\mathcal{L}}$ on the choice of Σ under certain restrictive assumptions. We let $\Sigma_{\min}$ denote the set consisting of primes v of K such that either $v|p$ or v is archimedean or the inertia subgroup of G_{K_v} acts nontrivially on $\mathcal{T}$. We assume that $L(K_v, \mathcal{D})$ has been defined in some way for all $v \in \Sigma_{\min}$, and call the corresponding specification $\mathcal{L}_{\min}$. For $v \notin \Sigma_{\min}$, we assume that $L(K_v, \mathcal{D}) = 0$. Furthermore, we make the following assumption.

HYPOTHESIS 3.3.1

$H^0(K_v, \mathcal{D})$ is a cotorsion Λ-module for all $v \notin \Sigma_{\min}$.

Assume that $v \notin \Sigma_{\min}$. By definition, the action of G_{K_v} on $\mathcal{D}$ is unramified. Let $H^1_{\mathrm{unr}}(K_v, \mathcal{D})$ denote $H^1(K_v^{\mathrm{unr}}/K_v, \mathcal{D})$, the kernel of the restriction map $H^1(K_v, \mathcal{D}) \to H^1(K_v^{\mathrm{unr}}, \mathcal{D}))$. It is straightforward to show that $H^0(K_v, \mathcal{D})$

and $H^1_{\mathrm{unr}}(K_v,\mathcal{D})$ have the same Λ-corank. If one assumes that $\mathcal{D}$ is a divisible Λ-module, then one finds that $H^1_{\mathrm{unr}}(K_v,\mathcal{D})$ vanishes if $H^0(K_v,\mathcal{D})$ is Λ-cotorsion. Thus, assuming that $\mathcal{D}$ is Λ-divisible, Hypothesis 3.3.1 means that $H^1_{\mathrm{unr}}(K_v,\mathcal{D})=0$ for all $v\notin\Sigma_{\mathrm{min}}$.

Suppose that Σ_1 and Σ_2 are finite sets of primes of K, both containing Σ_{min}. Assume also that $\Sigma_1\subseteq\Sigma_2$. The definition of $L(K_v,\mathcal{D})$ described above gives specifications $\mathcal{L}_1$ and $\mathcal{L}_2$ for the sets Σ_1 and Σ_2. Note that since the action of G_K on $\mathcal{D}$ factors through $\mathrm{Gal}(K_{\Sigma_1}/K)$, we have

$$H^1(K_{\Sigma_2}/K_{\Sigma_1},\mathcal{D})=\mathrm{Hom}\big(\mathrm{Gal}(K_{\Sigma_2}/K_{\Sigma_1}),\mathcal{D}\big).$$

We assume that Hypothesis 3.3.1 is satisfied. Since the inertia subgroups of $\mathrm{Gal}(K_{\Sigma_2}/K_{\Sigma_1})$ generate a dense subgroup, it follows that we have an exact sequence

$$0\longrightarrow H^1(K_{\Sigma_1}/K,\mathcal{D})\longrightarrow H^1(K_{\Sigma_2}/K,\mathcal{D})\xrightarrow{\beta}\bigoplus_{v\in\Sigma_2-\Sigma_1}H^1(K_v,\mathcal{D}).\tag{12}$$

We then obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & H^1(K_{\Sigma_1}/K,\mathcal{D}) & \to & H^1(K_{\Sigma_2}/K,\mathcal{D}) & \to & H^1(K_{\Sigma_2}/K,\mathcal{D})/H^1(K_{\Sigma_1}/K,\mathcal{D}) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_{\mathcal{L}_1}(K,\mathcal{D}) & \longrightarrow & Q_{\mathcal{L}_2}(K,\mathcal{D}) & \longrightarrow & Q_{\mathcal{L}_2}(K,\mathcal{D})/Q_{\mathcal{L}_1}(K,\mathcal{D}) & \longrightarrow & 0
\end{array}$$

where the rows are exact, the first two vertical maps are $\phi_{\mathcal{L}_1}$ and $\phi_{\mathcal{L}_2}$, respectively, and the third map is induced by the global-to-local map β.

The exactness of (12) implies the injectivity of the third vertical map. Applying the snake lemma to the above commutative diagram gives us the following proposition.

PROPOSITION 3.3.2

Assume that $\mathcal{D}$ is a divisible Λ-module, that Hypothesis 3.3.1 is satisfied, that $L(K_v,\mathcal{D})=0$ for all $v\notin\Sigma_{\mathrm{min}}$, and that $\Sigma_1\subseteq\Sigma_2$ are finite sets of primes of K containing Σ_{min}. Then the maps

$$\ker(\phi_{\mathcal{L}_1})\longrightarrow\ker(\phi_{\mathcal{L}_2}),\qquad \mathrm{coker}(\phi_{\mathcal{L}_1})\longrightarrow\mathrm{coker}(\phi_{\mathcal{L}_2})$$

are both injective. Furthermore, the first map is also surjective, and the cokernel of the second map is isomorphic to $\mathrm{coker}(\beta)$.

One can weaken the hypotheses somewhat. It suffices to make the assumption that $L(K_v,\mathcal{D})=0$ and that $H^0(K_v,\mathcal{D})$ is Λ-cotorsion just for the primes v in $\Sigma_2-\Sigma_1$.

One can regard the map β as the map $\phi_{\mathcal{M}_2}$, where $\mathcal{M}_2$ is the following specification for Σ_2:

$$M_2(K_v,\mathcal{D})=H^1(K_v,\mathcal{D})\quad\text{for } v\in\Sigma_1,\qquad M_2(K_v,\mathcal{D})=0\quad\text{for } v\in\Sigma_2-\Sigma_1.$$

Since $\ker(\phi_{\mathcal{M}_2}) = H^1(K_{\Sigma_1}/K, \mathcal{D})$, the third vertical map in the above diagram is injective. Its cokernel is precisely the cokernel of $\phi_{\mathcal{M}_2}$. We can examine $\mathrm{coker}(\phi_{\mathcal{M}_2})$ by using Proposition 3.1.1. Note that $M_2(K_v, \mathcal{D}) = H^1_{\mathrm{unr}}(K_v, \mathcal{D})$ for $v \in \Sigma_2 - \Sigma_1$. Its orthogonal complement $M_2^*(K_v, \mathcal{T}^*)$ is $H^1_{\mathrm{unr}}(K_v, \mathcal{T}^*)$. Therefore, just as for (12), we have an exact sequence

$$(13) \qquad 0 \longrightarrow H^1(K_{\Sigma_1}/K, \mathcal{T}^*) \longrightarrow H^1(K_{\Sigma_2}/K, \mathcal{T}^*) \longrightarrow \bigoplus_{v \in \Sigma_2 - \Sigma_1} H^1(K_v, \mathcal{T}^*).$$

For $v \in \Sigma_1$, we have $M_2^*(K_v, \mathcal{T}^*) = 0$. It follows that the corresponding Selmer group $S_{\mathcal{M}_2}(K, \mathcal{T}^*)$ is isomorphic to the image of $Ш^1(K, \Sigma_1, \mathcal{T}^*)$ under the inflation map in (13) and that

$$\widehat{\mathrm{coker}(\beta)} = \widehat{\mathrm{coker}(\phi_{\mathcal{M}_2})} \cong Ш^1(K, \Sigma_1, \mathcal{T}^*)/Ш^1(K, \Sigma_2, \mathcal{T}^*).$$

In particular, if we are in a situation where $Ш^1(K, \Sigma_1, \mathcal{T}^*) = 0$, then it follows that $\mathrm{coker}(\beta) = 0$.

The above observations and Proposition 3.3.2 have the following useful consequence.

PROPOSITION 3.3.3

Assume that $\mathcal{D}$ is a divisible Λ-module and that hypothesis 3.3.1 is satisfied. Consider the following global-to-local map:

$$\psi : H^1(K, \mathcal{D}) \longrightarrow \Big(\bigoplus_{v \in \Sigma_{\min}} Q_{\mathcal{L}}(K_v, \mathcal{D}) \Big) \oplus \Big(\bigoplus_{v \notin \Sigma_{\min}} H^1(K_v, \mathcal{D}) \Big).$$

Let Σ be a finite set of primes of K containing $\Sigma_{\min}$, and let $\mathcal{L}$ be the corresponding specification, as defined above. Then $\ker(\psi) \cong \ker(\phi_{\mathcal{L}})$. *Furthermore, if one assumes in addition that* $Ш^1(K, \Sigma_{\min}, \mathcal{T}^*)$ *vanishes, then* $\mathrm{coker}(\psi) \cong \mathrm{coker}(\phi_{\mathcal{L}})$.

Note that any element of $H^1(K, \mathcal{D})$ is unramified at all but finitely many primes v of K. Since we are assuming Hypothesis 3.3.1, it follows that the image of ψ is indeed contained in the direct sum.

Proof

The assumption that $Ш^1(K, \Sigma_{\min}, \mathcal{T}^*)$ vanishes implies that $Ш^1(K, \Sigma', \mathcal{T}^*)$ also vanishes for any finite set Σ' containing $\Sigma_{\min}$. If $\Sigma_1 \subseteq \Sigma_2$ are two such sets, then the fact that $\mathrm{coker}(\beta) = 0$ and Proposition 3.3.2 imply that the second map in Proposition 3.3.2 is an isomorphism. Let Σ' vary over all finite sets of primes of K containing Σ, ordered by inclusion. It follows that $\mathrm{coker}(\phi_{\mathcal{L}})$ is isomorphic to the direct limit of these cokernels. But that direct limit is precisely $\mathrm{coker}(\psi)$, and so the stated isomorphism for the cokernels follows. Similarly, the stated isomorphism for the kernels follows from the fact that the first map in Proposition 3.3.2 is an isomorphism. $\square$

Although we do not pursue this topic further, one can study what happens if $\text{Ш}^1(K, \Sigma_{\min}, \mathcal{T}^*)$ is nontrivial. A useful tool would be the analogue of Proposition 3.1.1 when the roles of $\mathcal{D}$ and $\mathcal{T}^*$ are reversed. However, in situations that come up naturally in Iwasawa theory, if the Krull dimension of Λ is at least 2, then one generally expects $\text{Ш}^1(K, \Sigma_{\min}, \mathcal{T}^*)$ to vanish (although exceptions can be constructed). When Λ has Krull dimension 1, it is not so uncommon for $\text{Ш}^1(K, \Sigma_{\min}, \mathcal{T}^*)$ to be nontrivial and even to have positive Λ-rank. This issue is discussed in some detail in [Gr3, Section 6, Part D]. We have some additional comments when $\Lambda = \mathbf{Z}_p$ in the next section, where we discuss the p-adic Tate module for an abelian variety.

3.4. Examples from Hida theory

Hida's theory of families of ordinary modular forms provides examples of Galois representations ρ of rank $n = 2$ over various complete Noetherian local rings R. We refer the reader to [Hid], [EPW], and [Och] for a discussion of these representations. In these examples, there is a canonical subring Λ of R. Its Krull dimension is either 2 (the one-variable case) or 3 (the two-variable case). All of these rings R are constructed somehow from Hida's universal ordinary Hecke algebra for a fixed level (or levels, as in [EPW]). These rings are not necessarily domains. However, one may replace R by $R/\mathfrak{a}$, where $\mathfrak{a}$ is a minimal prime ideal of R, obtaining a domain, and ρ by its reduction modulo $\mathfrak{a}$. Even if R is already a domain, one can replace R by various possibly larger rings in its field of fractions $\mathcal{K}$, for example, its reflexive hull as a Λ-module or its integral closure in $\mathcal{K}$. Both of those domains are also finitely generated as Λ-modules. This is clear for the reflexive hull. For the integral closure, this assertion follows from the theorem of Nagata mentioned in the introduction. In either case, [Coh, Theorem 7] then implies that the new ring is again a complete Noetherian local ring. Also, the residue field is still finite. One obtains a representation over the new ring from ρ by extending scalars.

The residual representation $\widetilde{\rho}$ is 2-dimensional over the residue field of R. We assume that $\widetilde{\rho}$ is irreducible. Proposition 2.2.1 and Remark 3.2.2 then imply that $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ vanishes. Suppose that $\mathcal{L}$ is a specification for ρ and Σ. Proposition 3.2.1(a) implies that $\phi_{\mathcal{L}}$ is surjective if one makes the assumption that $\mathrm{LEO}(\mathcal{D})$ is satisfied and that $\mathrm{coker}(\phi_{\mathcal{L}})$ is Λ-cotorsion.

Let $r = \mathrm{rank}_\Lambda(R)$. The discrete Galois module $\mathcal{D}$ has Λ-corank $2r$. There is a natural specification $\mathcal{L}$ in this situation. One can find a description of $\mathcal{L}$ in [Gr2] and also in [Och] with more detail. Theorem 3.10 in [Och] gives the surjectivity of $\phi_{\mathcal{L}}$, except for one case where the Selmer group $S_{\mathcal{L}}(K, \mathcal{D})$ may fail to be Λ-cotorsion. Ochiai refers to this as the *diagonal* case. Roughly speaking, in the diagonal case, $S_{\mathcal{L}}(K, \mathcal{D})$ may turn out to have positive Λ-corank if a certain root number is -1. In that situation, $\mathrm{coker}(\phi_{\mathcal{L}})$ would also turn out to have positive Λ-corank. We exclude this case in the rest of this discussion.

For all the examples mentioned above, apart from the diagonal case, one finds that $q_{\mathcal{L}}(K, \mathcal{D}) = r$. For the quantity $b_1(K, \mathcal{D})$ mentioned in the introduction, one also finds that $b_1(K,\mathcal{D})=r$. One verifies both of those assertions by a nontrivial *specialization* argument, reducing to a study of the 2-dimensional representations associated to modular forms of varying weight. Furthermore, by using theorems of Kato and Rohrlich, one shows that $S_{\mathcal{L}}(K,\mathcal{D})$ is a cotorsion Λ-module. This assertion is contained in [Och, Theorem 3, Proposition 3.4]. One therefore has equality in (4). It follows that $\mathrm{LEO}(\mathcal{D})$ is satisfied and that $\mathrm{coker}(\phi_{\mathcal{L}})$ is Λ-cotorsion. Consequently, under the assumption that $\widetilde{\rho}$ is irreducible, we can conclude that $\phi_{\mathcal{L}}$ is surjective.

4. The Tate module of an abelian variety

Assume that A is an abelian variety of dimension g defined over K. Let p be any prime. We illustrate the results of Sections 2 and 3 in the case where $R=\Lambda=\mathbf{Z}_p$ and $\mathcal{T}=T_p(A)$. Thus, $\mathcal{D}=A[p^\infty]$, the group of p-power torsion points on A. We can take Σ to be any finite set of primes of K containing the primes lying over p and ∞ and the primes where A has bad reduction. The minimal such set is denoted by $\Sigma_{\min}$, just as in Section 3.3. If we choose a $\mathbf{Z}_p$-module basis for $\mathcal{T}$, then we can take $\rho:\ \mathrm{Gal}(K_\Sigma/K)\to \mathrm{GL}_{2g}(\mathbf{Z}_p)$ to be the homomorphism giving the natural action of $\mathrm{Gal}(K_\Sigma/K)$ on $\mathcal{T}$. Note also that the Weil pairing shows that $\mathcal{T}^*\cong T_p(B)$, where B is the dual abelian variety for A. The results in Section 3 provide a proof of a well-known theorem of Cassels, as we discuss in Section 4.5.

4.1. Various ranks and coranks

We first determine the $\mathbf{Z}_p$-corank of $Q_{\mathcal{L}}(K,\mathcal{D})$. As in the introduction, the local specification $\mathcal{L}$ is defined as follows. For each $v\in\Sigma$, let $L(K_v,D)$ be the image of the local Kummer map κ_v. Thus, $L(K_v,D)$ is a divisible $\mathbf{Z}_p$-module for all $v\in\Sigma$. In fact, for $v\nmid p$, $H^1(K_v,D)$ is a finite group and we have $L(K_v,D)=0$. This is true even if v is archimedean. On the other hand, if $v|p$, then it is known that $A(\mathbf{Q}_v)$ contains a subgroup of finite index which is a free $\mathbf{Z}_p$-module of rank $g[K_v:\mathbf{Q}_p]$. Therefore, $A(\mathbf{Q}_v)\otimes(\mathbf{Q}_p/\mathbf{Z}_p)$ is a cofree $\mathbf{Z}_p$-module with corank $g[K_v:\mathbf{Q}_p]$. Since κ_v is injective, it follows that $L(K_v,D)$ has the same $\mathbf{Z}_p$-corank.

Now $H^1(K_v,\mathcal{D})$ is finite if $v\nmid p$ and has $\mathbf{Z}_p$-corank equal to $2[K_v:\mathbf{Q}_p]g$ if $v|p$. These facts are consequences of the formula for the local Euler-Poincaré characteristic for the $G_{\mathbf{Q}_v}$-module $\mathcal{D}$ (which involves the $\mathbf{Z}_p$-coranks of $H^i(K_v,\mathcal{D})$, where $0\le i\le 2$). It then follows that the $\mathbf{Z}_p$-corank of $Q_{\mathcal{L}}(K_v,\mathcal{D})$ is zero for $v\nmid p$ and is equal to $g[K_v:\mathbf{Q}_p]$ for $v|p$. Summing over all $v\in\Sigma$, we see that the $\mathbf{Z}_p$-corank of $Q_{\mathcal{L}}(K,D)$ is $[K:\mathbf{Q}]g$, as stated in the introduction. It was denoted there by $q_{\mathcal{L}}(K,\mathcal{D})$.

If v is a nonarchimedean prime, then the torsion subgroup of $B(K_v)$ is finite. In particular, $H^0(K_v,B[p^\infty])$ is finite. It follows that $H^0(K_v,\mathcal{T}^*)=0$ for all nonarchimedean primes v in Σ. This has the following consequences. By Proposition 2.1.1, $\text{Ш}^1(K,\Sigma,\mathcal{T}^*)$ is torsion free and $\text{Ш}^2(K,\Sigma,\mathcal{D})$ is divisible. Also, $H^2(K_v,\mathcal{D})=0$ for all nonarchimedean primes in Σ and for all primes in Σ if p is odd. Therefore, we have $H^2(K_\Sigma/K,\mathcal{D}) = Ш^2(K,\Sigma,\mathcal{D})$ when p is odd. For $p=2$, $H^2(K_\Sigma/K,\mathcal{D})/Ш^2(K,\Sigma,\mathcal{D})$ is a finite group of exponent 2.

We now discuss the $\mathbf{Z}_p$-corank of $H^1(K_\Sigma/K,\mathcal{D})$. The Euler-Poincaré characteristic for the $\mathrm{Gal}(K_\Sigma/K)$-module $\mathcal{D}$ is known to be $-[K:\mathbf{Q}]g$. Also, the torsion subgroup of $A(K)$ is finite, and so $H^0(K_\Sigma/K,\mathcal{D})$ is certainly finite. Now $H^2(K_\Sigma/K,\mathcal{D})$ and $Ш^2(K,\Sigma,\mathcal{D})$ have the same $\mathbf{Z}_p$-corank for any prime p. This gives the formula

$$\mathrm{corank}_{\mathbf{Z}_p}\big(H^1(K_\Sigma/K,\mathcal{D})\big) = [K:\mathbf{Q}]g + \mathrm{corank}_{\mathbf{Z}_p}\big(Ш^2(K,\Sigma,\mathcal{D})\big), \tag{14}$$

and so the quantity denoted by $b_1(K,\mathcal{D})$ in the introduction is equal to $[K:\mathbf{Q}]g$. Note that $b_1(K,\mathcal{D}) = q_{\mathcal{L}}(K,\mathcal{D})$. Also, it follows from (2) that

$$\mathrm{corank}_{\mathbf{Z}_p}\big(S_{\mathcal{L}}(K,\mathcal{D})\big) = \mathrm{corank}_{\mathbf{Z}_p}\big(Ш^2(K,\Sigma,\mathcal{D})\big) + \mathrm{corank}_{\mathbf{Z}_p}\big(\mathrm{coker}(\phi_{\mathcal{L}})\big). \tag{15}$$

4.2. The torsion subgroup of $H^1(K_\Sigma/K,\mathcal{T}^*)$ and $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$

Since $\Lambda = \mathbf{Z}_p$, if X is a Λ-module, then $X_{\Lambda\text{-tors}}$ is just the torsion subgroup X_{tors} of X. We have the following result.

PROPOSITION 4.2.1

With the above notation, we have the following equalities and isomorphism:

$$S_{\mathcal{L}^*}(K,\mathcal{T}^*)_{\mathrm{tors}} = H^1(K_\Sigma/K,\mathcal{T}^*)_{\mathrm{tors}} \cong H^0(K,B[p^\infty]) = B(K)_p.$$

In particular, $H^1(K_\Sigma/K,\mathcal{T}^*)$ and $S_{\mathcal{L}^*}(K,\mathcal{T}^*)$ are torsion free if and only if $B(K)_p = 0$.

Proof

The fact that $L(K_v,\mathcal{D})$ is divisible for all $v\in\Sigma$ together with Proposition 2.3.1 implies the first equality. One can apply Proposition 2.2.2 to $\mathcal{T}^* = T_p(B)$ and $\theta = p^t$ for t sufficiently large to obtain the isomorphism. Alternatively, one can also derive this directly from the following exact sequence. It involves the $\mathbf{Q}_p$-representation space $V_p(B) = T_p(B)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ for $\mathrm{Gal}(K_\Sigma/K)$.

$$0 \longrightarrow T_p(B) \longrightarrow V_p(B) \longrightarrow B[p^\infty] \longrightarrow 0. \tag{16}$$

The corresponding cohomology sequence proves that isomorphism since we have $H^0(K,V_p(B)) = 0$ and $H^1(K_\Sigma/K,V_p(B))$ is torsion free. By definition, we have $H^0(K,B[p^\infty]) = B(K)_p$. □

4.3. Hypothesis $\mathrm{LEO}(\mathcal{D})$

Proposition 2.2.1 implies that $Ш^2(K,\Sigma,\mathcal{D})$ is a divisible group. Therefore, $\mathrm{LEO}(\mathcal{D})$ means that $Ш^2(K,\Sigma,\mathcal{D}) = 0$. Equivalently, $H^2(K_\Sigma/K,\mathcal{D})$ has $\mathbf{Z}_p$-corank zero. This means that $H^2(K_\Sigma/K,\mathcal{D})$ vanishes if p is odd and is elementary abelian if $p=2$. The following result gives other equivalent versions of $\mathrm{LEO}(\mathcal{D})$. We let $\mathcal{D}^* = \mathcal{T}^*\otimes_{\mathbf{Z}_p}(\mathbf{Q}_p/\mathbf{Z}_p)$. One can identify $\mathcal{D}^*$ with $B[p^\infty]$.

PROPOSITION 4.3.1
Let $\mathcal{D} = A[p^\infty], \mathcal{D}^* = B[p^\infty]$, and $\mathcal{T}^* = T_p(B)$. The $\mathbf{Z}_p$-coranks of

$$Ш^1(K,\Sigma,\mathcal{D}), \qquad Ш^1(K,\Sigma,\mathcal{D}^*), \qquad Ш^2(K,\Sigma,\mathcal{D}), \qquad \textit{and} \qquad Ш^2(K,\Sigma,\mathcal{D}^*)$$

are all equal to the $\mathbf{Z}_p$-rank of $Ш^1(K,\Sigma,\mathcal{T}^*)$. In particular, LEO($\mathcal{D}$) *is satisfied if and only if any of the above groups is finite.*

We remark that $Ш^1(K,\Sigma,\mathcal{D})$ can be finite and still nontrivial, in contrast to $Ш^2(K,\Sigma,\mathcal{D})$ and $Ш^1(K,\Sigma,\mathcal{T}^*)$.

Proof
The fact that B is isogenous to A over K implies that the groups $Ш^i(K,\Sigma,\mathcal{D})$ and $Ш^i(K,\Sigma,\mathcal{D}^*)$ have the same $\mathbf{Z}_p$-corank for any $i \geq 0$. This is of interest only for $i \in \{1,2\}$ since those groups are trivial otherwise. By the pairing (6), we have $\mathrm{corank}_{\mathbf{Z}_p}(Ш^2(K,\Sigma,\mathcal{D})) = \mathrm{rank}_{\mathbf{Z}_p}(Ш^1(K,\Sigma,\mathcal{T}^*))$. It suffices then to show that $\mathrm{rank}_{\mathbf{Z}_p}(Ш^1(K,\Sigma,\mathcal{T}^*)) = \mathrm{corank}_{\mathbf{Z}_p}(Ш^1(K,\Sigma,\mathcal{D}^*))$. However, both of these quantities are equal to the $\mathbf{Q}_p$-dimension of $Ш^1(K,\Sigma,V_p(B))$. □

It is difficult to state a precise conjecture predicting when LEO($\mathcal{D}$) is satisfied. Of course, one sufficient condition is that $S_{\mathcal{L}}(K,\mathcal{D})$ be finite, as pointed out in the introduction. To state a more general criterion, we assume that $Ш_A(K)_p$, the p-primary subgroup of the Tate-Shafarevich group for A over K, is finite. Consider the Kummer homomorphism

$$\kappa : A(K) \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow H^1(K_\Sigma/K,\mathcal{D}).$$

Obviously, we have $Ш^1(K,\Sigma,\mathcal{D}) \subseteq S_{\mathcal{L}}(K,\mathcal{D})$. Our assumption about $Ш_A(K)_p$ means that $[S_{\mathcal{L}}(K,\mathcal{D}) : \mathrm{im}(\kappa)]$ is finite. Hence, $Ш^1(K,\Sigma,\mathcal{D})$ and the intersection $Ш^1(K,\Sigma,\mathcal{D}) \cap \mathrm{im}(\kappa)$ have the same $\mathbf{Z}_p$-corank. Since κ is injective, it follows that $Ш^1(K,\Sigma,\mathcal{D}) \cap \mathrm{im}(\kappa)$ is isomorphic to the kernel of the map

$$\varepsilon : A(K) \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow \bigoplus_{v|p} A(K_v) \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p).$$

Therefore, under the assumption that $Ш_A(K)_p$ is finite, we have

$$\mathrm{corank}_{\mathbf{Z}_p}\big(\ker(\varepsilon)\big) = \mathrm{corank}_{\mathbf{Z}_p}\big(Ш^1(K,\Sigma,\mathcal{D})\big).$$

In particular, LEO($\mathcal{D}$) is satisfied if and only if ε has finite kernel. One can view $A(K)$ as a subgroup of $\bigoplus_{v|p} A(K_v)$ by the diagonal embedding. The latter group contains a subgroup of finite index isomorphic to $\mathbf{Z}_p^{[K:\mathbf{Q}]g}$. If $r = \mathrm{rank}(A(K))$, then one can choose independent points $P_1,\ldots,P_r$ in $A(K)$ which are in that subgroup. One then sees easily that $\ker(\varepsilon)$ is finite if and only if $P_1,\ldots,P_r$ are $\mathbf{Z}_p$-independent.

Suppose now that A is defined over $\mathbf{Q}$ and that K is an abelian extension of $\mathbf{Q}$. One can regard $A(K) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p$ as a representation space over $\overline{\mathbf{Q}}_p$ for $\mathrm{Gal}(K/\mathbf{Q})$. For any character χ of $\mathrm{Gal}(K/\mathbf{Q})$, let $r_\chi(A)$ denote the multiplicity of χ in $A(K) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p$. As above, we continue to assume that $Ш_A(K)_p$ is finite.

The map ε is $\mathrm{Gal}(K/\mathbf{Q})$-equivariant. Suppose that $\mathcal{A} = A(K) \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p)$ and $\mathcal{B} = \bigoplus_{v|p} A(K_v) \otimes_{\mathbf{Z}} (\mathbf{Q}_p/\mathbf{Z}_p)$. If ε has finite kernel, then the adjoint map $\widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ has finite cokernel. Hence, we have a surjective map $\widehat{\mathcal{B}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \to \widehat{\mathcal{A}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ of representation spaces for $\mathrm{Gal}(K/\mathbf{Q})$. If $g = 1$, then $\widehat{\mathcal{B}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is isomorphic to the regular representation of $\mathrm{Gal}(K/\mathbf{Q})$ over $\mathbf{Q}_p$. It follows that if $\mathrm{LEO}(\mathcal{D})$ is satisfied for A and K, then we have $r_\chi(A) \leq 1$ for all χ. The following conjecture is the converse.

CONJECTURE 4.3.2

Suppose that A is an elliptic curve defined over $\mathbf{Q}$ and that K is an abelian extension of $\mathbf{Q}$. Then $\mathrm{LEO}(\mathcal{D})$ is satisfied if $r_\chi(A) \leq 1$ for all characters χ of $\mathrm{Gal}(K/\mathbf{Q})$.

If $K = \mathbf{Q}$, then Conjecture 4.3.2 is easily proven. One may assume that we have $r = \mathrm{rank}(A(\mathbf{Q})) = 1$. As we explain in Remark 4.4.3, the image of ε is then infinite. It follows that the kernel of ε is indeed finite. This argument can be extended to the case where $\mathrm{Gal}(K/\mathbf{Q})$ has exponent 2 or, more generally, where we have $[\mathbf{Q}(\chi) : \mathbf{Q}] = [\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ for all the characters χ of $\mathrm{Gal}(K/\mathbf{Q})$. Furthermore, Conjecture 4.3.2 can be proven if E is an elliptic curve with complex multiplication. This case follows from a result in transcendental number theory, a theorem of Bertrand [Ber, théorème 3] giving the analogue of the Baker-Brumer theorem for the formal group logarithm for E.

4.4. The cokernel of $\phi_{\mathcal{L}}$

We prove the following partial result.

PROPOSITION 4.4.1

The order of $\mathrm{coker}(\phi_{\mathcal{L}})/\mathrm{coker}(\phi_{\mathcal{L}})_{\mathrm{div}}$ is divisible by the order of $B(K)_p$. If $S_{\mathcal{L}}(K, \mathcal{D})$ is finite, then $\mathrm{coker}(\phi_{\mathcal{L}})$ is finite and is isomorphic to the Pontryagin dual of $B(K)_p$.

Proof

The fact that $\text{Ш}^1(K, \Sigma, \mathcal{T}^*)$ is torsion free and Proposition 3.1.1 imply that $\widehat{\mathrm{coker}(\phi_{\mathcal{L}})}$ has a subgroup isomorphic to $S_{\mathcal{L}^*}(K, \mathcal{T}^*)_{\mathrm{tors}}$. This group is isomorphic to $B(K)_p$ according to Proposition 4.2.1. The first assertion follows.

As explained in the introduction, if we assume that $S_{\mathcal{L}}(K, \mathcal{D})$ is finite, then $\text{Ш}^2(K, \Sigma, \mathcal{D})$ and $\mathrm{coker}(\phi_{\mathcal{L}})$ are both finite. Therefore, it follows that $S_{\mathcal{L}^*}(K, \mathcal{T}^*)$ is finite and that $\text{Ш}^1(K, \Sigma, \mathcal{T}^*) = 0$. The stated isomorphism then follows from Proposition 3.1.1. □

COROLLARY 4.4.2

If $B(K)_p \neq 0$, then $\phi_{\mathcal{L}}$ is not surjective.

REMARK 4.4.3

If $S_{\mathcal{L}}(K,\mathcal{D})$ is infinite, then it should also be true that $\operatorname{coker}(\phi_{\mathcal{L}})$ is infinite. One can at least show this if $A(K)$ is infinite. First of all, note that if $P\in A(K)$ has infinite order, then $\langle P\rangle\otimes_{\mathbf{Z}}(\mathbf{Q}_p/\mathbf{Z}_p)$ is an infinite subgroup of $A(K_v)\otimes_{\mathbf{Z}}(\mathbf{Q}_p/\mathbf{Z}_p)$ for any $v|p$. It follows that

$$\operatorname{corank}_{\mathbf{Z}_p}\big(\text{Ш}^1(K,\Sigma,\mathcal{D})\big)<\operatorname{corank}_{\mathbf{Z}_p}\big(S_{\mathcal{L}}(K,\mathcal{D})\big).$$

Therefore, using Proposition 4.3.1 together with (15), one then indeed has $\operatorname{corank}_{\mathbf{Z}_p}(\operatorname{coker}(\phi_{\mathcal{L}}))>0$. One also has the trivial upper bound $[K:\mathbf{Q}]g$ on the $\mathbf{Z}_p$-corank of $\operatorname{coker}(\phi_{\mathcal{L}})$, which is just $q_{\mathcal{L}}(K,\mathcal{D})$. In particular, suppose that $K=\mathbf{Q}$ and $g=1$. Then $\operatorname{coker}(\phi_{\mathcal{L}})$ has $\mathbf{Z}_p$-corank ≤ 1.

4.5. The classical definition of the Selmer group

One usually defines $\operatorname{Sel}_A(K)$ to be the kernel of the map

$$\phi_{K,A}: H^1\big(K,A(\overline{K})_{\text{tors}}\big)\longrightarrow\bigoplus_v H^1\big(K_v,A(\overline{K}_v)\big), \tag{17}$$

where v varies over *all* primes of K. The p-primary subgroup of $A(\overline{K})_{\text{tors}}$ is $\mathcal{D}=A[p^\infty]$, and $\operatorname{Sel}_A(K)_p$ is a subgroup of $H^1(K,\mathcal{D})$. We now explain why the inflation map from $H^1(K_\Sigma/K,\mathcal{D})$ to $H^1(K,\mathcal{D})$ induces an isomorphism from $S_{\mathcal{L}}(K,\mathcal{D})$ to $\operatorname{Sel}_A(K)_p$. This turns out to follow from Proposition 3.3.3. First of all, note that Hypothesis 3.3.1 is satisfied because $A(K_v)_{\text{tors}}$ is finite for every nonarchimedean prime v of K. Furthermore, $L(K_v,\mathcal{D})=0$ for all $v\nmid p$. Finally, note that for all primes v, we have an exact sequence

$$0\longrightarrow\operatorname{im}(\kappa_v)\longrightarrow H^1(K_v,\mathcal{D})\longrightarrow H^1\big(K_v,A(\overline{K}_v)\big)_p\longrightarrow 0$$

and therefore we have $\ker(\phi_{K,A})_p=\ker(\psi)$, where ψ is the map occurring in Proposition 3.3.3 for $\mathcal{D}=D$. We also obtain an isomorphism from $\operatorname{coker}(\phi_{K,A})_p$ to $\operatorname{coker}(\psi)$.

Proposition 3.3.3 implies that the map from $\ker(\phi_{\mathcal{L}})$ to $\ker(\psi)_p$ is always an isomorphism. This gives the identification of $S_{\mathcal{L}}(K,\mathcal{D})$ to $\operatorname{Sel}_A(K)_p$, as mentioned above. Proposition 3.3.3 implies that the injective map from $\operatorname{coker}(\phi_{\mathcal{L}})$ to $\operatorname{coker}(\psi)_p$ is an isomorphism if we assume that $\text{Ш}^2(K,\Sigma,\mathcal{D})=0$. In particular, this is so if $\operatorname{Sel}_A(K)_p=S_{\mathcal{L}}(K,\mathcal{D})$ is finite.

The theorem of Cassels alluded to previously states that if $\operatorname{Sel}_A(K)$ is finite, then the cokernel of $\phi_{K,A}$ is isomorphic to the Pontryagin dual of $B(K)_{\text{tors}}$. To prove this, it is enough to prove that the p-primary subgroups of those groups are isomorphic for every prime p, and that assertion follows from the second part of Proposition 4.4.1.

Cassels also proved a theorem including the case where $\operatorname{Sel}_A(K)_p$ is infinite, at least under the assumption that $\text{Ш}_A(K)_p$ is finite. This theorem asserts that the Pontryagin dual of $\operatorname{coker}(\phi_{K,A})_p$ is isomorphic to $B(K)\otimes_{\mathbf{Z}}\mathbf{Z}_p$. It follows that the $\mathbf{Z}_p$-corank of $\operatorname{coker}(\phi_{K,A})_p$ is equal to $\operatorname{rank}(B(K))=\operatorname{rank}(A(K))$. One finds a discussion and proof of this result in [Bas].

As a consequence, it is possible for $\mathrm{coker}(\phi_{K,A})_p$ and $\mathrm{coker}(\phi_{\mathcal{L}})$ to have different $\mathbf{Z}_p$-coranks. For example, consider the special case where $K=\mathbf{Q}$, $g=1$, and $r=\mathrm{rank}(A(\mathbf{Q}))\geq 2$. Assume that $Ш_A(K)_p$ is finite. Thus, by Remark 4.4.3, $\mathrm{coker}(\phi_{\mathcal{L}})$ has $\mathbf{Z}_p$-corank 1. That is, we have $\mathrm{coker}(\phi_{\mathcal{L}})_{\mathrm{div}}\cong\mathbf{Q}_p/\mathbf{Z}_p$. This is true for any finite set Σ containing $\Sigma_{\min}$. However, $\mathrm{coker}(\phi_{K,A})_p$ is the direct limit of the groups $\mathrm{coker}(\phi_{\mathcal{L}})$ as Σ varies over all those finite sets. Thus, if one assumes that $Ш_A(K)_p$ is finite, then that direct limit turns out to have $\mathbf{Z}_p$-corank r. Evidently, the finite groups $\mathrm{coker}(\phi_{\mathcal{L}})/\mathrm{coker}(\phi_{\mathcal{L}})_{\mathrm{div}}$ have unbounded exponent as Σ varies if $r\geq 2$.

5. Twist deformations

Suppose that K_∞/K is a Galois extension and that $\Gamma=\mathrm{Gal}(K_\infty/K)\cong\mathbf{Z}_p^m$ for some $m\geq 1$. Let $\Lambda=\mathbf{Z}_p[[\Gamma]]$, the completed group algebra for Γ over $\mathbf{Z}_p$. If $\{\gamma_1,\ldots,\gamma_m\}$ is a set of topological generators for Γ, then one can define an isomorphism from the formal power series ring $\mathbf{Z}_p[[x_1,\ldots,x_m]]$ to Λ by sending x_i to γ_i-1 for $1\leq i\leq m$. It follows that Λ is a domain and has Krull dimension $m+1$. One can regard Γ as a subgroup of $\Lambda^\times$, and hence, one has a natural representation

$$\kappa:\Gamma\longrightarrow\mathrm{GL}_1(\Lambda).$$

We let $\Lambda(\kappa)$ denote the free Λ-module of rank 1 with this action of Γ.

Suppose now that T is a free $\mathbf{Z}_p$-module of rank n with a $\mathbf{Z}_p$-linear action of $\mathrm{Gal}(K_\Sigma/K)$. Let $V=T\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$ and $D=T\otimes_{\mathbf{Z}_p}(\mathbf{Q}_p/\mathbf{Z}_p)$. Let $T_\Lambda=T\otimes_{\mathbf{Z}_p}\Lambda$, a free Λ-module of rank n. This Λ-module has a Λ-linear action of $\mathrm{Gal}(K_\Sigma/K)$, where the action is just through the first factor. Since we have $K_\infty\subset K_\Sigma$, we can regard κ as a representation of $\mathrm{Gal}(K_\Sigma/K)$ over Λ of rank 1. We now consider $\mathcal{T}=T_\Lambda\otimes_\Lambda\Lambda(\kappa)$, which is also a free Λ-module of rank n, but with a new Λ-linear action of $\mathrm{Gal}(K_\Sigma/K)$. If we choose a basis for $\mathcal{T}$, then we obtain a representation

$$\rho:\ \mathrm{Gal}(K_\Sigma/K)\longrightarrow\mathrm{GL}_n(\Lambda).$$

The underlying Galois module is $\mathcal{T}$. As in the introduction, the corresponding discrete Galois module is $\mathcal{D}=\mathcal{T}\otimes_\Lambda\widehat{\Lambda}$. We think of $\mathcal{T}$ as the twist of T_Λ, or of T, by the $\Lambda^\times$-valued character κ. For brevity, we sometimes denote $\mathcal{T}$ by $T\otimes\kappa$. Similarly, we sometimes write $D\otimes\kappa$ for $\mathcal{D}$. Note also that $\mathcal{T}^*$ is isomorphic to $T^*\otimes\kappa^{-1}$, where $T^*=\mathrm{Hom}(D,\mu_{p^\infty})$.

Suppose that $\varphi:\Gamma\to\overline{\mathbf{Q}}_p^\times$ is a continuous group homomorphism. Let $\mathbf{Z}_p[\varphi]$ denote the ring $\mathbf{Z}_p[\varphi(\gamma_1),\ldots,\varphi(\gamma_m)]$. This ring is an order in some finite extension of $\mathbf{Q}_p$. It is clear that φ has values in the group of principal units of $\mathbf{Z}_p[\varphi]$. We can define an action of $\mathrm{Gal}(K_\Sigma/K)$ on the free $\mathbf{Z}_p[\varphi]$-module $T\otimes_{\mathbf{Z}_p}\mathbf{Z}_p[\varphi]$, where $\mathrm{Gal}(K_\Sigma/K)$ has the given action on the first factor and acts by φ on the second. We denote this Galois module by $T\otimes\varphi$ and refer to it as the twist of T by φ. The corresponding discrete module is denoted by $D\otimes\varphi$.

We call the Galois module $\mathcal{T}$ defined above, or the corresponding representation ρ, a *twist deformation* for the following reason. If φ is as in the previous paragraph, then we can naturally extend φ to a continuous ring homomorphism from Λ to $\overline{\mathbf{Q}}_p$, which we also denote simply by φ. In effect, we are identifying $\mathrm{Hom}_{\mathrm{cont}}(\Gamma, \overline{\mathbf{Q}}_p^\times)$ with $\mathrm{Hom}_{\mathrm{cont}}(\Lambda, \overline{\mathbf{Q}}_p)$. The kernel $\mathfrak{p}_\varphi$ of φ is in $\mathrm{Spec}_{ht=m}(\Lambda)$. The image of φ is the ring $\mathbf{Z}_p[\varphi]$. Of course, φ induces a continuous homomorphism $\lambda_\varphi: \mathrm{GL}_n(\Lambda) \to \mathrm{GL}_n(\mathbf{Z}_p[\varphi])$, and composing this with ρ gives the representation $\rho_\varphi = \lambda_\varphi \circ \rho$ which describes the action of $\mathrm{Gal}(K_\Sigma/K)$ on the twisted Galois module $T \otimes \varphi$. That is, we have an isomorphism $\mathcal{T}/\mathfrak{p}_\varphi \mathcal{T} \cong T \otimes \varphi$ as Galois modules. Note, however, that $\mathcal{T}^*/\mathfrak{p}_\varphi \mathcal{T}^* \cong T^* \otimes \varphi^{-1}$.

5.1. The torsion Λ-submodule of $H^1(K_\Sigma/K, \mathcal{T}^*)$

We prove the following general results.

PROPOSITION 5.1.1

If $m \geq 2$, then $H^1(K_\Sigma/K, \mathcal{T}^*)$ is torsion free as a Λ-module.

PROPOSITION 5.1.2

If $m = 1$, then $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ is a free $\mathbf{Z}_p$-module of finite rank.

PROPOSITION 5.1.3

If K_∞ is the cyclotomic $\mathbf{Z}_p$-extension of K, then $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}} \neq 0$ if and only if $D = T \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)$ has a quotient isomorphic to μ_{p^∞} for the action of G_{K_∞}.

The proofs of the above propositions will follow easily from the results proven in Section 2 together with the following lemma.

LEMMA 5.1.4

We have $H^0(K, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*) = 0$ for all but finitely many $\mathfrak{p} \in \mathrm{Spec}_{ht=m}(\Lambda)$. If $\Lambda/\mathfrak{p}$ has characteristic p, then $H^0(K, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*) = 0$. If we assume that $m \geq 2$, then we have $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*) = 0$ for all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. For any $m \geq 1$, we have $H^0(K, \mathcal{T}^*) = 0$.

Proof

If $\mathfrak{p} \in \mathrm{Spec}_{ht=m}(\Lambda)$ and $\Lambda/\mathfrak{p}$ has characteristic zero, then $\Lambda/\mathfrak{p}$ is isomorphic to an order in some finite extension of $\mathbf{Q}_p$. Thus, the ring homomorphism $\Lambda \to \Lambda/\mathfrak{p}$ induces a continuous group homomorphism $\varphi: \Gamma \to \overline{\mathbf{Q}}_p^\times$, and $\mathfrak{p} = \mathfrak{p}_\varphi$. We then have $\mathcal{T}^*/\mathfrak{p}\mathcal{T}^* \cong T^* \otimes \varphi$. Now $H^0(K, T^* \otimes \varphi) \neq 0$ implies that the representation space $T^* \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p$ for G_K has a subspace on which G_K acts by φ^{-1}. This can happen for only finitely many φ's.

Assume that $\mathfrak{p} \in \mathrm{Spec}_{ht=m}(\Lambda)$ and $\Lambda/\mathfrak{p}$ has characteristic p. We show that $H^0(K, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*)$ vanishes for all such $\mathfrak{p}$. The action of G_K on T^*/pT^* factors through $\mathrm{Gal}(L/K)$, where L is a finite Galois extension of K. Thus, the action of G_K on $\mathcal{T}^*/\mathfrak{p}\mathcal{T}^*$ factors through $\mathrm{Gal}(LK_\infty/K)$. It is enough to prove that $H^0(L, \mathcal{T}^*/\mathfrak{p}\mathcal{T}^*) = 0$. Now $\mathcal{T}^*/\mathfrak{p}\mathcal{T}^*$ is a free module over $\Lambda/\mathfrak{p}$, and G_L acts by the restriction of κ. Therefore, it is enough to show that $H^0(\Gamma', \Lambda/\mathfrak{p}) = 0$, where Γ' is a subgroup of Γ with finite index. Furthermore, we can assume that $\Gamma' = \Gamma^{p^t}$ for some $t \geq 0$.

Note that the maximal ideal $\mathfrak{m}$ in Λ is generated by $\{p, \gamma_1 - 1, \ldots, \gamma_m - 1\}$. Its image in $\Lambda/\mathfrak{p}$ is the maximal ideal in that local ring, which we denote by $\mathfrak{m}_\mathfrak{p}$. It is a nonzero ideal because $\mathfrak{p}$ has height m and $\mathfrak{m}$ has height $m+1$. Since we are assuming that $p \in \mathfrak{p}$, $\mathfrak{m}_\mathfrak{p}$ is generated by the images of $\gamma_1 - 1, \ldots, \gamma_m - 1$ in $\Lambda/\mathfrak{p}$, and hence, at least one of those images is nonzero. Note also that for $1 \leq i \leq m$, the images of $\gamma_i^{p^t} - 1$ and $(\gamma_i - 1)^{p^t}$ in $\Lambda/\mathfrak{p}$ are the same. Therefore, for some $\gamma' \in \Gamma'$, the image of $\gamma' - 1$ in $\Lambda/\mathfrak{p}$ is a nonzero element α in $\Lambda/\mathfrak{p}$. Thus, we have

$$H^0(\Gamma', \Lambda/\mathfrak{p}) \subseteq (\Lambda/\mathfrak{p})[\alpha],$$

which vanishes because $\Lambda/\mathfrak{p}$ is a domain.

The statement about the vanishing of $H^0(K, \mathcal{T}^*/\Pi\mathcal{T}^*)$ now follows immediately from Lemmas 2.2.6 and 2.2.8. The final statement also follows immediately. $\square$

Proof of Propositions 5.1.1, 5.1.2, and 5.1.3

Proposition 5.1.1 now follows immediately from Proposition 2.2.7. For Proposition 5.1.2, one can verify that $H^1(K_\Sigma/K, \mathcal{T}^*)[p] = 0$ by using the exact sequence (7) for $\theta = p$ and Lemma 5.1.4 for $\mathfrak{p} = (p)$. The stated assertion then follows because $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}}$ is a finitely generated, torsion Λ-module.

For proving Proposition 5.1.3, note that the statement about D means that $U = H^0(K_\infty, T^*)$ has positive $\mathbf{Z}_p$-rank. The action of G_K on U factors through Γ. Hence, $\mathrm{rank}_{\mathbf{Z}_p}(U) > 0$ if and only if there exists a $\varphi \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \overline{\mathbf{Q}}_p^\times)$ such that $H^0(K, T^* \otimes \varphi^{-1}) \neq 0$. That statement is in turn equivalent to the nonvanishing of $H^1(K_\Sigma/K, \mathcal{T}^*)[\mathfrak{p}_\varphi]$. Now if $\mathfrak{p} \in \mathrm{Spec}_{ht=1}(\Lambda)$ and $\Lambda/\mathfrak{p}$ has characteristic zero, then $\mathfrak{p} = \mathfrak{p}_\varphi$ for some φ as above. Therefore, by Propositions 2.2.2 and 5.1.2, it indeed follows that $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda\text{-tors}} \neq 0$ if and only if $\mathrm{rank}_{\mathbf{Z}_p}(U) > 0$. $\square$

5.2. The validity of $\mathrm{LEO}(\mathcal{D})$

It is reasonable to conjecture that $\mathrm{LEO}(\mathcal{D})$ is always satisfied when $\mathcal{D}$ has the form $\mathcal{D} = D \otimes \kappa$. We emphasize that here D is simply a $\mathrm{Gal}(K_\Sigma/K)$-module which is isomorphic as a group to $(\mathbf{Q}_p/\mathbf{Z}_p)^n$ for some $n \geq 1$.

CONJECTURE 5.2.1

Assume that $\mathcal{D} = D \otimes \kappa$ as defined above. Then $\text{Ш}^2(K, \Sigma, \mathcal{D}) = 0$.

An equivalent version of this conjecture was stated in the introduction to [Gr3, page 364]. It was called Conjecture **L** there and asserts that $\text{Ш}^2(K_\infty, \Sigma, D) = 0$. The equivalence of these formulations is discussed briefly in [Gr3] and will be explained in detail in [Gr5].

Proposition 5.2.3 below states that $\text{Ш}^2(K,\Sigma,\mathcal{D})$ is a divisible Λ-module. Hence, the vanishing of $\text{Ш}^2(K,\Sigma,\mathcal{D})$ is equivalent to the validity of $\text{LEO}(\mathcal{D})$. The proposition follows immediately from Proposition 2.1.1 and the following lemma.

LEMMA 5.2.2

Suppose that $v \in \Sigma$ and that the decomposition subgroup of Γ for v is nontrivial. Then $H^0(K_v,\mathcal{T}^*)=0$. In particular, $H^0(K_v,\mathcal{T}^*)$ vanishes for at least one $v|p$.

Proof

We use the analogue of Lemma 2.2.6 for K_v in place of K. The proof of that lemma still works because if L_v is a finite extension of K_v and M_v is the maximal pro-p extension of L_v, then $\text{Gal}(M_v/L_v)$ is topologically finitely generated. This follows from the Burnside Basis Theorem.

Let Γ_v be the decomposition subgroup of Γ for v. The assumption about Γ_v implies that it is infinite. If $\psi : \Gamma_v \to \overline{\mathbf{Q}}_p^\times$ is any continuous homomorphism, then choose some continuous homomorphism $\varphi : \Gamma \to \overline{\mathbf{Q}}_p^\times$ such that $\varphi|_{\Gamma_v} = \psi$. We can regard $T^* \otimes_{\mathbf{Z}_p} \overline{\mathbf{Q}}_p$ as a representation space for G_{K_v}. It has a nontrivial subspace on which G_{K_v} acts by ψ for only finitely many ψ's. Thus, we can choose ψ so that $H^0(K_v,T^* \otimes \varphi^{-1})=0$. This means that $H^0(K_v,\mathcal{T}^*/\mathfrak{p}_\varphi\mathcal{T}^*)=0$. The analogue of Lemma 2.2.6 then implies that $H^0(K_v,\mathcal{T}^*)=0$. □

PROPOSITION 5.2.3

Assume that $\mathcal{D}=D\otimes\kappa$ as above. Then $\text{Ш}^2(K,\Sigma,\mathcal{D})$ is a divisible Λ-module.

Archimedean primes always split completely in K_∞/K. The hypothesis in the next result is that the nonarchimedean primes in Σ do not split completely. This assumption is satisfied if K_∞ contains the cyclotomic $\mathbf{Z}_p$-extension of K. However, if K is not totally real and if v is any nonarchimedean prime not lying over p, then one can always find at least one $\mathbf{Z}_p$-extension of K in which v splits completely.

PROPOSITION 5.2.4

Suppose that Γ_v is nontrivial for all nonarchimedean $v \in \Sigma$. If p is odd, then $\text{Ш}^2(K,\Sigma,\mathcal{D}) = H^2(K_\Sigma/K,\mathcal{D})$. If $p=2$, then $H^2(K_\Sigma/K,\mathcal{D})/\text{Ш}^2(K,\Sigma,\mathcal{D})$ has exponent 2.

Proof

Lemma 5.2.2 and the local duality theorem imply that $H^2(K_v,\mathcal{D})=0$ for all nonarchimedean $v\in\Sigma$. If $p>2$, the same is true for $v|\infty$. If $p=2$, then $H^2(K_v,\mathcal{D})$ may be nonzero but has exponent 2. The stated assertions clearly follow. □

Under the assumptions of Proposition 5.2.4, one can use [Gr3, Proposition 4.3] to give a simple formula for the quantity $b_1(K,\mathcal{D})$ mentioned in the introduction. It just involves T. For every real prime v of K, we can write

$$n = \mathrm{rank}_{\mathbf{Z}_p}(T) = n_v^+ + n_v^-,$$

where $n_v^{\pm}$ is the dimension of the (± 1)-eigenspace for a generator of G_{K_v} acting on $T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Then we have

$$b_1(K,\mathcal{D}) = \sum_{v|\infty} \mathrm{rank}_\Lambda\big(H^0(K_v, \mathcal{T}^*)\big) = r_2 n + \sum_{v \text{ real}} n_v^-, \tag{18}$$

where r_2 denotes the number of complex primes of K. For the last equality, one uses the fact that if v is archimedean, then Γ_v is trivial. It follows that $\mathrm{rank}_\Lambda(H^0(K_v,\mathcal{T}^*)) = \mathrm{rank}_{\mathbf{Z}_p}(H^0(K_v,T^*))$ for all $v|\infty$.

Proposition 2.1.4 provides one possible way to verify that $\mathrm{LEO}(\mathcal{D})$ is satisfied in many interesting cases. It is an inductive argument. We suppose that K_∞ contains the cyclotomic $\mathbf{Z}_p$-extension of K, which we now denote by C_∞. We can choose a sequence of extensions $K_\infty^{(i)}$ for $1 \le i \le m$ such that $K_\infty^{(1)} = C_\infty$, $K_\infty^{(m)} = K_\infty$, and $\mathrm{Gal}(K_\infty^{(i)}/K) \cong \mathbf{Z}_p^i$. Let $\Gamma^{(i)} = \mathrm{Gal}(K_\infty^{(i)}/K)$, and let $\Lambda^{(i)}$ denote $\mathbf{Z}_p[[\Gamma^{(i)}]]$. Thus, $\Lambda^{(i)}$ has Krull dimension $i+1$. Let $\kappa_i : \Gamma^{(i)} \to \mathrm{GL}_1(\Lambda^{(i)})$ be the corresponding representation. For each i, we have a Galois module $\mathcal{D}^{(i)} = D \otimes \kappa_i$. In particular, $\mathcal{D} = \mathcal{D}^{(m)}$.

There is also a surjective ring homomorphism $\Lambda^{(i)} \to \Lambda^{(i-1)}$ for each $i \ge 2$. The kernel of that homomorphism is a prime ideal $\Pi^{(i)}$ of height 1 in $\Lambda^{(i)}$. One has $\mathcal{D}^{(i-1)} \cong \mathcal{D}^{(i)}[\Pi^{(i)}]$. According to Proposition 5.2.4, for $1 \le j \le m$, $\mathrm{LEO}(\mathcal{D}^{(j)})$ means that $H^2(K_\Sigma/K, \mathcal{D}^{(j)})$ is $\Lambda^{(j)}$-cotorsion. Proposition 2.1.4 then shows that if $2 \le i \le m$ and $\mathrm{LEO}(\mathcal{D}^{(i-1)})$ is satisfied, then so is $\mathrm{LEO}(\mathcal{D}^{(i)})$. Therefore, it is enough to verify that $\mathrm{LEO}(\mathcal{D}^{(1)})$ is satisfied.

Assume that $H^2(K_\Sigma/K, D \otimes \varphi)$ is finite for some φ in $\mathrm{Hom}_{\mathrm{cont}}(\Gamma^{(1)}, \overline{\mathbf{Q}}_p^\times)$. We can identify φ with an element of $\mathrm{Hom}_{\mathrm{cont}}(\Lambda^{(1)}, \overline{\mathbf{Q}}_p)$, and then $\Pi = \ker(\varphi)$ is in $\mathrm{Spec}_{ht=1}(\Lambda^{(1)})$. Since $D \otimes \varphi \cong \mathcal{D}^{(1)}[\Pi]$, Proposition 2.1.4 again shows that $H^2(K_\Sigma/K, \mathcal{D}^{(1)})$ is $\Lambda^{(1)}$-cotorsion, and so $\mathrm{LEO}(\mathcal{D}^{(1)})$ is satisfied. These considerations prove the following result.

PROPOSITION 5.2.5

Assume that K_∞ contains the cyclotomic $\mathbf{Z}_p$-extension C_∞ of K. Assume that $H^2(K_\Sigma/K, D \otimes \varphi)$ is finite for some $\varphi \in \mathrm{Hom}(\mathrm{Gal}(C_\infty/K), \overline{\mathbf{Q}}_p^\times)$. Then $\mathrm{LEO}(\mathcal{D})$ *is satisfied.*

We now discuss two important special cases as illustrations.

ILLUSTRATION 5.2.6

The action of G_K on μ_{p^∞} is given by a homomorphism $\chi : G_K \to \mathbf{Z}_p^\times$ which factors through $\mathrm{Gal}(K(\mu_{p^\infty})/K)$. Let $w = [K(\mu_p) : K]$ if p is odd, and let $w = [K(\mu_4) : K]$ if $p = 2$. Note that χ^w factors through $\mathrm{Gal}(C_\infty/K)$. Let j be a fixed integer. Suppose that $T = \mathbf{Z}_p(j)$, which we regard as a $\mathrm{Gal}(K_\Sigma/K)$-module. The Galois action is by χ^j. One has $T^* \cong \mathbf{Z}_p(1-j)$. If $j=1$, then $D = \mu_{p^\infty}$, and one shows easily that $Ш^2(K,\Sigma,D)=0$. If $j \neq 1$ and p is odd, then the local H^2's vanish and so we have $Ш^2(K,\Sigma,D) = H^2(K_\Sigma/K,D)$. It is a conjecture of Schneider that this group vanishes (see [Sch1, page 192]). In general, one would conjecture that $Ш^2(K,\Sigma,D)=0$ for all j and all p. Satz 3 in Section 6 of [Sch1] proves this vanishing for all but finitely many j's. This theorem suffices to verify the hypothesis in Proposition 5.2.5. For if one takes any $j' \equiv j \pmod{w}$, then $\mathbf{Z}_p(j') \cong T \otimes \varphi$, where $\varphi = \chi^{j'-j}$. Note that φ is in $\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(C_\infty/K), 1+p\mathbf{Z}_p)$. One can choose j' so that $Ш^2(K,\Sigma,D\otimes\varphi)$ vanishes. It follows that $\mathrm{LEO}(\mathcal{D})$ is satisfied if K_∞ contains C_∞.

ILLUSTRATION 5.2.7

Assume now that $T = T_p(E)$, where E is an elliptic curve defined over $\mathbf{Q}$, and that $K/\mathbf{Q}$ is abelian. We have $D = E[p^\infty]$. For $n \geq 0$, let C_n denote the unique subfield of C_∞ such that $[C_n : K] = p^n$. Theorems of Kato and Rohrlich then imply that the $\mathbf{Z}_p$-corank of $\mathrm{Sel}_E(C_n)_p$ is bounded as $n \to \infty$. This follows easily from [Kat, Theorem 17.4]. Thus, for some n_0, we have

$$\mathrm{corank}_{\mathbf{Z}_p}\big(\mathrm{Sel}_E(C_n)_p\big) = \mathrm{corank}_{\mathbf{Z}_p}\big(\mathrm{Sel}_E(C_{n_0})_p\big)$$

for all $n \geq n_0$. One can then show that if φ is a character of $\mathrm{Gal}(C_\infty/K)$ of order p^n, where $n > n_0$, then $H^2(K_\Sigma/K, D\otimes\varphi)$ is finite (and zero if p is odd). Therefore, one can again conclude from Proposition 5.2.5 that $\mathrm{LEO}(\mathcal{D})$ is satisfied if K_∞ contains C_∞.

5.3. Surjectivity

Propositions 5.3.1 and 5.3.3 give sufficient conditions for the surjectivity of $\phi_{\mathcal{L}}$. Proposition 5.3.2 is an interesting remark about the cokernel when it is nonzero. Those results are consequences of the propositions in Section 5.1 and Proposition 3.1.1. The hypotheses imply that $S_{\mathcal{L}^*}(K,\mathcal{T}^*) \subseteq H^1(K_\Sigma/K,\mathcal{T}^*)_{\Lambda\text{-tors}}$ and that $Ш^1(K,\Sigma,\mathcal{T}^*)=0$. For Proposition 5.3.3, the assumption that $\mathcal{L}$ is Λ-divisible means that $L(K_v,\mathcal{D})$ is a divisible Λ-module for all $v \in \Sigma$. If that is so, then Proposition 2.3.1 implies that $\mathrm{coker}(\phi_{\mathcal{L}})$ is dual to $H^1(K_\Sigma/K,\mathcal{T}^*)_{\Lambda\text{-tors}}$. The stated result then follows from Proposition 5.1.3.

PROPOSITION 5.3.1

Assume that $m \geq 2$, that $\mathrm{LEO}(\mathcal{D})$ is satisfied and that $\mathrm{coker}(\phi_{\mathcal{L}})$ is Λ-cotorsion. Then $\phi_{\mathcal{L}}$ is surjective.

PROPOSITION 5.3.2

Assume that $m = 1$, that $\mathrm{LEO}(\mathcal{D})$ is satisfied, and that $\mathrm{coker}(\phi_{\mathcal{L}})$ is Λ-cotorsion. Then $\mathrm{coker}(\phi_{\mathcal{L}}) \cong (\mathbf{Q}_p/\mathbf{Z}_p)^c$ for some $c \geq 0$.

PROPOSITION 5.3.3

Assume that K_∞ is the cyclotomic $\mathbf{Z}_p$-extension of K, that $\mathrm{LEO}(\mathcal{D})$ *is satisfied, that the specification $\mathcal{L}$ is Λ-divisible, and that* $\mathrm{coker}(\phi_{\mathcal{L}})$ *is Λ-cotorsion. Then $\phi_{\mathcal{L}}$ is surjective if and only if $H^0(K_\infty, T^*) = 0$.*

Illustrations

To continue Illustration 5.2.6, assume that K_∞ contains C_∞ and that we choose the specification $\mathcal{L}$ so that $L(K_v, \mathcal{D}) = 0$ for all $v \in \Sigma$. Thus, $\mathrm{LEO}(\mathcal{D})$ is satisfied. Also, by definition, $S_{\mathcal{L}}(K, \mathcal{D}) = \text{Ш}^1(K, \Sigma, \mathcal{D})$. In general, [Gr3, Proposition 4.4] implies that $\mathrm{corank}_\Lambda(\text{Ш}^1(K, \Sigma, \mathcal{D}))$ and $\mathrm{rank}_\Lambda(\text{Ш}^1(K, \Sigma, \mathcal{T}))$ are equal. The argument in Illustration 5.2.6 for $\mathbf{Z}_p(1-j)$ instead of $\mathbf{Z}_p(j)$ shows that $\text{Ш}^1(K, \Sigma, \mathcal{T})$ has Λ-rank zero. It follows that $\mathrm{corank}_\Lambda(S_{\mathcal{L}}(K, \mathcal{D})) = 0$.

By (2), we have $\mathrm{corank}_\Lambda(\mathrm{coker}(\phi_{\mathcal{L}})) = 0$ if and only if $h_1(K, \mathcal{D}) = q_{\mathcal{L}}(K, \mathcal{D})$. One finds that $q_{\mathcal{L}}(K, \mathcal{D}) = [K : \mathbf{Q}]$. This follows from [Gr2, Proposition 4.2], which is a consequence of the formulas for the local Euler-Poincaré characteristic for the Λ-module $\mathcal{D}$ and for all $v \in \Sigma$. The only nonzero contribution to $q_{\mathcal{L}}(K, \mathcal{D})$ comes from $v|p$ and is then $[K_v : \mathbf{Q}_p]$. By (18), together with the fact that $\mathrm{LEO}(\mathcal{D})$ holds, we have

$$h_1(K, \mathcal{D}) = b_1(K, \mathcal{D}) = \begin{cases} r_2 & \text{if } j \text{ is even,} \\ r_1 + r_2 & \text{if } j \text{ is odd.} \end{cases}$$

The above remarks show that $\mathrm{corank}_\Lambda(\mathrm{coker}(\phi_{\mathcal{L}})) = 0$ if and only if j is odd and K is totally real.

However, according to Leopoldt's conjecture, if K is totally real, then the cyclotomic $\mathbf{Z}_p$-extension of K should be the only one, and hence, we should have $K_\infty = C_\infty$ and $m = 1$. One can then apply Proposition 5.3.3 to conclude that $\phi_{\mathcal{L}}$ is surjective if and only if $j \not\equiv 1 \pmod{w}$.

Assume now that $j \equiv 1 \pmod{w}$. One then gets the isomorphism

$$\mathrm{coker}(\phi_{\mathcal{L}}) \cong \widehat{\Lambda}[\mathfrak{p}_\varphi], \quad \text{where } \varphi = \chi^{1-j}. \tag{19}$$

This is an isomorphism of discrete Λ-modules. We have $\mathrm{coker}(\phi_{\mathcal{L}}) \cong \mathbf{Q}_p/\mathbf{Z}_p$ as groups. To justify (19), recall that $\mathcal{T}^*/\mathfrak{p}_\varphi \mathcal{T}^*$ is isomorphic to $T^* \otimes \varphi^{-1}$ for any φ. Thus, $H^0(K, \mathcal{T}^*/\mathfrak{p}_\varphi \mathcal{T}^*) \neq 0$ only for $\varphi = \chi^{1-j}$. For that φ, we have $H^1(K_\Sigma/K, \mathcal{T}^*)[\mathfrak{p}_\varphi] \cong \Lambda/\mathfrak{p}_\varphi$. This group is isomorphic to $\mathbf{Z}_p$. Furthermore, one can easily show that $H^0(K, \mathcal{T}^*/\mathfrak{p}_\varphi^2 \mathcal{T}^*)$ is also isomorphic to $\mathbf{Z}_p$. One then uses Propositions 2.2.2 and 3.1.1 to prove that $\widehat{\mathrm{coker}(\phi_{\mathcal{L}})} \cong \Lambda/\mathfrak{p}_\varphi$ and hence that (19) holds.

We briefly discuss another choice of specification for $\mathcal{T}$, where $T = \mathbf{Z}_p(j)$. Suppose that $L(K_v, \mathcal{D}) = H^1(K_v, \mathcal{D})_{\Lambda\text{-div}}$ for all $v \in \Sigma$. In particular, for $v \nmid p$, $L(K_v, \mathcal{D}) = 0$. Obviously, we now have $q_{\mathcal{L}}(K, \mathcal{D}) = 0$. The cokernel of $\phi_{\mathcal{L}}$ is always Λ-cotorsion. Propositions 5.3.1 and 5.3.3 imply that $\phi_{\mathcal{L}}$ is surjective if and only if either $m \geq 2$ or $m = 1$ and $j \not\equiv 1 \pmod{w}$.

To continue Illustration 5.2.7. we still assume that $T = T_p(E)$, that $K/\mathbf{Q}$ is abelian, and that $C_\infty \subseteq K_\infty$. Since LEO($\mathcal{D}$) is satisfied, we have the equality $h_1(K,\mathcal{D}) = b_1(K,\mathcal{D})$. Now (18) implies that $b_1(K,\mathcal{D}) = [K:\mathbf{Q}]$. If $\mathcal{L}$ is any specification for $\mathcal{D}$ and Σ, it follows that

$$c_{\mathcal{L}}(K,\mathcal{D}) = 0 \qquad \Longleftrightarrow \qquad \operatorname{corank}_\Lambda\big(S_{\mathcal{L}}(K,\mathcal{D})\big) = [K:\mathbf{Q}] - q_{\mathcal{L}}(K,\mathcal{D}).$$

If $c_{\mathcal{L}}(K,\mathcal{D}) = 0$, then $\phi_{\mathcal{L}}$ is surjective. This follows from Proposition 5.3.1 if $m \geq 2$. For $m = 1$, note that Remark 3.2.7 allows one to replace $\mathcal{L}$ by $\mathcal{L}_{\mathrm{div}}$. Furthermore, it is known that $H^0(C_\infty, E[p^\infty])$ is finite. Since $T^* \cong T_p(E)$, it follows that $H^0(C_\infty, T^*) = 0$. One can then use Proposition 5.3.3 to see that $\phi_{\mathcal{L}}$ is surjective.

If A is an arbitrary abelian variety, K is an arbitrary number field, and K_∞ is ramified at all the primes of K lying above p, then there is a natural choice for the specification $\mathcal{L}$. As in Section 4, the definition involves the local Kummer maps. One simply takes $L(K_v,\mathcal{D}) = 0$ for $v \nmid p$. There is a relatively simple description of $L(K_v,\mathcal{D})$ even for $v|p$. This is based on the results in [CG] and will be discussed in [Gr5]. An important feature of the definition is that the Λ-corank of $L(K_v,\mathcal{D})$ depends on the reduction type of A at v. In general, one has only the inequality $q_{\mathcal{L}}(K,\mathcal{D}) \leq [K:\mathbf{Q}]g$. One conjectures that the equality

$$\operatorname{corank}_\Lambda\big(S_{\mathcal{L}}(K,\mathcal{D})\big) = [K:\mathbf{Q}]g - q_{\mathcal{L}}(K,\mathcal{D})$$

always holds. In the case where $K_\infty = C_\infty$, this conjecture was made by Mazur [Maz, page 184] when A has good ordinary reduction at all $v|p$. In that case, one has $q_{\mathcal{L}}(K,\mathcal{D}) = [K:\mathbf{Q}]g$, and the conjecture is that $S_{\mathcal{L}}(K,\mathcal{D})$ is Λ-cotorsion. Schneider stated such a conjecture when A is just assumed to have good reduction at all $v|p$ (see [Sch2, Conjecture, page 348; Lemma 2, page 344]). If the reduction is not ordinary for at least one prime $v|p$, then one has $q_{\mathcal{L}}(K,\mathcal{D}) < [K:\mathbf{Q}]g$, and $S_{\mathcal{L}}(K,\mathcal{D})$ cannot be Λ-cotorsion. As in the above discussion, those conjectures imply that $\phi_{\mathcal{L}}$ is surjective.

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