On volume-preserving complex structures on real tori

Fabrizio Catanese, Keiji Oguiso, and Thomas Peternell

Abstract A basic problem in the classification theory of compact complex manifolds is to give simple characterizations of complex tori. It is well known that a compact Kähler manifold X homotopy equivalent to a complex torus is biholomorphic to a complex torus.

The question whether a compact complex manifold X diffeomorphic to a complex torus is biholomorphic to a complex torus has a negative answer due to a construction by Blanchard and Sommese.

Their examples, however, have negative Kodaira dimension; thus it makes sense to ask whether a compact complex manifold X with trivial canonical bundle which is homotopy equivalent to a complex torus is biholomorphic to a complex torus.

In this article we show that the answer is positive for complex threefolds satisfying some additional condition, such as the existence of a nonconstant meromorphic function.

1. Introduction

The Enriques-Kodaira classification of compact complex surfaces implies in particular that a compact complex surface homotopy equivalent to a complex torus of dimension 2 is biholomorphic to a complex torus of dimension 2. The corresponding result in dimension 1 was already known in the nineteenth century.

Surprisingly, the analogous result in dimension 3 is no longer true, as shown by Sommese, using results of Blanchard (see [So, p. 213], (E) written after [Bl1]; see also [Ca2, Section 5], [Ca3, Section 7]).

Indeed, there are countably many families of complex manifolds even diffeomorphic to a complex torus of dimension 3 which are not biholomorphic to a complex torus.

These are constructed as follows. Let L be a line bundle on a curve C, generated by global sections (if C is an elliptic curve, it suffices that the degree of L be at least 2). Let $s_1, s_2 \in H^0(C, L)$ be two sections without common zeros, so that $s := (s_1, s_2)$ is a nowhere-vanishing section of the rank two vector bundle $L \oplus L$. Identifying the fiber $\mathbb{C}^2$ with the quaternions, one finds that s, is, js, ks yield four sections $\in H^0(C, L \oplus L)$ giving an $\mathbb{R}$-basis over each point (hence the total space of $L \oplus L$ is diffeomorphic to a product $C \times \mathbb{R}^4$).

Defining X as the quotient of the total space of $L \oplus L$ by the free abelian subgroup $\mathbb{Z}^4$ generated by the four sections, X is then diffeomorphic to a torus, yet its canonical bundle K_X has the property $K_X = p^*(-2L)$ (see [Ca3, Remark 7.3]); in particular, $h^0(X, -K_X) = h^0(C, 2L)$ which, in the case where C is an elliptic curve, equals $2 \deg(L) \geq 4$. Hence X is not a complex torus for which K_X is a trivial divisor.

It is now natural to ask which kind of additional conditions are sufficient to characterize complex tori as complex manifolds. The simplest among such conditions, under a weak Kähler assumption (Theorem 2.3) requires us to have the same integral cohomology algebra. However, if one drops the Kähler condition, the problem becomes much more difficult, and, so far, not much is known (see, however, some characterizations in [Ca1] and in [Ca3], especially [Ca3, Proposition 2.9]).

The examples of Blanchard and Sommese lead Catanese ([Ca3, p. 269]) to ask the following question:

> Are there compact complex manifolds X with trivial canonical bundle K_X which are diffeomorphic but not biholomorphic to a complex torus?

In response to this question, we prove the following theorem as a corollary of more general results (see Theorems 2.3, 3.1, 3.2, 4.1, 5.1).

THEOREM 1.1

Let X be a compact complex threefold subject to the following conditions.

(1) *X is homotopy equivalent to a complex torus of dimension 3.*

(2) *X has a dominant meromorphic map to a compact complex analytic space Y of smaller dimension, that is, with $0 < \dim Y < 3$.*

(3) *X has a trivial canonical divisor, that is, $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$.*

Then X is biholomorphic to a complex torus.

We should remark that condition (2) is of course not a necessary condition for X to be a complex torus. However, the only known examples of threefolds which are homeomorphic but not biholomorphic to a complex torus are Blanchard and Sommese's examples, and they are all fibered over elliptic curves, as one can see from the construction described above. So they satisfy conditions (1) and (2) (but not (3)).

As a special case of Theorem 1.1, we obtain the following.

COROLLARY 1.2

Let X be a compact complex threefold such that

(i)   *X is homotopy equivalent to a complex torus of dimension 3;*

(ii) *X has a nontrivial map $\alpha : X \to T$ to a positive dimensional complex torus T;*

(iii) *$\mathcal{O}_X(K_X) \cong \mathcal{O}_X$.*

Then X is biholomorphic to a complex torus (of dimension 3).

This is a direct consequence of Theorem 1.1 using [Ue1, Lemma 10.1, Theorem 10.3] in the case $\dim a(X) = 3$.

COROLLARY 1.3

Let X be a compact complex threefold such that

(1) X is homotopy equivalent to a complex torus of dimension 3;
(2) either $a(X) > 0$, that is, X has a nonconstant meromorphic function, or the Albanese torus $\mathrm{Alb}(X)$ is nontrivial;
(3) $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$.

Then X is biholomorphic to a complex torus.

The remaining case where X has no nonconstant meromorphic function and also no meromorphic map to a surface without meromorphic functions seems difficult.

If, however, the tangent or the cotangent bundle have some sections, the situation becomes amenable.

THEOREM 1.4

Let X be a smooth compact complex threefold with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$, which is homotopy equivalent to a torus. If $h^0(T_X) \geq 3$ or if $h^0(\Omega^1_X) \geq 3$, then X is biholomorphic to a torus.

2. Preliminaries

We start with some notation. Let X be an irreducible compact complex space. Then $a(X)$ denotes the algebraic dimension of X (see [Ue1, Definition 3.2]), the maximal number of algebraically independent meromorphic functions. If $a(X) = \dim X$, then X is called a Moishezon manifold.

We also recall that for a compact complex manifold X, the Albanese torus of X is the complex torus defined by

$$\mathrm{Alb}(X) = H^0(X, d\mathcal{O}_X)^\vee / \Lambda,$$

where Λ is the minimal closed complex Lie group containing $\mathrm{Im}(H_1(X, \mathbb{Z}) \to H^0(X, d\mathcal{O}_X)^\vee)$.

We then have the Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ (see [Ue1, pp. 101–104]), assigning to each point x the class of the linear functional $\int_{x_0}^x$ on $H^0(X, d\mathcal{O}_X)$, obtained by integrating on a path from x_0 to x.

PROPOSITION 2.1

Let $f : X \longrightarrow Y$ be a surjective morphism with connected fibers from a compact (connected) complex manifold X with $\pi_1(X) \simeq \mathbb{Z}^k$ to a complex manifold Y. Let F be a general fiber of f. Then there exists an exact sequence of groups

$$0 \longrightarrow A \longrightarrow \pi_1(X) \simeq \mathbb{Z}^k \xrightarrow{f_*} \pi_1(Y) \longrightarrow 0,$$

where A contains $\mathrm{Im}(\pi_1(F) \to \pi_1(X))$ as a finite-index subgroup. In particular, $\pi_1(Y)$ is a finitely generated abelian group of rank $\leq k$, and there is an inequality of Betti numbers

$$b_1(F) + b_1(Y) \geq k = b_1(X).$$

Proof

The following proof is very close to those of [No, Lemma 1.5] and [CKO, Lemma 3].

Let F be a general fiber of f, and let $U \subset Y$ be the maximal Zariski open subset such that f is smooth over U. Consider the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & \pi_1(f^{-1}(U)) & \xrightarrow{(f_U)_*} & \pi_1(U) & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle b_F} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow \\
1 & \longrightarrow & \mathrm{Ker}\, f_* & \longrightarrow & \pi_1(X) \simeq \mathbb{Z}^k & \xrightarrow{f_*} & \pi_1(Y) & \longrightarrow & 1
\end{array}
$$

Here $G = \mathrm{Im}\big(\pi_1(F) \to \pi_1(f^{-1}(U))\big)$. Since b is surjective, the snake lemma yields the exact sequence

$$\mathrm{Ker}\, b \to \mathrm{Ker}\, c \to \mathrm{Coker}\, b_F \to 0.$$

Thus

$$\mathrm{Ker}\, c / (f_U)_*(\mathrm{Ker}\, b) \simeq \mathrm{Coker}\, b_F = \mathrm{Ker}\, f_* / \mathrm{Im}\, b_F.$$

Since $\mathrm{Ker}\, f_* \subset \mathbb{Z}^k$, it follows that

$$\mathrm{Ker}\, c / (f_U)_*(\mathrm{Ker}\, b) \simeq \mathrm{Coker}\, b_F$$

is a finitely generated abelian group. On the other hand, each $[\gamma] \in \mathrm{Ker}\, c$ is represented by the product of conjugates of elements represented by a closed circle γ contained in $D \cap (Y \setminus U)$ with a base point x, where $D \simeq \Delta$ is a small disk on Y transversal to $Y \setminus U$ in a point s_0.

However, since $\pi_1(Y)$ is abelian, we see that $\mathrm{Ker}\, c$ is generated by such elements.

For each such element take a small disk $\tilde{D} \simeq \Delta$ in X such that $f(\tilde{D}) = D$, and let d be the degree of the finite branched cover $\tilde{D} \to D$.

The preimage of γ in $\tilde{D}$ is a closed circle $\tilde{\gamma}$ such that $f(\tilde{\gamma}) = d\gamma$. Thus $\mathrm{Ker}\, c / (f_U)_*(\mathrm{Ker}\, b)$ is a torsion group. Hence $\mathrm{Coker}\, b_F$ is a finite abelian group. The last statement is clear from the fact that $\pi_1(X)$, $\pi_1(Y)$, and A are all abelian. $\qquad\square$

From [Ca3, Proposition 2.9] (see also [Ca1, Corollary C], [Ca2, Proposition 4.8]), we cite the following.

THEOREM 2.2

Let X be a compact complex manifold of dimension n such that

(1) *the cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to the cohomology ring of the n-dimensional complex torus;*

(2) *$H^0(X, d\mathcal{O}_X) = n$; that is, there are exactly n linearly independent d-closed holomorphic 1-forms.*

Then X is biholomorphic to a complex torus.

If X is bimeromorphically equivalent to a Kähler manifold, our main problem is easily answered.

THEOREM 2.3

Let X be a compact complex manifold such that

(1) *the cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to the cohomology ring of the n-dimensional complex torus (for instance, X is homotopy equivalent to a complex torus of dimension n);*

(2) *X is in the Fujiki class $\mathcal{C}$; that is, X is bimeromorphic to a compact Kähler manifold.*

Then X is biholomorphic to a complex torus of dimension n.

Proof
We apply Theorem 2.2 to our X. The first condition in Theorem 2.2 holds by assumption. In particular, $b_1(X) = 2n$. As X is in class $\mathcal{C}$, every holomorphic form is d-closed and the Hodge decomposition holds for X (see [Ue1, Corollaries 9.3, 9.5]; see also [Fj, Corollary 1.7]). Thus the second condition in Theorem 2.2 also holds, and an application of Theorem 2.2 implies the result. □

A special case of Theorem 2.3 is the following.

COROLLARY 2.4

A Moishezon manifold X homotopy equivalent to a complex torus of dimension n is biholomorphic to an abelian variety.

Recall that a compact complex manifold is said to be a *Moishezon* manifold if the algebraic dimension is maximal: $a(X) = \dim X$.

3. Complex torus bundles over a complex torus

In this section we prove two general results on submersions of special manifolds (Theorems 3.1, 3.2). These results are used in our proof of our main theorem (Theorem 1.1). The crucial point in both results is that we do *not* assume the total space X to be Kähler.

THEOREM 3.1

Let $f : X \to Y$ be a holomorphic submersion with connected fibers between compact (connected) complex manifolds and assume the following:

(1) *X has complex dimension $n + m$ and $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$;*
(2) *Y has complex dimension m and also $\mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y$;*
(3) *every fiber X_y $(y \in Y)$ is Kähler;*
(4) *the monodromy action of $\pi_1(Y)$ on $H^n(X_y, \mathbb{Z})$ is trivial.*

Then all the fibers X_y are biholomorphic, and f is a holomorphic fiber bundle.

Proof
By (4), $R^n f_* \mathbb{Z}_X$ is not only locally constant but also *globally constant* on Y. Thus, for the $\mathbb{Z}_Y$ dual local system, we have

$$(R^n f_* \mathbb{Z}_X)^* \simeq H_n(X_b, \mathbb{Z})_f \times Y.$$

Here $b \in Y$ is any base point, and $H_n(X_b, \mathbb{Z})_f$ denotes the free part of $H_n(X_b, \mathbb{Z})$. The same abbreviation is applied for other points $y \in Y$. Let

$$\gamma_{1,b}, \ldots, \gamma_{k,b}$$

be a basis of $H_n(X_b, \mathbb{Z})_f$, and let

$$\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k$$

be the corresponding flat basis of $(R^n f_* \mathbb{Z}_X)^*$ over Y.

Then the elements $\tilde{\gamma}_{i,y}$ $(1 \le i \le k)$ form a free basis of $H_n(X_y, \mathbb{Z})_f$ for each $y \in Y$.

Now, following Fujita [Fu1, pp. 780–781], we construct a family of holomorphic n-forms on the fibers, say $\{\varphi_y\}_{y \in Y}$, which varies holomorphically with respect to $y \in Y$.

Since

$$\omega_{X/Y} = \mathcal{O}(K_X) \otimes f^* \mathcal{O}(K_Y)^\vee \simeq \mathcal{O}_X,$$

we obtain

$$f_*(\omega_{X/Y}) \cong \mathcal{O}_Y.$$

We are done by the exact sequence

$$0 \to f^*(\Omega_Y^1) \to \Omega_X^1 \to \Omega_{X|Y}^1 \to 0$$

and since, by definition,

$$\omega_{X/Y} := \det(\Omega_{X|Y}^1) = \Lambda^n(\Omega_{X|Y}^1).$$

Hence a global generator of $f_*(\omega_{X/Y}) \cong \mathcal{O}_Y$ gives the desired family of holomorphic n-forms on the fibers, yielding a nowhere-vanishing form on each fiber.

Note that φ_y is d-closed, being a top holomorphic form.

Now we consider the *nonprojectivized, global* period map:

$$\tilde{p}_Y : Y \to \mathbb{C}^k, \qquad y \mapsto \left(\int_{\tilde{\gamma}_{i,y}} \varphi_y \right)_{i=1}^k.$$

This map is holomorphic by a fundamental result of Griffiths [Gr]. Indeed, to be able to apply [Gr, Theorem 1.1], we need that the fibers X_y be Kähler, but we do not need that the total space X be Kähler.

On the other hand, since Y is compact, the global *holomorphic* functions on Y are constant. Thus all functions

$$y \mapsto \int_{\tilde{\gamma}_{i,y}} \varphi_y$$

are constant on Y. Hence the usual period map $p_Y : Y \to \mathbb{P}^{k-1}$, which is just the projectivization of the target domain $\mathbb{C}^k$ of $\tilde{p}_Y$, is also constant as well.

As all the fibers X_y $(y \in Y)$ are compact Kähler manifolds with trivial canonical class, the local Torelli theorem holds for them; that is, the period map from the Kuranishi space to the period domain is injective (see, e.g., [GHJ, Theorem 16.9, p. 109]; the proof given there is written only for Calabi-Yau threefolds, but the proof in the general case is exactly the same).

Since p_Y is constant and Y is connected, it follows that all the fibers X_y are biholomorphic. Hence f is locally analytically trivial by the fundamental result of Grauert and Fischer (or by Kuranishi's theorem). This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

THEOREM 3.2

Let $f : X \to Y$ be a holomorphic submersion with connected fibers between compact (connected) complex manifolds, and assume that

(1) *X is homotopy equivalent to a complex torus of dimension $n + m$;*
(2) *Y is a complex torus of dimension m;*
(3) *some fiber X_y is biholomorphic to a complex torus.*

Then $f : X \to Y$ is a principal holomorphic torus bundle, and X is biholomorphic to a complex torus.

Proof

By [Ca3, Theorem 2.1], every fiber X_y is isomorphic to a complex torus of dimension n. Let $F = X_y$ be one of the fibers of f. Since $\pi_2(Y) = 0$, we have the following exact sequence:

$$0 \to \pi_1(F) \simeq \mathbb{Z}^{2m} \to \pi_1(X) \to \pi_1(Y) \simeq \mathbb{Z}^{2n} \to 0.$$

Since $\pi_1(X) \simeq \mathbb{Z}^{2(n+m)}$ by (1), this sequence splits and $\pi_1(Y)$ acts on $\pi_1(F)$ as the identity. Then, by the proof of Theorem 3.1, f is a holomorphic fiber bundle. In particular, the Kodaira-Spencer map

$$T_{Y,y} \to H^1(X_y, T_{X_y})$$

of f is zero at every point $y \in Y$. Then, by [Ca3, Proposition 3.2] and its proof, f is a principal fiber bundle with structure group F, that is, a fiber bundle whose transition functions are given by translations by local holomorphic sections of F over Y. We want to show that they can actually be chosen to be locally constant.

To verify this, we follow [BHPV, p. 196]. Set $\Gamma = H_1(F, \mathbb{Z}) \simeq \mathbb{Z}^{2n}$.

Consider the following commutative diagram of exact sequences of abelian sheaves on Y:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma = \Gamma_Y & \longrightarrow & \mathbb{C}_Y^n & \longrightarrow & F_Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma = \Gamma_Y & \longrightarrow & \mathcal{O}_Y^n & \longrightarrow & \mathcal{F}_Y & \longrightarrow & 0
\end{array}
$$

Here F_Y is the abelian sheaf of locally constant sections with values in F_Y, and $\mathcal{F}_Y$ is the abelian sheaf of holomorphic sections with values in F.

Taking the corresponding cohomology sequences yields the diagram

$$
\begin{array}{ccccc}
H^1(Y, \mathbb{C}^n) & \xrightarrow{\gamma} & H^1(Y, F_Y) & \longrightarrow & H^2(Y, \Gamma) \\
{\scriptstyle \beta_1} \downarrow & & {\scriptstyle \beta_2} \downarrow & & \downarrow {\scriptstyle =} \\
H^1(Y, \mathcal{O}_Y^n) & \xrightarrow{\alpha} & H^1(Y, \mathcal{F}_Y) & \xrightarrow{c} & H^2(Y, \Gamma)
\end{array}
$$

Let $\eta \in H^1(Y, \mathcal{F}_Y)$ be the class representing the principal holomorphic bundle structure of $f : X \to Y$. Set $\epsilon = c(\eta)$. Note that f is topologically trivial since the exact sequence of the fundamental group splits trivially. Thus $\epsilon = 0$, and therefore $\eta = \alpha(\eta_1)$ for some $\eta_1 \in H^1(Y, \mathcal{O}_Y^n)$. Since Y is Kähler, the map β_1 is the one induced by the natural projection under the Hodge decomposition

$$
H^1(Y, \mathbb{C}) = H^1(\mathcal{O}_Y) \oplus H^0(\Omega_Y^1).
$$

In particular, β_1 is surjective. Thus $\eta_1 = \beta_1(\eta_2)$ for some $\eta_2 \in H^1(Y, \mathbb{C}^n)$. Hence

$$
\eta = \alpha \beta_1(\eta_2) = \beta_2 \gamma(\eta_2) = \beta_2(\eta_3),
$$

where $\eta_3 = \gamma(\eta_2) \in H^1(Y, F_Y)$. This means that the transition functions defining the principal bundle structure $f : X \to Y$ can be chosen to be *locally constant*.

Let $Y = \bigcup_{i \in I} U_i$ be a sufficiently small open covering of Y with trivializations

$$
\varphi_i : X_{U_i} \simeq F \times U_i
$$

such that the transition functions $\varphi_i^{-1} \circ \varphi_j$ are all constant on $U_i \cap U_j$. Let τ_Y be a standard Kähler form on Y, and let τ_F be a standard Kähler form on F.

Set $\tau_i = \tau_Y|U_i$. Then $\tilde{\tau}_i := \varphi_i^*(\tau_i \wedge \tau_F)$ gives a Kähler form on X_{U_i}. As $\varphi_i^{-1} \circ \varphi_j$ is a translation by some *constant* element of F over $U_i \cap U_j$, it follows that $\tilde{\tau}_i = \tilde{\tau}_j$ on $X_{U_i} \cap X_{U_j}$. Hence $\{\tilde{\tau}_i\}_{i \in I}$ defines a global Kähler form on X. In particular, X is Kähler, and therefore X is biholomorphic to a complex torus by Theorem 2.3. $\qquad \square$

4. A characterization of complex tori: The case fibered by curves

The goal of this section is the following.

THEOREM 4.1

Let X be a compact complex manifold subject to the following conditions:

(1) *X is homotopy equivalent to a complex torus of dimension $m + 1$;*

(2) *there is a dominant meromorphic map* $f : X \dashrightarrow Y$ *to a compact complex manifold* Y *with* $\dim Y = m$;

(3) *either* $m \le 2$ *or* Y *is in the Fujiki class* $\mathcal{C}$ *with* $\kappa(Y) \ge 0$;

(4) $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$.

Then X *is biholomorphic to a complex torus of dimension* $m + 1$.

In the rest of this section, we prove Theorem 4.1 and always assume the situation of Theorem 4.1. Take a resolution of indeterminacies $\nu : \tilde{X} \longrightarrow X$ of f, yielding a surjective morphism

$$\tilde{f} : \tilde{X} \longrightarrow Y.$$

By considering the Stein factorization, we may assume that $\tilde{f}$ has connected fibers; loosely speaking, f has connected fibers. In the case where Y is in the Fujiki class $\mathcal{C}$, we may replace Y by a suitable bimeromorphic model and therefore may assume Y to be Kähler. Finally, F always denotes a smooth fiber of $\tilde{f}$.

LEMMA 4.2

(1) *If* $\kappa(Y) \ge 0$, *all smooth fibers of* $\tilde{f}$ *are isomorphic to a single elliptic curve, say* E, *and* $\kappa(Y) = 0$.

(2) *If, moreover,* Y *is in the Fujiki class* $\mathcal{C}$, *then* Y *is bimeromorphic to a complex torus of dimension* m. *More precisely, the Albanese map* $a : Y \to \mathrm{Alb}\, Y$ *is a bimeromorphic surjective morphism.*

Proof

(1) Since $K_{\tilde{X}}$ is effective, the fiber F has genus $g(F) \ge 1$. Then by [Ue2, Theorem 2.1], F must actually be an elliptic curve. Moreover, by [Ue2, Theorem 2.2],

$$0 = \kappa(X) = \kappa(\tilde{X}) \ge \max\big(\kappa(Y), \mathrm{var}(\tilde{f})\big) \ge 0,$$

where $\mathrm{var}(\tilde{f})$ denotes the variation of $\tilde{f}$. Thus $\kappa(Y) = 0$ and $\mathrm{var}(f) = 0$, and the first assertion is proven.

(2) For the second assertion, assume now that Y is Kähler. Note that $\pi_1(\tilde{X}) \simeq \mathbb{Z}^{2(m+1)}$ since X is homotopy equivalent to a complex torus of dimension $m + 1$. Thus, applying Proposition 2.1,

$$\pi_1(Y) \simeq \mathbb{Z}^n$$

(up to torsion) for some integer n such that $2m \le n \le 2(m + 1)$. Since Y is a Kähler manifold, Hodge decomposition gives either $n = 2m$ or $n = 2(m + 1)$ and $h^1(\mathcal{O}_Y) = m$ or $h^1(\mathcal{O}_Y) = m + 1$. Since $\kappa(Y) = 0$, a fundamental result due to Kawamata ([Ka, Theorem 24 and its corollary]) yields

$$h^1(\mathcal{O}_Y) = m (= \dim Y),$$

and also the fact that the Albanese morphism $a : Y \longrightarrow \mathrm{Alb}\, Y$ is bimeromorphic. This completes the proof. $\qquad\square$

LEMMA 4.3

Assume that $m \leq 2$. Then X is either a complex torus or the following two statements hold (recall that we assume f to have connected fibers).

    (1) *All smooth fibers are isomorphic to a fixed elliptic curve, say E.*

    (2) *Y is bimeromorphic to a complex torus of dimension 2. More precisely, the Albanese map $a : Y \to \mathrm{Alb}\, Y$ is a bimeromorphic surjective morphism.*

Proof

When $m = 1$, we have $\dim X = 2$. Then by classification, X is a complex torus since $K_X = \mathcal{O}_X$ and $b_1(X) = 4$. So from now we assume that $m = 2$. We may also assume that Y is a minimal surface.

    Suppose first that $\kappa(Y) \geq 0$; hence $\kappa(Y) = 0$ by Lemma 4.2(1), and F is a fixed elliptic curve. If Y would not be a complex torus, then $b_1(Y) \leq 3$ by classification. Then, however,

$$b_1(F) + b_1(Y) \leq 5 < 6 = b_1(X) = b_1(\tilde{X}),$$

a contradiction to Proposition 2.1.

    It remains to consider the case where $\kappa(Y) = -\infty$. If in addition Y is Kähler, then Y is projective (rational or birationally ruled) by classification. We also have $b_1(Y) \leq 2$ by the fact that $\pi_1(Y)$ is abelian (see Proposition 2.1). Then $b_1(F) \geq 4$ for the general fiber F of $\tilde{f} : \tilde{X} \to Y$, again by Proposition 2.1. Therefore $g(F) \geq 2$, where $g(F)$ is the genus of the curve F. Then we have a relative pluricanonical map $\hat{X} \dashrightarrow Z$ of $\tilde{X}$ over Y (see [Ue1, Theorem 12.1 and its proof]). As Y is projective, Z is a projective threefold by the construction given there. Hence

$$a(X) = a(\tilde{X}) = a(Z) = 3,$$

and we conclude that X is biholomorphic to a complex torus by Theorem 2.3 or by Corollary 2.4, and we are done.

    If $\kappa(Y) = -\infty$ and Y is not Kähler, then Y is a minimal surface of class VII. In particular, Y is not covered by rational curves and $b_1(Y) = 1$. Now, observe that f is almost holomorphic in the sense that f is proper holomorphic over some Zariski dense open subset of Y. Indeed, otherwise, the exceptional locus of the resolution of indeterminacies $\tilde{X} \to X$ dominates Y, so that Y would be dominated by a uniruled surface contradicting the assumption that Y is of class VII. Now f being almost holomorphic, the general fiber F of $\tilde{f}$ is an elliptic curve by adjunction. Thus

$$b_1(F) + b_1(Y) = 3 < 6 = b_1(X) = b_1(\tilde{X}),$$

a contradiction to Proposition 2.1. This completes the proof.   □

The upshot of the preceding two lemmata is that we may assume Y to be a torus. In particular, the meromorphic map $f : X \dashrightarrow Y$ (from our *original X*) is holomorphic, and all smooth fibers are isomorphic to a fixed elliptic curve E.

LEMMA 4.4

    (1) *The map f is smooth in codimension 1; that is, the set of critical values of f is of codimension ≥ 2 on Y.*

    (2) *The map f is equidimensional, or equivalently, f is a flat morphism.*

Proof

(1.a) Let us first consider the case where Y is projective. Then we take a general complete intersection curve C on Y, that is, a complete intersection of $m-1$ general hyperplanes of Y. So by Bertini's theorem, C is a smooth curve and $X_C = f^{-1}(C)$ is a smooth surface. Let $f_C : X_C \longrightarrow C$ be the induced morphism; then it suffices to show that f_C is a smooth morphism. By the adjunction formula, by $K_X = \mathcal{O}_X$ and $K_Y = \mathcal{O}_Y$, we obtain

$$K_{X_C} = f_C^*(K_C),$$

that is, $K_{X_C/C} = \mathcal{O}_{X_C}$. Then the canonical bundle formula for an elliptic surface (see, e.g., [BHPV, Theorem 12.3, p. 213]) gives the smoothness of f_C.

    (1.b) It remains to consider the case where $\dim Y = 2$ with Y not projective. If $a(Y) = 0$, then Y has no complete curve and f is smooth in codimension 1. If $a(Y) = 1$, then the algebraic reduction $a : Y \to C$ of Y is a smooth elliptic fibration over an elliptic curve C and all curves on Y are fibers of a. Thus the 1-dimensional part of the critical values form a normal crossing divisor, and we can apply the canonical bundle formula (see [Ue2, Theorem 2.4], [Fu2, Theorem 2.15]) to our elliptic threefold $f : X \to Y$. As a result, if the set of critical values is *not* of codimension ≥ 2, then there are fibers C_i $(1 \leq i \leq k)$ of a and positive integers n_i and M such that we have a bijection

$$|MK_X| \leftrightarrow \left| f^*\left(MK_Y + \sum_{i=1}^{k} n_i C_i \right) \right|.$$

This, however, is absurd because the left-hand side is an empty set by $K_X = \mathcal{O}_X$, but the right-hand side is a nonempty set since $K_Y = \mathcal{O}_Y$ and $n_i > 0$.

    This completes the proof of (1).

    (2) To begin, notice that equidimensionality and flatness are equivalent, X and Y being smooth. We denote the union of all irreducible components of dimension ≥ 2 in the fibers of f by N_0. Assuming $N_0 \neq \emptyset$, we derive a contradiction. To do that, let

$$N = f^{-1}f(N_0).$$

    First of all, N must be of pure codimension 1 in X. In fact, otherwise we take a general small m-dimensional disk Δ centered at a general point P of a 1-dimensional component of N. Then Δ dominates Y at $f(P)$ and $f|\Delta : \Delta \to Y$ is a generically finite surjective morphism around $f(P)$ branched in codimension ≥ 2 on Δ. However, this is impossible by the purity of the branch loci. Thus N is a divisor.

    Choose an irreducible component B of N.

By Hironaka's flattening theorem ([Hi1, main result]), there is a successive sequence of blowups $\mu : \hat{Y} \to Y$ such that the induced morphism

$$f_1 : X_1 := X \times_Y \hat{Y} \to \hat{Y}$$

is a flat morphism. Let

$$E'_i \quad (1 \le i \le k)$$

be the exceptional divisors of $\mu : \hat{Y} \to Y$. Since flatness is preserved under base change, we may assume that $\sum_{i=1}^k E'_i$ is a normal crossing divisor, possibly performing further blowups of $\hat{Y}$. Consider the normalization $X_2 \to X_1$ of X_1, and perform a resolution of singularities (see [Hi2, main result]) of X_2, say, $X_3 \to X_2$, and then, finally, take a resolution of indeterminacies (see [Hi2]) of $X \dashrightarrow X_3$, say, $\pi : \hat{X} \to X$. Let $\hat{f} : \hat{X} \to \hat{Y}$ be the induced morphism.

Let E_j $(1 \le j \le \ell)$ be the exceptional divisors of $\pi : \hat{X} \to X$, and let $\hat{B}$ be the proper transform of B on $\hat{X}$. Since B is of codimension 1 on X, necessarily $\hat{B} \ne E_j$ for any j. On the other hand, the fact that we have flattened f means that $\hat{f}(\hat{B})$ is one of the E'_i, say, E'_1.

We apply the canonical bundle formula for $\hat{f}$ in [Ue2, Theorem 2.4] (or [Fu2, Theorem 2.15]). Note that

$$K_{\hat{X}} = \sum_{j=1}^{\ell} a_j E_j$$

with every $a_j > 0$ since $K_X = \mathcal{O}_X$. For the same reason,

$$K_{\hat{Y}} = \sum_{i=1}^k b_i E'_i$$

with every $b_i > 0$. As f is smooth in codimension 1, the discriminant divisor of $\hat{f}$ is supported in $\bigcup_{i=1}^k E'_k$. Thus for a large multiple $M > 0$, we obtain

$$M \sum_{j=1}^{\ell} a_j E_j = M K_{\hat{X}} = \hat{f}^* \left(M K_{\hat{Y}} + \sum_{i=1}^k c_i E'_i \right) + D_1 - D_2$$

$$= \hat{f}^* \left(\sum_{i=1}^k (b_i + c_i) E'_i \right) + D_1 - D_2,$$

where D_1 is an effective divisor such that no multiple of D_1 moves, D_2 is an effective divisor such that $\hat{f}(D_2)$ is of codimension ≥ 2, and each c_i is a nonnegative integer. Notice that $b_i + c_i > 0$ for all i. Moreover, by [Ue2, Theorem 2.4] (especially statement (6) there), every element of $|M K_X|$ is uniquely written, as a divisor, in the form of the sum of an element of

$$\left| \hat{f}^* \left(\sum_{i=1}^k (b_i + c_i) E'_i \right) \right| = \hat{f}^* \left| \sum_{i=1}^k (b_i + c_i) E'_i \right| = \left\{ \hat{f}^* \left(\sum_{i=1}^k (b_i + c_i) E'_i \right) \right\}$$

and the divisor $D_1 - D_2$. Thus we have the following equality as a *divisor*:

$$M \sum_{j=1}^{\ell} a_j E_j + D_2 = \hat{f}^* \left(\sum_{i=1}^{k} (b_i + c_i) E_i' \right) + D_1.$$

In this equality of divisors, the prime divisor $\hat{B}$ appears in the right-hand side because $\hat{f}(\hat{B}) = E_1'$. But it does not appear in the left-hand side, since—as already observed—$\hat{B}$ is not π-exceptional, nor is $\hat{f}(\hat{B})$ of codimension ≥ 2 in $\hat{Y}$. This contradiction concludes the equidimensionality of f and the flatness of f as well. $\qquad\square$

LEMMA 4.5

The map $f : X \to Y$ is smooth.

Proof

Let $y \in Y$ be any point of Y, and suppose that the fiber X_y is singular. If X_y has a nonreduced component C, necessarily of dimension 1 by Lemma 4.4, we choose an m-dimensional general disk Δ centered at a general nonreduced point $P \in C$. Then $f|\Delta : \Delta \to Y$ is a generically finite surjective morphism around $f(P)$ whose branch locus in Δ is of codimension ≥ 2 (since f is smooth in codimension 1), a contradiction to the purity of branch loci.

Hence X_y is reduced. Now, take a local section D at a general point of X_y. Once we have chosen D, we can describe the fibration by the Weierstrass equation locally near y:

$$y^2 = x^3 + a(t)x + b(t),$$

where $a(t)$, $b(t)$ are holomorphic functions around y. Then the critical locus of the original f around y is given by the equation

$$4a(t)^3 + 27b(t)^2 = 0.$$

In particular, it is of pure codimension 1 on Y unless it is empty. As f is smooth in codimension 1, it follows that the critical locus is empty; that is, f is smooth. $\qquad\square$

Finally, we can apply Theorem 3.2 to conclude the proof of Theorem 4.1. $\qquad\square$

REMARK 4.6

In the case $\dim X = 3$, Lemma 4.5 can be proved without using Weierstrass models in the following way. We suppose that we already have shown in the first part of the proof of Lemma 4.5, without using the Weierstrass normal form, that f can have finitely many singularities, say, $x_1, \ldots, x_N$. Choose a general holomorphic 1-form ω on Y. Then $f^*(\omega)$ vanishes exactly at $x_1, \ldots, x_N$; hence $c_3(\Omega_X^1) > 0$. But $c_3(X) = \chi_{\text{top}}(X) = 0$ since X is homotopy equivalent to a torus.

REMARK 4.7

As the referees pointed out to us, it is likely true that if $f : X \longrightarrow Y$ is a proper holomorphic surjective map between complex manifolds with connected fibers and with trivial relative dualizing sheaf $\omega_{X/Y} = \mathcal{O}(K_{X/Y})$, then f is equidimensional. However, the authors can neither prove nor disprove this in general.

5. A characterization of complex tori: The case fibered over a curve

In this section we prove the following.

THEOREM 5.1

Let X be a compact complex threefold such that

(1) *X is homotopy equivalent to a complex torus of dimension 3;*
(2) *there is a dominant meromorphic map $f : X \dashrightarrow Y$ to a smooth compact curve;*
(3) *$\mathcal{O}_X(K_X) \cong \mathcal{O}_X$.*

Then X is biholomorphic to a complex torus.

Observe that in condition (2), by possibly taking the Stein factorization, we may assume f to have connected fibers.

We start with the folllowing.

LEMMA 5.2

Let X be a compact complex manifold subject to the assumptions in Theorem 5.1. Then either X is biholomorphic to a complex torus of dimension 3 or Y is an elliptic curve, f is a holomorphic map, and the general smooth fiber of f is a complex torus of dimension 2.

Proof
By Proposition 2.1, $\pi_1(Y)$ is an abelian group. Hence Y is either an elliptic curve or $\mathbb{P}^1$.

Consider first the case where Y is an elliptic curve, so that f is holomorphic. Let F be a general fiber of f. Then $K_F = \mathcal{O}_F$ by the adjunction formula. Since

$$b_1(F) + b_1(Y) \geq b_1(X) = 6$$

by Proposition 2.1, it follows from the classification of compact complex surfaces with $\kappa = 0$ (see, e.g., [BHPV, Table 10, p. 244]) that $b_1(F) = 4$ and F is a complex torus of dimension 2, so that we are done.

In the case $Y = \mathbb{P}^1$, let $\tilde{f} : \tilde{X} \to Y$ be a resolution of indeterminacies of f, and let F be a general fiber of $\tilde{f}$. Then F is smooth, and $\kappa(F) \geq 0$ by the adjunction formula.

If $\kappa(F) \geq 1$, then we can take a relative pluricanonical map $\varphi : X \dashrightarrow Z$ from X over Y (see [Ue1, Theorem 12.1] and its proof). As Y is projective and Z is

projective over Y, it follows that Z is projective. We have also $\dim Z \geq 2$. Thus X is biholomorphic to a complex torus by Theorem 4.1.

If $\kappa(F) = 0$, then $b_1(F) \leq 4$ again by classification of compact complex surfaces with $\kappa = 0$. Then, however,

$$b_1(F) + b_1(Y) \leq 4 < 6 = b_1(X) = b_1(\tilde{X}),$$

contradicting Proposition 2.1.

This completes the proof. □

From now on we may assume that we have a surjective holomorphic map $f : X \to Y$ over an elliptic curve Y with connected fibers.

The next two propositions of a more topological nature are applicable to many other situations.

PROPOSITION 5.3

Let X (resp., Y) be a topological space which is homotopy equivalent to a real torus A of real dimension, N, respectively, homotopy equivalent to a real torus B of real dimension r. Let $f : X \to Y$ be a continuous map which is dominant in the sense that $f^* : H^r(Y, \mathbb{Z}) \to H^r(X, \mathbb{Z})$ is nonzero. Let $u : \hat{Y} \to Y$ be the universal covering map, and let $\hat{X} = X \times_Y \hat{Y}$ be the fiber product. Then $\hat{X}$ is homotopy equivalent to a real torus of real dimension $N - r$; in particular, we have $H^{N-r}(\hat{X}, \mathbb{Z}) = \mathbb{Z}$.

Proof
Notice that we have natural isomorphisms $\pi_1(X) \simeq H_1(X, \mathbb{Z})$ and $\pi_1(Y) \simeq H_1(Y, \mathbb{Z})$ and—by our assumptions—they are isomorphic as abstract groups to $\mathbb{Z}^N$ and $\mathbb{Z}^r$, respectively. Let us consider the homomorphism

$$f_* : \pi_1(X) \simeq \mathbb{Z}^N \to \pi_1(Y) \simeq \mathbb{Z}^r,$$

induced by f. Under the above isomorphisms, this homomorphism is the same as the homomorphism

$$f_* : H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^N \to H_1(Y, \mathbb{Z}) \simeq \mathbb{Z}^r.$$

The dual homomorphism

$$(f_1)^* : H^1(Y, \mathbb{Z}) \simeq \mathbb{Z}^r \to H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^N$$

is a part of the homomorphism of algebras given by the pullback

$$f^* : H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z}).$$

Again, since X and Y are homotopy equivalent to real tori of dimensions N and r, respectively, we know that

$$\bigwedge^r H^1(Y, \mathbb{Z}) \simeq H^r(Y, \mathbb{Z}) \simeq \mathbb{Z}, \qquad \bigwedge^r H^1(X, \mathbb{Z}) \simeq H^r(X, \mathbb{Z}),$$

and the natural homomorphism

$$(f_r)^* : H^r(Y, \mathbb{Z}) \simeq \mathbb{Z} \to H^r(X, \mathbb{Z})$$

is simply $\bigwedge^r (f_1)^*$. As this is not zero by assumption, it follows that $(f_1)^*$ is injective. Hence $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective up to a finite cokernel.

Now f factors through the finite unramified cover Y' of Y corresponding to $f_*(\pi_1(X))$, whence we may replace Y by Y' and assume that f_* is surjective.

We have then $\operatorname{Ker} f_* \simeq \mathbb{Z}^{N-r}$, and $\pi_1(X)$ splits as

$$\pi_1(X) \simeq \mathbb{Z}^N \simeq \mathbb{Z}^r \oplus \mathbb{Z}^{N-r} \simeq \pi_1(Y) \oplus \operatorname{Ker} f_*.$$

Hence under the universal covering maps $u_B : \mathbb{R}^r \to B$ (corresponding to $\pi_1(Y) \simeq \pi_1(B)$) and $u_A : \mathbb{R}^N \to A$ (corresponding to $\pi_1(X) \simeq \pi_1(A)$), it follows that X' is homotopy equivalent to

$$\mathbb{R}^N / \operatorname{Ker} f_* \simeq \mathbb{R}^r \times (\mathbb{R}^{N-r}/\mathbb{Z}^{N-r}) \simeq \mathbb{R}^r \times T^{N-r},$$

where T^{N-r} is a real torus of dimension $N - r$. The result is now obvious. $\qquad\square$

PROPOSITION 5.4

Let X be a compact complex manifold homotopy equivalent to a complex torus of dimension $n + 1$, let Y be an elliptic curve, and let $f : X \to Y$ be a surjective holomorphic map with connected fibers.

Then all analytic sets $f^{-1}(y)$ $(y \in Y)$ are irreducible. Moreover, if Z is the reduction of a singular fiber and $\tilde{Z}$ is a resolution of singularities of Z, then there is a real torus of real dimension $2n$ and a surjective differentiable map $\rho : \tilde{Z} \to T^{2n}$ such that the induced homomorphism

$$\rho_* : \pi_1(\tilde{Z}) \to \pi_1(T^{2n})$$

is surjective.

Proof
As f is proper holomorphic and Y is compact, the critical values of f consist of finitely many points, say,

$$B := \{b_1, b_2, \ldots, b_k\}.$$

Let $u : \mathbb{C} \to Y$ be the universal cover of Y. We consider the fiber product $\hat{X} = X \times_Y \mathbb{C}$ and let $\hat{f} : \hat{X} \to \mathbb{C}$ be the induced holomorphic map. The set of critical values of $\hat{f}$ is $\hat{B} = u^{-1}(B)$. This is a discrete set of points of $\mathbb{C}$. By an appropriate choice of the origin in $\mathbb{C}$, we may assume that $0 \notin \hat{B}$. Then $u^{-1}(u(0))$ forms a lattice Λ such that $Y = \mathbb{C}/\Lambda$. We choose generators of Λ, say, v_1 and v_2. Then each region

$$U_{n,m} := \{\alpha v_1 + \beta v_2 \mid n \leq \alpha < n+1, m \leq \beta < m+1\}$$

$(n, m \in \mathbb{Z})$ forms a fundamental domain for Y. We first take the contractible graph

$$\Gamma_0 := \mathbb{R}v_1 \cup \bigcup_{k \in \mathbb{Z}} (\mathbb{R}v_2 + kv_1)$$

in $\mathbb{C}$. Then connect each $nv_1 + mv_2 \in \Gamma_0$ to each point $b_{k,n,m}$, of $\hat{B} \cap U_{n,m}$ by a simple path, say, $\gamma_{k,n,m}$ in $U_{n,m}$, so that they are mutually disjoint. Then

$$\Gamma := \Gamma_0 \cup \bigcup_{k,n,m} \gamma_{k,n,m}$$

becomes a contractible tree connecting 0 with the end points, which are the critical values of $\hat{f}$.

We next remove from Γ all the end points $b_{k,n,m}$ and denote the resulting space by

$$\Gamma' := \Gamma \setminus \bigcup_{k,n,m} \{b_{k,n,m}\}.$$

Finally, for each of the removed end points $b_{k,n,m}$ we fill in a small ball $B_{k,n,m}$ centered at $b_{k,n,m}$ and denote the resulting space by

$$\tilde{\Gamma} := \Gamma \cup \bigcup_{k,n,m} B_{k,n,m} = \Gamma' \cup \bigcup_{k,n,m} B_{k,n,m}.$$

We put

$$\hat{X}_\Gamma = \hat{f}^{-1}(\Gamma), \qquad \hat{X}_{\Gamma'} = \hat{f}^{-1}(\Gamma'), \qquad \hat{X}_{k,n,m} = \hat{f}^{-1}(B_{k,n,m}),$$

$$\hat{X}_{\tilde{\Gamma}} = \hat{f}^{-1}(\tilde{\Gamma}), \qquad F_{k,n,m} = \hat{f}^{-1}(b_{k,n,m}).$$

As Γ is tree, one can choose a neighborhood $U \subset \mathbb{C}$ of Γ such that Γ is a deformation retract of U and U is also a deformation retract of $\mathbb{C}$. Then we obtain a deformation retract from $\hat{X}$ to $\hat{X}_{\tilde{\Gamma}}$ and then to $\hat{X}_\Gamma$.

Combining this with Proposition 5.3, we obtain

$$H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z}) \simeq H^{2n}(\hat{X}, \mathbb{Z}) \simeq \mathbb{Z}.$$

On the other hand, $H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z})$ can be also computed as follows. We notice that Γ' is contractible and that the fiber over Γ' is smooth and homeomorphic to F_0. Here F_0 is the fiber over the base point zero. Thus

$$H^*(\hat{X}_{\tilde{\Gamma}'}, \mathbb{Z}) \simeq H^*(F_0, \mathbb{Z}).$$

As each $F_{k,n,m}$ is a deformation retract of $\hat{X}_{k,n,m}$ and since all the $\hat{X}_{k,n,m}$'s are mutually disjoint ($B_{k,n,m}$ being sufficiently small), it follows that

$$H^*\Big(\bigcup_{k,n,m} \hat{X}_{k,n,m}, \mathbb{Z} \Big) \simeq \bigoplus_{k,n,m} H^*(F_{k,n,m}, \mathbb{Z}).$$

Moreover, since each $B_{k,n,m} \cap \Gamma'$ is contractible and since the fibers over this set are homeomorphic to F_0, we also have

$$H^*\Big(\hat{X}_{\tilde{\Gamma}'} \cap \Big(\bigcup_{k,n,m} \hat{X}_{k,n,m} \Big), \mathbb{Z} \Big) \simeq \bigoplus_{k,n,m} H^*(F_0, \mathbb{Z}).$$

Thus by the Mayer-Vietoris exact sequence, we obtain

$$H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z}) = H^{2n}\Big(\hat{X}_{\tilde{\Gamma}'} \cup \Big(\bigcup_{k,n,m} \hat{X}_{k,n,m} \Big), \mathbb{Z} \Big)$$

$$\simeq \Big(H^{2n}(F_0, \mathbb{Z}) \oplus \bigoplus_{k,n,m} H^{2n}(F_{k,n,m}, \mathbb{Z}) \Big) \Big/ \Big(\bigoplus_{k,n,m} H^{2n}(F_0, \mathbb{Z}) \Big)$$

$$\simeq H^{2n}(F_0, \mathbb{Z}) \oplus \bigoplus_{k,n,m} \big(H^{2n}(F_{k,n,m}, \mathbb{Z})/H^{2n}(F_0, \mathbb{Z}) \big).$$

Since $H^{2n}(\hat{X}_{\tilde{\Gamma}}, \mathbb{Z}) \simeq \mathbb{Z}$, it follows that

$$H^{2n}(F_{k,n,m}, \mathbb{Z}) \simeq H^{2n}(F_0, \mathbb{Z})$$

for each singular fiber $F_{k,n,m}$. Since $H^{2n}(F_0, \mathbb{Z}) \simeq \mathbb{Z}$, it follows that

$$H^{2n}(F_{k,n,m}, \mathbb{Z}) \simeq \mathbb{Z}.$$

This implies the irreducibility of $F_{k,n,m}$ because $F_{k,n,m}$ is a compact connected complex space of pure dimension n (hence of real dimension $2n$), so that the rank of $H^{2n}(F_{k,n,m}, \mathbb{Z})$ is the cardinality of the set of irreducible components of $F_{k,n,m}$.

Let $y \in Y$, and let $Z = (X_y)_{\mathrm{red}}$ be the reduction of the fiber X_y of the original fibration f. Now we know that Z is irreducible. Moreover, by the proof of Proposition 2.1, we also know that the image of the natural map

$$\pi_1(Z) \to \mathrm{Ker}\big(\pi_1(X) \to \pi_1(Y) \big) \simeq \mathbb{Z}^{2n}$$

has finite cokernel. Thus the image is isomorphic to $\mathbb{Z}^{2n}$ as well. Consequently, we have a surjective homomorphism

$$\pi_1(Z) \to \mathbb{Z}^{2n}.$$

Since $\mathbb{Z}^{2n}$ is isomorphic to the fundamental group of a real torus of dimension $2n$, say, T^{2n}, the surjective morphism above is induced by a dominant continuous map

$$a : \pi_1(Z) \to \pi_1(T^{2n}).$$

Since $\pi_1(T^{2n})$ is commutative, this naturally induces a surjective morphism

$$a : H_1(Y, \mathbb{Z}) \to H_1(T^{2n}, \mathbb{Z}).$$

Passing to the dual, we obtain an injective morphism

$$a^* : H^1(T^{2n}, \mathbb{Z}) \simeq \mathbb{Z}^{2n} \to H^1(Z, \mathbb{Z}).$$

Let $\nu : \tilde{Z} \to Z$ be a resolution of singularities of the complex space Z (see [Hi2, main result]). Then the composition of a and ν defines a continuous map $\tilde{a}$ such that its action on the first homology is the composition $\tilde{a}_*$ of $\nu_* : H_1(\tilde{Z}, \mathbb{Z}) \to H_1(Z, \mathbb{Z})$ with $a_* : H_1(Z, \mathbb{Z}) \to H_1(T^{2n}, \mathbb{Z})$.

Let $\langle \varphi_i \rangle_{i=1}^{2n}$ be a basis of $H^1(T^{2n}, \mathbb{Z})$, and consider their inverse images $\tilde{a}^*(\varphi_i)$ as being represented by d-closed differential forms. Then the map given by integration

$$\tilde{Z} \ni x \mapsto \Big(\int_{x_0}^{x} \tilde{a}^*(\varphi_i) \Big)_{i=1}^{2n}$$

gives a differentiable map $\rho : \tilde{Z} \to T^{2n}$ such that the induced morphism $\rho_* : H_1(\tilde{Z}, \mathbb{Z}) \to H_1(T^{2n}, \mathbb{Z})$ is the homomorphism $\tilde{a}_*$.

Since we have isomorphisms

$$H^{2n}(\tilde{Z}, \mathbb{Z}) \cong H^{2n}(Z, \mathbb{Z}) \cong H^{2n}(T^{2n}, \mathbb{Z}),$$

it follows that $\tilde{a}$ is dominant and that we have a surjective homomorphism $\pi_1(\tilde{Z}) \to \mathbb{Z}^{2n}$. □

Let us return to the proof of Theorem 5.1 and recall that by Lemma 5.2, we may assume Y to be an elliptic curve. We need only show that f is smooth; then Theorem 5.1 follows from Theorem 3.2.

So assume that a fiber Z of f is singular. We already know that Z is irreducible by Proposition 5.3. Denote by m the multiplicity of Z so that $Z = mZ_{\mathrm{red}}$. Since X is smooth, hence Z_{red} is also a Cartier divisor on X. In particular, the dualizing sheaf $\omega_{Z_{\mathrm{red}}}$ is invertible. More precisely, by the adjunction formula and by $K_X = \mathcal{O}_X$, we have

$$\omega_{Z_{\mathrm{red}}} = \mathcal{O}_X(Z_{\mathrm{red}}) \otimes \mathcal{O}_{Z_{\mathrm{red}}}$$

and therefore

$$\omega_{Z_{\mathrm{red}}}^{\otimes m} \simeq \mathcal{O}_{Z_{\mathrm{red}}}.$$

Since $K_X = \mathcal{O}_X$, the multiplicity m is nothing but the minimal positive integer satisfying this isomorphism (see, e.g., [BHPV, Lemma 8.3, p. 111]).

Let $\tilde{Z}$ be the minimal resolution of the normalization Z' of Z_{red}. Since Z_{red} is Gorenstein, the conductor ideal of $Z' \to Z_{\mathrm{red}}$ is of pure dimension 1 (if Z_{red} is not normal). Moreover, since $\tilde{Z}$ is a minimal resolution, the canonical divisors of Z_{red} and $\tilde{Z}$ differ by an effective divisor, classically called the subadjunction divisor. We conclude with the well-known formula

$$\omega_{\tilde{Z}}^{\otimes m} \simeq \mathcal{O}_{\tilde{Z}}(-D),$$

where D is an effective divisor, possibly zero (if and only if Z_{red} is normal with at most rational double points as singularities).

First, suppose $D = 0$; hence $\kappa(\tilde{Z}) = 0$. Since $\pi_1(\tilde{Z})$ maps onto $\pi_1(T) \simeq \mathbb{Z}^4$ by Proposition 5.3, it follows again from surface classification that $\tilde{Z}$ is a complex torus of dimension 2. Since a complex torus of dimension 2 has no curve with negative self-intersection, it has no nontrivial crepant contraction to a normal surface Z'. Hence the three surfaces

$$\tilde{Z}, Z', Z_{\mathrm{red}}$$

are all isomorphic.

In particular, Z_{red} is also a smooth complex torus (of dimension 2), and $\omega_{Z_{\mathrm{red}}} \simeq \mathcal{O}_{Z_{\mathrm{red}}}$. This implies that $m = 1$ and $Z = Z_{\mathrm{red}}$ is smooth.

If $D \neq 0$, then $\kappa(\tilde{Z}) = -\infty$. From the classification of compact complex surfaces with $\kappa = -\infty$ (see, e.g., [BHPV, Table 10, p. 244]), $\tilde{Z}$ is either birationally

ruled, say, over a curve C, or a surface of class VII. If it is birationally ruled, then $H^1(\tilde{Z}, \mathbb{Z})$ is the pullback of $H^1(C, \mathbb{Z})$. This, however, implies that

$$\wedge^4 H^1(\tilde{Z}, \mathbb{Z}) = 0,$$

a contradiction to the proven injectivity of $\wedge^4 H^1(T, \mathbb{Z}) \to \wedge^4 H^1(\tilde{Z}, \mathbb{Z})$.

If $\tilde{Z}$ is of class VII, then $b_1(\tilde{Z}) = 1$. This again contradicts the surjectivity of

$$\pi_1(\tilde{Z}) \to \pi_1(T) \simeq \mathbb{Z}^4$$

in Proposition 5.3.

Hence $f : X \to Y$ is smooth. This finishes the proof of Theorem 5.1.   □

Theorem 1.1 now follows from Theorems 2.3, 4.1, and 5.1.   □

6. Threefolds without meromorphic functions

Instead of assuming the existence of meromorphic functions, we require in this short concluding section the existence of some holomorphic tangent vectors or holomorphic 1-forms.

THEOREM 6.1

Let X be a smooth compact complex threefold which is homotopy equivalent to a torus. If the tangent bundle T_X is trivial, then X is biholomorphic to a torus.

Proof

By our assumption, $X \simeq G/\Gamma$, where G is a simply connected complex 3-dimensional Lie group and $\Gamma \simeq \pi_1(X)$ is cocompact. By Lemma 6.2, $G \simeq \mathbb{C}^3$ or $G \simeq \mathrm{SL}(2, \mathbb{C})$ as groups. But $\mathrm{SL}(2, \mathbb{C})$ is not contractible; in fact,

$$b_3\big(\mathrm{SL}(2, \mathbb{C})\big) = 3$$

(see, e.g., [Ko]). Hence $X \simeq \mathbb{C}^3/\mathbb{Z}^6$, and our claim follows.   □

The article [Ko] was communicated to us by I. Radloff. For discussions concerning the first part of the following lemma, which is of course well known to the experts, we thank J. Winkelmann and, in particular, A. Huckleberry.

LEMMA 6.2

Let G be a simply connected 3-dimensional complex Lie group. Then

   (1) *either $G \simeq \mathrm{SL}(2, \mathbb{C})$ as Lie group, or G is solvable and $G \simeq \mathbb{C}^3$ as complex manifold;*

   (2) *if G is solvable and if G contains a lattice Γ (i.e., a discrete subgroup such that G/Γ has bounded volume) such that Γ is abelian, then G is abelian and therefore $G \simeq \mathbb{C}^3$ as Lie group.*

Proof

(1) By the Levi-Malcev decomposition, G is either semisimple or solvable by reasons of dimension. In the semisimple case, $G \simeq \mathrm{SL}(2, \mathbb{C})$. If G is solvable, then $G \simeq \mathbb{C}^3$ (even in any dimension); see, e.g., [Na, Proposition 1.4]).

(2) Since Γ is abelian, so is G by, for example, [Wi, (3.14.6)]. Hence $G \simeq \mathbb{C}^3$ as Lie group. $\qquad \square$

LEMMA 6.3

Let X be a compact complex manifold with algebraic dimension $a(X) = 0$. Let V be a holomorphic rank r bundle on X. Then the evaluation homomorphism

$$\mathrm{ev} : H^0(X, V) \otimes \mathcal{O}_X \to V$$

is injective. In particular, all the global sections of V are carried by the trivial rank h subsheaf $H^0(X, V) \otimes \mathcal{O}_X$.

In particular, $h^0(X, V) = h \leq r$, and if $h = r$ and $\det(V) \cong \mathcal{O}_X$, then V is trivial.

Proof

For each point $x \in X$ we have a linear map of $\mathbb{C}$-vector spaces

$$\mathrm{ev}_x : H^0(X, V) \to V(x),$$

where $V(x) := V_x / \mathfrak{M}_x V_x$ is the fiber of the vector bundle over the point x.

We claim that ev_x is injective for a general point $x \in X$.

Otherwise, let m be the generic rank of ev_x: then we get a meromorphic map

$$k : X \dashrightarrow \mathrm{Grass}\bigl(h - m, H^0(X, V)\bigr)$$

associating to x the subspace $\mathrm{Ker}(\mathrm{ev}_x)$.

By the projectivity of the Grassmann manifold, k must be constant. But a section vanishing at a general point is identically zero, which proves our assertion that $H^0(X, V) \otimes \mathcal{O}_X$ yields a subsheaf of V.

Moreover, if $h = r$, the homomorphism ev induces a nonconstant homomorphism $\Lambda^r(\mathrm{ev}) : \mathcal{O}_X \to \det(V)$. Thus if $\det(V)$ is trivial, $\Lambda^r(\mathrm{ev})$ is invertible; hence ev is an isomorphism. $\qquad \square$

COROLLARY 6.4

Let X be a compact complex manifold of dimension n with algebraic dimension $a(X) = 0$ and with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$. Then

$$h^0(\Omega_X^1) \geq n \Leftrightarrow h^0(T_X) \geq n \Leftrightarrow T_X \cong \mathcal{O}_X^n \Leftrightarrow \Omega_X^1 \cong \mathcal{O}_X^n.$$

THEOREM 6.5

Let X be a smooth compact complex threefold with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$, which is homotopy equivalent to a torus. If $h^0(X, \Omega_X^1) \geq 3$ or if $h^0(X, T_X) \geq 3$, then X is biholomorphic to a torus.

Proof

By our main theorem (Theorem 1.1), we may assume $a(X) = 0$, and applying the previous Corollary 6.4, we get $\Omega_X^1 \cong \mathcal{O}_X^3$. Now we conclude by Theorem 6.1.

$\square$

REMARK 6.6

It already seems difficult to exclude the case $h^0(\Omega_X^1) = 2$. Taking a basis ω_1, ω_2, we are able to exclude the case when both ω_i are nonclosed. Since X is not necessarily Kähler, the existence of a closed holomorphic 1-form does not lead to a nontrivial Albanese map, which is the obstacle to conclude.

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Catanese: Lehrstuhl Mathematik VIII, Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany; fabrizio.catanese@uni-bayreuth.de

Oguiso: Department of Mathematics, Osaka University, Toyonaka, 560-0043 Osaka, Japan; oguiso@math.sci.osaka-u.ac.jp

Peternell: Lehrstuhl Mathematik I, Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth, Germany; thomas.peternell@uni-bayreuth.de