

Cyclic symmetry and adic convergence in Lagrangian Floer theory

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Abstract In this article we use a continuous family of multisections of the moduli space of pseudoholomorphic discs to partially improve the construction of the Lagrangian Floer cohomology of [11] in the case of \mathbb{R} coefficient. Namely, we associate a *cyclically symmetric* filtered A_∞ -algebra to every relatively spin Lagrangian submanifold. We use the same trick to construct a local rigid analytic family of filtered A_∞ -structures associated to a (family of) Lagrangian submanifolds. We include the study of homological algebra of pseudoisotopy of cyclic (filtered) A_∞ -algebras.

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1. Introduction

In this article we study the de Rham version of Lagrangian Floer theory and use it to improve some of the results in [11] over \mathbb{R} coefficient. In particular, in this article, we prove [11] Conjectures 3.6.46, 3.6.48, and part of Conjecture T over

Kyoto Journal of Mathematics, Vol. 50, No. 3 (2010), 521–590

DOI 10.1215/0023608X-2010-004, © 2010 by Kyoto University

Received May 8, 2008. Revised May 1, 2009. Accepted January 5, 2010.

Mathematics Subject Classification: Primary 53D40; Secondary 53D12, 53D37.

Author's work supported by Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research number 18104001 and Global Centers of Excellence Program G08.

\mathbb{R} coefficient. Let L be a relatively spin Lagrangian submanifold in a symplectic manifold (M, ω) . In this article we always assume that L is compact and M is either compact or convex at infinity.

The universal Novikov ring $\Lambda_{0, \text{nov}}^{\mathbb{R}}$ is defined in [11] (see also Definition 6.3). Let $H(L; \mathbb{R})$ be the (de Rham) cohomology group of L over \mathbb{R} coefficient. We put $H(L; \Lambda_{0, \text{nov}}^{\mathbb{R}}) = H(L; \mathbb{R}) \otimes_{\mathbb{R}} \Lambda_{0, \text{nov}}^{\mathbb{R}}$.

THEOREM 1.1

$H(L; \Lambda_{0, \text{nov}}^{\mathbb{R}})$ has a structure of unital filtered cyclic A_{∞} -algebra which is well defined up to isomorphism.

Hereafter we work over \mathbb{R} coefficient, so we write $\Lambda_{0, \text{nov}}$ in place of $\Lambda_{0, \text{nov}}^{\mathbb{R}}$. We denote by $\Lambda_{0, \text{nov}}^+$ its ideal $\bigcup_{E_0 > 0} \bigcup_G F^{E_0} \Lambda_0^G$ (see Definition 6.3).

Let us explain the notions appearing in Theorem 1.1. We consider a (graded) antisymmetric inner product $\langle \cdot \rangle$ on $H(L; \mathbb{R})[1]$ by

$$(1.1) \quad \langle u, v \rangle = (-1)^{\deg u \deg v + \deg u} \int_L u \wedge v.$$

Filtered A_{∞} -structure defines a map

$$\mathbf{m}_k : B_k(H(L; \Lambda_{0, \text{nov}})[1]) \rightarrow H(L; \Lambda_{0, \text{nov}})[1]$$

of degree one (for $k \geq 0$) such that

$$(1.2) \quad \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} (-1)^* \mathbf{m}_{k_1}(\mathbf{x}_1, \dots, \mathbf{m}_2(\mathbf{x}_i, \dots, \mathbf{x}_{i+k_2-1}), \dots, \mathbf{x}_k) = 0,$$

$* = \deg \mathbf{x}_1 + \dots + \deg \mathbf{x}_{i-1} + i - 1$. (We require $\mathbf{m}_0(1) \equiv 0 \pmod{\Lambda_{0, \text{nov}}^+}$.)

The filtered A_{∞} -structure is said to be *unital* if $\mathbf{e} = 1 \in H^0(L; \mathbb{R})$ satisfies

$$(1.3a) \quad \mathbf{m}_k(\mathbf{x}_1, \dots, \mathbf{x}_i, \mathbf{e}, \mathbf{x}_{i+2}, \dots, \mathbf{x}_k) = 0$$

for $k \neq 2$ and

$$(1.3b) \quad \mathbf{x} = \mathbf{m}_2(\mathbf{e}, \mathbf{x}) = (-1)^{\deg \mathbf{x}} \mathbf{m}_2(\mathbf{x}, \mathbf{e}).$$

The filtered A_{∞} -structure is said to be *cyclically symmetric* or *cyclic* if

$$(1.4) \quad \langle \mathbf{m}_k(\mathbf{x}_1, \dots, \mathbf{x}_k), \mathbf{x}_0 \rangle = (-1)^* \langle \mathbf{m}_k(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}), \mathbf{x}_k \rangle.$$

$* = (\deg \mathbf{x}_0 + 1)(\deg \mathbf{x}_1 + \dots + \deg \mathbf{x}_k + k)$. A cyclically symmetric filtered A_{∞} -algebra is said to be a *cyclic filtered A_{∞} -algebra* (see Remark 8.1 for the definition of isomorphism of cyclic filtered A_{∞} -algebra).

REMARK 1.1

(1) Except for the statement on cyclicity, Theorem 1.1 was proved in [11, Theorem A]. Actually in that case we may take \mathbb{Q} in place of \mathbb{R} . The author does not know how to generalize Theorem 1.1 to \mathbb{Q} (or $\Lambda_{0, \text{nov}}^{\mathbb{Q}}$) coefficient. See also [6, Section 9].

(2) The formula (1.4) is slightly different from [11, Proposition 8.4.8]. Also, the sign in (1.1) is different from those explained in [11, Remark 8.4.7(1)]. If we use sign convention in (1.1), then (1.4) becomes equivalent to one in [11, Proposition 8.4.8] (see Lemma 6.1). The author thanks C.-H. Cho (see [3]) for this remark.

We next state our result about adic convergence. We take a basis \mathbf{e}_i of $H(L; \mathbb{R})$ ($i = 0, \dots, b$) such that $\mathbf{e} = \mathbf{e}_0 = 1 \in H^0(L; \mathbb{R})$, $\mathbf{e}_1, \dots, \mathbf{e}_{b_1}$ are the basis of $H^1(L; \mathbb{R})$ and other \mathbf{e}_i 's ($i = b_1 + 1, \dots, b$) are the basis of $H^k(L; \mathbb{R})$, $k \geq 2$. For $\mathbf{x} \in H^{\text{odd}}(L; \Lambda_{0, \text{nov}})$, we put

$$(1.5) \quad \mathbf{x} = \sum x_i \mathbf{e}_i.$$

We introduce y_i ($i = 1, \dots, b_1$) by

$$(1.6) \quad y_i = e^{x_i} = \sum_{k=0}^{\infty} \frac{1}{k!} x_i^k$$

(see Section 10 for a discussion of the convergence of the right-hand side). We consider

$$\sum_{k=0}^N \mathfrak{m}_k(\mathbf{x}, \dots, \mathbf{x}) = P_N(\mathbf{x}).$$

We remark that $\lim_{N \rightarrow \infty} P_N(\mathbf{x})$ converges in T -adic topology if $x_i \in \Lambda_{0, \text{nov}}^+$. We used this fact to define the Maurer-Cartan equation and its solution (bounding cochain) in [14]. We improve it as follows.

THEOREM 1.2

- (1) If $x_i \in \Lambda_{0, \text{nov}}$ then $\lim_{N \rightarrow \infty} P_N(\mathbf{x})$ converges. We denote its limit by $\mathfrak{m}(e^{\mathbf{x}})$.
- (2) $\mathfrak{m}(e^{\mathbf{x}})$ depends only on $y_1, \dots, y_{b_1}, x_{b_1+1}, \dots, x_b$.
- (3) There exists $\delta > 0$ such that $\mathfrak{m}(e^{\mathbf{x}})$ extends to

$$(1.7) \quad \left\{ (x_0, y_1, \dots, y_{b_1}, x_{b_1+1}, \dots, x_b) \mid \delta > v(y_i) > -\delta, \right. \\ \left. v(x_{b_1+1}), \dots, v(x_b) \geq 0 \right\}.$$

- (4) Let $\mathcal{M}(L)_\delta$ be the set of $(x_0, y_1, \dots, y_{b_1}, x_{b_1+1}, \dots, x_b)$ in the domain (1.7) such that $\mathfrak{m}(e^{\mathbf{x}}) = 0$. Then there exists a family of strict and unital cyclic filtered A_∞ -algebras parameterized by $\mathcal{M}(L)_\delta$.

Here $v : \Lambda \rightarrow \mathbb{R}$ is a valuation defined by

$$v\left(\sum a_i T^{\lambda_i}\right) = \inf\{\lambda_i \mid a_i \neq 0\}.$$

(Here we assume that $a_i \in \mathbb{R}[e, e^{-1}]$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.)

A filtered A_∞ -algebra is said to be *strict* if $\mathfrak{m}_0 = 0$.

In [14] we considered the case $\mathbf{x} \equiv 0 \pmod{\Lambda_{0,\text{nov}}^+}$. In that case, we defined

$$(1.8) \quad \mathfrak{m}_k^{\mathbf{x}}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \sum_{m_0, \dots, m_k=0}^{\infty} \mathfrak{m}_{k+\sum_{i=0}^k m_i}(\mathbf{x}^{\otimes m_0}, \mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}^{\otimes m_k}).$$

Then $(H(L; \Lambda_{0,\text{nov}}), \mathfrak{m}_k^{\mathbf{x}})$ is a filtered A_{∞} -algebra. (This A_{∞} -algebra may not be strict; namely, $\mathfrak{m}_*^{\mathbf{x}}(1)$ may be nonzero. The equality $\mathfrak{m}_*^{\mathbf{x}}(1) = 0$ holds if and only if $\sum_{k=0}^{\infty} \mathfrak{m}_k(\mathbf{x}^k) = 0$.) It is unital or cyclic if $(H(L; \Lambda_{0,\text{nov}}), \mathfrak{m}_k)$ is unital or cyclic, respectively. It is strict if and only if $\mathfrak{m}(e^{\mathbf{x}}) = 0$ (see [11, Proposition 3.6.10]).

Using the technique of [2], we can relax the condition $\mathbf{x} \equiv 0 \pmod{\Lambda_{0,\text{nov}}^+}$ to $\mathbf{x} \equiv 0 \in H^1(L; \Lambda_{0,\text{nov}})$ (see [12], where the case when M is toric is discussed).

Theorem 1.2(4) says that we can further extend this story to the case when \mathbf{x} is contained in a larger domain. The convergence of (1.8) is one on T -adic topology. The convergence in Theorem 1.2(1) is different from T -adic topology and is a mixture of Archimedean and non-Archimedean topology (see Definition 13.1).

Our main theorems, Theorems 1.1 and 1.2, are related as follows. In Theorem 1.2 it is essential to be able to change variables from x_i to $y_i = e^{x_i}$. This becomes possible after we perform the whole construction of Kuranishi structure and its perturbation in a way compatible with the forgetful map. Then, for example, $\mathfrak{m}_{3,\beta}(\mathbf{e}_i, \mathbf{e}_i, \mathbf{x}) + \mathfrak{m}_{3,\beta}(\mathbf{e}_i, \mathbf{x}, \mathbf{e}_i) + \mathfrak{m}_{3,\beta}(\mathbf{x}, \mathbf{e}_i, \mathbf{e}_i)$ is related to $\mathfrak{m}_{1,\beta}(\mathbf{x})$ by the formula

$$(1.9) \quad \mathfrak{m}_{3,\beta}(\mathbf{e}_i, \mathbf{e}_i, \mathbf{x}) + \mathfrak{m}_{3,\beta}(\mathbf{e}_i, \mathbf{x}, \mathbf{e}_i) + \mathfrak{m}_{3,\beta}(\mathbf{x}, \mathbf{e}_i, \mathbf{e}_i) = \frac{1}{2!}(\beta \cap \mathbf{e}_i)^2 \mathfrak{m}_{1,\beta}(\mathbf{x}).$$

Here \mathbf{e}_i is a degree 1 cohomology class and β is an element of $H_2(X; \mathbb{Z})$. The map $\mathfrak{m}_{k,\beta}$ is the contribution of the pseudoholomorphic disc of homology class β to \mathfrak{m}_k . Equation (1.9) and a similar formula make the change of coordinates $y_i = e^{x_i}$ possible.

Compatibility with the forgetful map (which is the reason why (1.9) is correct) is also used to prove cyclic symmetry (1.4). (See also [6, Section 9].) It is used also to prove unitality (1.3).

Thus the main technical part of this article works out the way to construct Kuranishi structure and (an abstract multivalued continuous family of) perturbations on it, which is invariant of the process of forgetting boundary marked points. This construction is performed in Sections 2–5. We use it to prove a version of Theorem 1.1 (namely, a version modulo T^E) in Sections 6–7. To go from this version to Theorem 1.1, we use a trick similar to the one in [11, Chapter 7]. We need to discuss some homological algebra of cyclic filtered A_{∞} -algebras, which is in Sections 8–10. In Section 11, we work out the one-parameter family version of the construction of Sections 2–5, which is used to prove well-definedness of the structure up to pseudoisotopy. Theorem 1.1 then is proved in Section 12. (The proof of independence of choices is completed in Section 14.) In Section 13 we prove Theorem 1.2.

Our main result of this article has two applications.

One is to define a numerical invariant of a (special) Lagrangian submanifold in a Calabi-Yau 3-fold, which is a rational homology sphere. Roughly speaking, it counts the number of pseudoholomorphic discs with appropriate weights. This is

invariant of perturbation and other choices but depends on almost complex structure. Existence of such an invariant was expected by several people, especially by D. Joyce. For the rigorous construction it is essential to find a perturbation that is compatible with forgetful maps. Such a perturbation is provided in this article. We will use it to define this invariant and discuss its properties in [8].

The other application is to define a (rigid analytic) family of Floer cohomologies. Actually the space $\mathcal{M}(L)_\delta$ in Theorem 1.2 is a chart of certain rigid analytic spaces. We can glue them up to define appropriate rigid analytic space. Then, by extending the construction of Theorem 1.2 to a pair (or triple, etc.) of Lagrangian submanifolds, we can show that another Lagrangian submanifold L' gives an object of derived category of coherent sheaves on this rigid analytic space. This is a proof of a part of [11, Conjecture U] and a step to realizing the project to construct a homological mirror functor by using family of Floer cohomology. (This project was started around 1998 in [5]; see also [4]. Its rigid analytic version was first proposed by [19].) The construction of the rigid analytic family of Floer cohomology will be given by extending the construction of this article in [9]; there its application to mirror symmetry of a torus in a form more general than [5] and [19] will be given. The story will be further generalized to include the case of singular fibers in [10], in dimension 2.

2. Kuranishi structure on the moduli space of pseudoholomorphic discs:

Review

For the purpose of this article, we need to take a Kuranishi structure on the moduli space of pseudoholomorphic discs with some additional properties. We construct such a Kuranishi structure in Section 3. In this section we review the definition of a Kuranishi structure and the construction of it on the moduli space of pseudoholomorphic discs, which was due to [13].

Let \mathcal{M} be a compact space. A Kuranishi chart is $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ which satisfies the following:

- (1) V_α is a smooth manifold (with boundaries or corners), and Γ_α is a finite group acting effectively on V_α ;
- (2) $\text{pr}_\alpha : E_\alpha \rightarrow V_\alpha$ is a finite-dimensional vector bundle on which Γ_α acts so that pr_α is Γ_α -equivariant;
- (3) s_α is a Γ_α -equivariant section of E_α ;
- (4) $\psi_\alpha : s_\alpha^{-1}(0)/\Gamma_\alpha \rightarrow \mathcal{M}$ is a homeomorphism to its image, which is an open subset.

We call E_α an *obstruction bundle*, and we call s_α a *Kuranishi map*. If $\mathbf{p} \in \psi_\alpha(s_\alpha^{-1}(0)/\Gamma_\alpha)$, we call $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ a *Kuranishi neighborhood* of \mathbf{p} .

Let $\psi_{\alpha_1}(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2}) \neq \emptyset$. A *coordinate transformation* from $(V_{\alpha_1}, E_{\alpha_1}, \Gamma_{\alpha_1}, \psi_{\alpha_1}, s_{\alpha_1})$ to $(V_{\alpha_2}, E_{\alpha_2}, \Gamma_{\alpha_2}, \psi_{\alpha_2}, s_{\alpha_2})$ is $(\hat{\phi}_{\alpha_2\alpha_1}, \phi_{\alpha_2\alpha_1}, h_{\alpha_2\alpha_1})$ such that

- (1) $h_{\alpha_2\alpha_1}$ is an injective homomorphism $\Gamma_{\alpha_1} \rightarrow \Gamma_{\alpha_2}$;

(2) $\phi_{\alpha_2\alpha_1} : V_{\alpha_1\alpha_2} \rightarrow V_{\alpha_2}$ is an $h_{\alpha_2\alpha_1}$ -equivariant smooth embedding from a Γ_{α_1} -invariant open set $V_{\alpha_1\alpha_2}$ to V_{α_1} , such that the induced map $\underline{\phi}_{\alpha_2\alpha_1} : V_{\alpha_1\alpha_2}/\Gamma_{\alpha_1} \rightarrow V_{\alpha_2}/\Gamma_{\alpha_2}$ is injective;

(3) $(\hat{\phi}_{\alpha_2\alpha_1}, \phi_{\alpha_2\alpha_1})$ is an $h_{\alpha_2\alpha_1}$ -equivariant embedding of vector bundles $E_{\alpha_1}|_{V_{\alpha_1\alpha_2}} \rightarrow E_{\alpha_2}$;

(4) $\hat{\phi}_{\alpha_2\alpha_1} \circ s_{\alpha_1} = s_{\alpha_2} \circ \phi_{\alpha_2\alpha_1}$;

(5) $\psi_{\alpha_1} = \psi_{\alpha_2} \circ \underline{\phi}_{\alpha_2\alpha_1}$ on $(s_{\alpha_2}^{-1}(0) \cap V_{\alpha_1\alpha_2})/\Gamma_{\alpha_2}$ (here $\underline{\phi}_{\alpha_2\alpha_1}$ is as in (2));

(6) the map $h_{\alpha_2\alpha_1}$ restricts to an isomorphism $(\Gamma_{\alpha_1})_x \rightarrow (\Gamma_{\alpha_1})_{\phi_{\alpha_2\alpha_1}(x)}$ between isotopy groups, for any $x \in V_{\alpha_1\alpha_2}$;

(7) $\psi_{\alpha_1}(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2}) = \psi_{\alpha_1}((s_{\alpha_1}^{-1}(0) \cap V_{\alpha_1\alpha_2})/\Gamma_{\alpha_1})$.

A *Kuranishi structure* on \mathcal{M} assigns a Kuranishi neighborhood $(V_{\mathbf{p}}, E_{\mathbf{p}}, \Gamma_{\mathbf{p}}, \psi_{\mathbf{p}}, s_{\mathbf{p}})$ to each $\mathbf{p} \in \mathcal{M}$, such that if $\mathbf{q} \in \psi_{\mathbf{p}}(V_{\mathbf{p}}/\Gamma_{\mathbf{p}})$, then there exists a coordinate transformation $(\hat{\phi}_{\mathbf{p}\mathbf{q}}, \phi_{\mathbf{p}\mathbf{q}}, h_{\mathbf{p}\mathbf{q}})$ from $(V_{\mathbf{q}}, E_{\mathbf{q}}, \Gamma_{\mathbf{q}}, \psi_{\mathbf{q}}, s_{\mathbf{q}})$ to $(V_{\mathbf{p}}, E_{\mathbf{p}}, \Gamma_{\mathbf{p}}, \psi_{\mathbf{p}}, s_{\mathbf{p}})$. We assume appropriate compatibility conditions among coordinate transformations, which we omit, and refer the reader to [16].

Let \mathcal{M} have a Kuranishi structure. We consider the normal bundle $N_{\phi_{\mathbf{p}\mathbf{q}}(V_{\mathbf{q}})}V_{\mathbf{p}}$. We take the fiber derivative of the Kuranishi map $s_{\mathbf{p}}$ and obtain a homomorphism

$$d_{\text{fiber } s_{\mathbf{p}}} : N_{\phi_{\mathbf{p}\mathbf{q}}(V_{\mathbf{q}})}V_{\mathbf{p}} \rightarrow E_{\mathbf{p}}|_{\text{Im } \phi_{\mathbf{p}\mathbf{q}}},$$

which is an $h_{\mathbf{p}\mathbf{q}}$ -equivariant bundle homomorphism. We say that the space with Kuranishi structure \mathcal{M} has the tangent bundle if $d_{\text{fiber } s_{\mathbf{p}}}$ induces a bundle isomorphism

$$(2.1) \quad N_{\phi_{\mathbf{p}\mathbf{q}}(V_{\mathbf{q}})}V_{\mathbf{p}} \cong \frac{E_{\mathbf{p}}|_{\text{Im } \phi_{\mathbf{p}\mathbf{q}}}}{\hat{\phi}_{\mathbf{p}\mathbf{q}}(E_{\mathbf{q}})}.$$

We call a space with Kuranishi structure which has a tangent bundle a *Kuranishi space* or *K-space*.

We call a Kuranishi space *oriented* if $V_{\mathbf{p}}$ and $E_{\mathbf{p}}$ are oriented and if (2.1) is orientation preserving.

Let (M, ω) be a symplectic manifold of (real) dimension $2n$. We take an almost complex structure J , which is tamed by ω . For each $\alpha \in H_2(M; \mathbb{Z})$ and $\ell \geq 0$, we denote by $\mathcal{M}_{\ell}^{\text{cl}}(\alpha; J)$ the moduli space of stable J -holomorphic curves of genus zero with ℓ marked points and of homology class α . (In this article we use only genus zero pseudoholomorphic curves.) We sometimes write it as $\mathcal{M}_{\ell}^{\text{cl}}(\alpha)$. It has a Kuranishi structure of dimension $2(n + \ell - 3 + c^1(M) \cap \alpha)$ (see [16]).

Let L be a relatively spin Lagrangian submanifold of M . For each $\beta \in H_2(M, L; \mathbb{Z})$ and $\ell \geq 0, k \geq 0$, we denote by $\mathcal{M}_{\ell, k}(\beta; L; J)$ the moduli space of bordered stable J -holomorphic curve of genus zero with ℓ interior marked points and k boundary marked points, one boundary component, and of homology class β . We sometimes write it as $\mathcal{M}_{\ell, k}(\beta)$. In the case $\ell = 0$ we write $\mathcal{M}_k(\beta)$ also. It has a Kuranishi structure with corner and boundary of dimension $2n + 2\ell + k - 3 + \mu(\beta)$, where $\mu : H_2(M, L; \mathbb{Z}) \rightarrow \mathbb{Z}$ is a Maslov index (see [13]).

REMARK 2.1

In [13] we used compatible almost complex structures. However, all the constructions there work for tame almost complex structures.

We review the construction of these Kuranishi structures, since we need to modify them so that they have some additional properties, in the next section.

Let $\mathbf{p} = (\Sigma, v) \in \mathcal{M}_{\ell,k}(\beta; L; J)$ (resp., $(\Sigma, v) \in \mathcal{M}_{\ell}^1(\alpha; J)$). Here Σ is a semistable marked bordered Riemann surface of genus zero with one boundary component, and $v : (\Sigma, \partial\Sigma) \rightarrow (M, L)$ is a J -holomorphic map (resp., Σ is a semistable marked Riemann surface of genus zero, and $v : \Sigma \rightarrow M$ is a J -holomorphic map). We assume that the enumeration of the boundary marked points respects the counterclockwise cyclic order of the boundary of Σ .

REMARK 2.2

In [11] we wrote $\mathcal{M}_{\ell,k}^{\text{main}}(\beta; L; J)$. Here “main” means the compatibility of the enumeration of the boundary marked points with the counterclockwise cyclic order of the boundary of Σ . We omit this symbol in this article since we consider only such Σ .

We consider the decomposition

$$(2.2) \quad \Sigma = \bigcup_{a \in A} \Sigma_a$$

to irreducible components. Here Σ_a is either a disc or a sphere component (resp., is a sphere component). Let $\Gamma_{\mathbf{p}}$ be a finite group consisting of biholomorphic maps $\varphi : \Sigma \rightarrow \Sigma$ such that $v \circ \varphi = v$.

We choose an open subset U_a for each Σ_a . We may choose them so that they are $\Gamma_{\mathbf{p}}$ -invariant in an obvious sense. We also assume that the closure of U_a does not intersect with the boundary marked points or singular points. Let Λ^{01} be a bundle on the disjoint union $\bigsqcup \Sigma_a$ of $(0, 1)$ -forms. We take a finite-dimensional vector space

$$(2.3) \quad E_a \subset C_0^\infty(U_a; v^*TM \otimes \Lambda^{01})$$

and take its direct sum $E_a^0 = \bigoplus_{a \in A} E_a$ as a tentative choice of the obstruction bundle. The explanation of this choice of E_a is in order. (The right-hand side of (2.3) denotes the space of compactly supported smooth sections of the pullback bundle v^*TM on U_a .)

We take a positive integer p and consider the space of sections $W^{1,p}(\Sigma_a, v^*TM)$ of the pullback bundle v^*TM on Σ_a of $W^{1,p}$ -class. We choose p sufficiently large so that the element of $W^{1,p}(\Sigma_a, v^*TM)$ is continuous on Σ_a . In the case when Σ_a has a boundary, we put boundary condition

$$(2.4) \quad \xi|_{\partial\Sigma_a} \subset W^{1-1/p,p}(\partial\Sigma_a, v^*TL)$$

and denote by $W^{1,p}(\Sigma_a, v^*TM; v^*TL)$ the space of ξ satisfying (2.4). By taking p large we may assume that the restriction to $\partial\Sigma_a$ is also continuous. We

write $W^{1,p}(\Sigma_a, v^*TM; v^*TL) = W^{1,p}(\Sigma_a, v^*TM)$ in the case when Σ_a is a sphere component.

We now consider the sum

$$(2.5) \quad \bigoplus_a W^{1,p}(\Sigma_a, v^*TM; v^*TL).$$

We require the following additional condition at the singular points. Let p be a singular point. There exist Σ_{a_1} and Σ_{a_2} ($a_1 \neq a_2$) with $p \in \Sigma_{a_1} \cap \Sigma_{a_2}$. (Using the fact that Σ has only a nodal point as a singularity and that the genus is zero, there exist such a_1, a_2 uniquely.) We now require that $(\xi_a) \in \bigoplus_a W^{1,p}(\Sigma_a, v^*TM; v^*TL)$ satisfy

$$(2.6) \quad \xi_{a_1}(p) = \xi_{a_2}(p).$$

We denote by

$$(2.7) \quad W^{1,p}(\Sigma, v^*TM; v^*TL)$$

the subspace of (2.5) of the elements (ξ_a) satisfying (2.6) at every singular point p .

Let Λ^{01} be a bundle on $\bigcup \Sigma_a$ of $(0, 1)$ forms. We put

$$(2.8) \quad L^p(\Sigma, v^*TM \otimes \Lambda^{01}) = \bigoplus_a L^p(\Sigma_a, v^*TM \otimes \Lambda^{01}).$$

The linearization of the pseudoholomorphic curve equation defines an operator

$$(2.9) \quad D_v \bar{\partial} : W^{1,p}(\Sigma, v^*TM; v^*TL) \rightarrow L^p(\Sigma, v^*TM).$$

(2.9) is a Fredholm operator.

DEFINITION 2.1

We say that (Σ, v) is *Fredholm regular* if (2.9) is surjective.

In the case when (Σ, v) is Fredholm regular, we can take its neighborhood in $\mathcal{M}_{\ell,k}(\beta; L; J)$ (resp., $\mathcal{M}_\ell(\alpha; J)$) so that it is an orbifold with boundary/corner (resp., orbifold). In general we need to take the obstruction bundle $E_{\mathbf{p}}$. We consider the finite-dimensional subspaces E_a as in (2.3). We require that $\bigoplus_a E_a$ be $\Gamma_{\mathbf{p}}$ invariant.

DEFINITION 2.2

We say that $((\Sigma, v), (E_a))$ is *Fredholm regular* if

$$(2.10) \quad \text{Im } D_v \bar{\partial} + \bigoplus_a E_a = L^p(\Sigma, v^*TM).$$

Under this assumption we construct a Kuranishi neighborhood of \mathbf{p} . We introduce the following notation (Definition 2.3) for convenience. We attach each of the tree of sphere components to the disc component to which the component is rooted. Let

$$(2.11) \quad \Sigma = \bigcup_{b \in B} \Sigma_b = \bigcup_{b \in B} \bigcup_{a \in A_b} \Sigma_a$$

be the resulting decomposition. (Here $\bigcup_b A_b = A$.)

DEFINITION 2.3

The choice of (E_a) is said to be *component-wise* if E_a depends only on $(\Sigma_a, v|_{\Sigma_a})$ and is independent of other components or the restriction of v to other components.

The choice of (E_a) is said to be *disc-component-wise* if E_a depends only on $(\Sigma_{a'}, v|_{\Sigma_{a'}})$ with $a, a' \in A_b$ for some b and is independent of the components $(\Sigma_{a'}, v|_{\Sigma_{a'}})$ with $a' \notin A_b$, where $a \in A_b$.

Before reviewing the construction of the Kuranishi neighborhood for Fredholm regular $((\Sigma, v), (E_a))$, we need to discuss the stability of the domain. We here follow [16, appendix]. Let Σ_a be a component of Σ . We remark that we include marked or singular points in the notation Σ_a . Namely, Σ_a is a marked disc or marked sphere where its marked points are either singular or a marked point of Σ which is on Σ_a . We say that Σ_a is *stable* if its automorphism group is of finite order. (In the case of disc components, it is equivalent to $k_a + 2\ell_a \geq 3$, where k_a is the number of boundary marked points and ℓ_a is the number of interior marked points. In the case of sphere components, it is equivalent to $\ell_a \geq 3$.)

If Σ_a is not stable, we add some interior marked points and make it stable. We denote it by Σ_a^+ . Note that $v|_{\Sigma_a}$ is nontrivial (in the case when Σ_a is unstable), since (Σ, v) is stable. Therefore $v|_{\Sigma_a}$ is an immersion at the generic point. We assume that v is an immersion at the additional marked points. We glue them and obtain Σ^+ . Let $\text{mkadd}(\Sigma^+)$ be the set of the marked points we add.

We require that $\text{mkadd}(\Sigma^+)$ be invariant of the $\Gamma_{\mathbf{p}}$ -action. Namely, we assume that none of the nontrivial elements of $\Gamma_{\mathbf{p}}$ fixes any of $p \in \text{mkadd}(\Sigma^+)$ and that $\Gamma_{\mathbf{p}}$ exchanges elements of $\text{mkadd}(\Sigma^+)$.

For each added marked point $p \in \text{mkadd}(\Sigma^+)$, we take a $(2n - 2)$ -dimensional submanifold $XM_p \subset M$ such that $XM_p \subset M$ intersect with $v : \Sigma_a \rightarrow M$ transversally at $v(p)$. (Here Σ_a is a component containing p .) We also require that if $\gamma \in \Gamma_{\mathbf{p}}$ then $XM_p = XM_{\gamma(p)}$.

Now we start with (Σ, v) and consider (Σ^+, v) . We take $E(\Sigma^+, v) = \bigoplus E_a(\Sigma^+, v)$ such that $((\Sigma^+, v), E(\Sigma^+, v))$ is Fredholm regular. We remark that all the components of Σ^+ are now stable. Let $U(\text{defresolv}; \Sigma^+)$ be a neighborhood of Σ^+ in the moduli space of marked bordered Riemann surface of genus zero and with one boundary component (resp., marked Riemann surface of genus zero). It is an orbifold with boundary or corners. We write

$$U(\text{defresolv}; \Sigma^+) = V(\text{defresolv}; \Sigma^+) / \text{Aut}(\Sigma^+),$$

where $V(\text{defresolv}; \Sigma^+)$ is a manifold with boundary or corner and $\text{Aut}(\Sigma^+)$ is a finite group of automorphisms of Σ^+ . By our choice of $\text{mkadd}(\Sigma^+)$, the group $\Gamma_{\mathbf{p}}$ of automorphisms of (Σ, v) is a subgroup of $\text{Aut}(\Sigma^+)$.

Let $\mathbf{v} \in V(\text{defresolv}; \Sigma^+)$, and let $\Sigma^+(\mathbf{v})$ be the corresponding marked bordered Riemann surface of genus zero. $\Sigma^+(\mathbf{v})$ minus a small neighborhood of

“neck region” is canonically isomorphic to Σ^+ minus a small neighborhood of singular points (see [16, Section 12]). Therefore the support U_a of E_a for each $E_a = E_a(\Sigma^+, v)$ has a canonical embedding to $\Sigma^+(\mathbf{v})$. (Note that we use here the fact that \mathbf{v} is an element of $V(\text{defresolv}; \Sigma^+)$, not just an element of $U(\text{defresolv}; \Sigma^+)$.) We consider $v' : (\Sigma^+(\mathbf{v}), \partial\Sigma^+(\mathbf{v})) \rightarrow (M, L)$, which is C^0 -close to v .

To define this C^0 -closedness precisely, we proceed as follows. We decompose

$$\Sigma^+ = \Sigma_{\text{reg}}^+ \cup \Sigma_{\text{sin}}^+,$$

where the second term is a small neighborhood (ϵ -neighborhood) of the singular point. We then decompose

$$\Sigma^+(\mathbf{v}) = \Sigma_{\text{reg}}^+(\mathbf{v}) \cup \Sigma_{\text{sin}}^+(\mathbf{v}),$$

where $\Sigma_{\text{reg}}^+(\mathbf{v})$ is biholomorphic to Σ_{reg}^+ . (For each ϵ we may take $V(\text{defresolv}; \Sigma^+)$ small enough that such decomposition exists.) We may assume that $U_a(\Sigma^+, v) \subset \Sigma_{\text{reg}}^+$ and hence that $U_a(\Sigma^+, v) \subset \Sigma_{\text{reg}}^+(\mathbf{v})$. Now v' is said to be ϵ -close to v if

- (1) $\text{dist}(v(x), v'(x)) \leq \epsilon$ if $x \in \Sigma_{\text{reg}}^+ = \Sigma_{\text{reg}}^+(\mathbf{v})$;
- (2) the diameter of the v' -image of each connected component of $\Sigma_{\text{sin}}^+(\mathbf{v})$ is smaller than ϵ .

Now for v' that is ϵ -close to v we define $E(\Sigma^+(\mathbf{v}), v')$ as follows. For U_a we consider the isomorphism

$$(2.12) \quad C^\infty(U_a; v^*TM) \cong C^\infty(U_a; v'^*TM)$$

by taking parallel transport along the minimal geodesic joining $v(x)$ to $v'(x)$. (If ϵ is smaller than the injectivity radius of M , such a minimal geodesic exists uniquely.) Using (2.12), we regard $E_a(\Sigma^+, v)$ as a subspace of $L^p(\Sigma^+(\mathbf{v}); v'^*TM)$. Now we consider

$$(2.13) \quad V^+(\mathbf{p}) = \{(\mathbf{v}, v') \mid v' : (\Sigma^+(\mathbf{v}), \partial\Sigma^+(\mathbf{v})) \rightarrow (M, L) \text{ is } \epsilon\text{-close to } v \text{ and satisfies (2.14)}\},$$

$$(2.14) \quad \bar{\partial}v' \in \bigoplus_a E_a(\Sigma^+, v).$$

The following is a consequence of standard glueing analysis (see [13]).

PROPOSITION 2.1

If $((\Sigma^+, v), E(\Sigma^+, v))$ is Fredholm regular and ϵ is sufficiently small, then $V^+(\mathbf{p})$ is a smooth manifold with boundary and corner.

We next define an evaluation map at an added marked point. Namely, we define

$$(2.15) \quad \text{ev}^{\text{int,add}} : V^+(\mathbf{p}) \rightarrow M^{\#(\text{mkadd}(\Sigma^+))}$$

by

$$(\mathbf{v}, v') \mapsto (v'(p_1), \dots, v'(p_{\text{mkadd}(\Sigma^+)})).$$

Using the fact that v is immersion and is transversal to XM_p for each $p \in \text{mkadd}(\Sigma^+)$, it follows that the fiber product

$$(2.16) \quad V_{\mathbf{p}} = V_{\mathbf{p}}^+ \times_{\text{ev}^{\text{int}, \text{add}}} \times_{M^{\#}(\text{mkadd}(\Sigma^+))} \prod_{p \in \text{mkadd}(\Sigma^+)} XM_p$$

is transversal. In particular, $V_{\mathbf{p}}$ is a smooth manifold with corners. We define $E_{\mathbf{p}}$ by

$$E_{\mathbf{p}}(\mathbf{v}, v') = \bigoplus_a E_a(\Sigma^+, v)$$

using the isomorphism (2.12). The section (Kuranishi map) $s_{\mathbf{p}}$ is defined by

$$s_{\mathbf{p}}(\mathbf{v}, v') = \bar{\partial}v' \in E_{\mathbf{p}}(\mathbf{v}, v').$$

They are $\Gamma_{\mathbf{p}}$ -equivariant by construction.

Let $s_{\mathbf{p}}(\mathbf{v}, v') = 0$. Then $v' : (\Sigma^+(\mathbf{v}), \partial\Sigma^+(\mathbf{v})) \rightarrow (M, L)$ is pseudoholomorphic. We forget the added marked point and obtain $\bar{v}' : (\Sigma(\mathbf{v}), \partial\Sigma(\mathbf{v})) \rightarrow (M, L)$. We put

$$\psi_{\mathbf{p}}(\mathbf{v}, v') = (\Sigma(\mathbf{v}), \bar{v}') \in \mathcal{M}_{\ell, k}(\beta).$$

We thus described a construction of a Kuranishi neighborhood for each given choice of $E_a(\Sigma^+, v)$, $\text{mkadd}(\Sigma^+)$. We next review how to glue them.

For each element $\mathbf{p} \in \mathcal{M}_{\ell, k}(\beta)$, we fix $E_{\mathbf{p}}$ and $\text{mkadd}(\mathbf{p})$. We write them as $E_{\mathbf{p}}^0$ and $\text{mkadd}^0(\mathbf{p})$ since later we change them. We also take sufficiently small ϵ depending on \mathbf{p} and construct a (tentative) Kuranishi neighborhood, which we denote by $(V_{\mathbf{p}}^0, E_{\mathbf{p}}^0, \Gamma_{\mathbf{p}}^0, s_{\mathbf{p}}^0, \psi_{\mathbf{p}}^0)$. We consider the covering

$$\mathcal{M}_{\ell, k}(\beta; L; J) = \bigcup_{\mathbf{p}} \psi_{\mathbf{p}}^0((s_{\mathbf{p}}^0)^{-1}(0)/\Gamma_{\mathbf{p}}^0).$$

We now take a finite set \mathbf{p}_a , $\mathbf{a} \in \mathfrak{A}$, and a closed subset $W(\mathbf{p}_a)$ of $\psi_{\mathbf{p}_a}^0((s_{\mathbf{p}_a}^0)^{-1}(0)/\Gamma_{\mathbf{p}_a}^0)$, such that

$$\bigcup_{\mathbf{a} \in \mathfrak{A}} \text{Int } W(\mathbf{p}_a) = \mathcal{M}_{\ell, k}(\beta; L; J).$$

Now, for each $\mathbf{p} = (\Sigma, v) \in \mathcal{M}_{\ell, k}(\beta; L; J)$, we choose $E_{\mathbf{p}}$ and $\text{mkadd}(\mathbf{p})$ as follows. We put

$$\mathfrak{A}(\mathbf{p}) = \{\mathbf{a} \in \mathfrak{A} \mid \mathbf{p} \in W(\mathbf{p}_a)\}.$$

Then

$$(2.17) \quad \text{mkadd}(\mathbf{p}) = \bigcup_{\mathbf{a} \in \mathfrak{A}(\mathbf{p})} \text{mkadd}^0(\mathbf{p}_a)$$

and

$$(2.18) \quad E_{\mathbf{p}} = \bigoplus_{\mathbf{a} \in \mathfrak{A}(\mathbf{p})} E_{\mathbf{p}_a}^0.$$

We remark that we may regard $\text{mkadd}^0(\mathbf{p}_a) \subset \Sigma$ and $E_{\mathbf{p}}^0 \subset L^p(\Sigma, v^*TM \otimes \Lambda^{01})$ by using $\psi_{\mathbf{p}_a}$. We can use them to define a Kuranishi neighborhood $(V_{\mathbf{p}}, E_{\mathbf{p}}, \Gamma_{\mathbf{p}}, s_{\mathbf{p}}, \psi_{\mathbf{p}})$ of each point in $\mathcal{M}_{\ell,k}(\beta; L; J)$.

Using the closedness of the set $W(\mathbf{p}_a)$, we can prove that if $\mathbf{q} \in \psi_{\mathbf{p}}((s_{\mathbf{p}})^{-1}(0)/\Gamma_{\mathbf{p}})$, then $E_{\mathbf{q}} \subseteq E_{\mathbf{p}}$ and $\text{mkadd}^0(\mathbf{q}) \subseteq \text{mkadd}^0(\mathbf{p})$. We can use it to construct coordinate change and then obtain Kuranishi structure.

We remark that here we construct Kuranishi structure on $\mathcal{M}_{\ell,k}(\beta; L; J)$ for each of ℓ, k, β individually. We actually need to construct them so that they are related to each other at their boundaries. In the next section we do it in a way slightly different from [11]. Namely, in [11] the constructions are not compatible with the forgetful map. We modify it so that it is compatible with the forgetful map of boundary marked points. For this purpose we include $k = 0$ (the case of no boundary marked points). This is the technical heart of the whole construction of this article.

We use the following notions in the next section.

DEFINITION 2.4

Let \mathcal{M} be a Kuranishi space, and let N be a smooth manifold. A *strongly continuous smooth* map $f : \mathcal{M} \rightarrow N$ is a family $f = \{f_{\mathbf{p}}\}$ of $\Gamma_{\mathbf{p}}$ -equivariant smooth maps $f_{\mathbf{p}} : V_{\mathbf{p}} \rightarrow N$ which induces $f_{\mathbf{p}} : V_{\mathbf{p}}/\Gamma_{\mathbf{p}} \rightarrow N$ and such that $f_{\mathbf{p}} \circ \bar{\varphi}_{\mathbf{p}\mathbf{q}} = f_{\mathbf{q}}$ on $V_{\mathbf{q}\mathbf{p}}/\Gamma_{\mathbf{q}}$.

We say that $\{f_{\mathbf{p}}\}$ is *weakly submersive* if each of $f_{\mathbf{p}}$ is a submersion.

In the case when Kuranishi space has boundary or corner, for the map to be submersive, we require that the restriction to each stratum be submersive.

We consider the moduli spaces $\mathcal{M}_{\ell,k}(\beta; L; J)$ and $\mathcal{M}_{\ell}^{\text{cl}}(\alpha; J)$. The evaluation at marked points induces maps

$$\begin{aligned} \text{ev} &= (\text{ev}^{\text{int}}, \text{ev}) = ((\text{ev}_1^{\text{int}}, \dots, \text{ev}_{\ell}^{\text{int}}), (\text{ev}_0, \dots, \text{ev}_{k-1})) : \mathcal{M}_{\ell,k}(\beta) \rightarrow M^{\ell} \times L^k, \\ \text{ev} &= \text{ev}^{\text{int}} = (\text{ev}_1^{\text{int}}, \dots, \text{ev}_{\ell}^{\text{int}}) : \mathcal{M}_{\ell}^{\text{cl}}(\tilde{\beta}) \rightarrow M^{\ell}. \end{aligned}$$

These maps are strongly continuous. In [11] the Kuranishi structure is chosen so that they are weakly submersive. In the next section we do *not* choose so. We explain the reason in Remark 3.2.

We next review fiber product of Kuranishi structures. Let $\mathcal{M}_1, \mathcal{M}_2$ be Kuranishi spaces, and let $f = \{f_{\mathbf{p}}\}, g = \{g_{\mathbf{p}'}\}$ be strongly continuous maps from them to a manifold N . We consider

$$(2.19) \quad \mathcal{M}_{1f} \times_g \mathcal{M}_2 = \{(x, y) \in \mathcal{M}_1 \times \mathcal{M}_2 \mid f(x) = g(y)\}.$$

LEMMA 2.1

If either f or g are weakly submersive then the fiber product (2.19) has a Kuranishi structure.

See [11, Section A1.3] for the proof. Let $h = \{h_{\mathbf{p}'}\} : \mathcal{M}_2 \rightarrow N'$ be another strongly continuous map. It induces a strongly continuous map

$$(2.20) \quad \mathcal{M}_{1f} \times_g \mathcal{M}_2 \rightarrow N'$$

in an obvious way.

LEMMA 2.2

If f and h are weakly submersive, then the map (2.20) is also weakly submersive.

The proof is immediate from the corresponding statement for the submersion of smooth maps between manifolds and from the construction of fiber product in [11].

3. Forgetful-map-compatible Kuranishi structure

Let (M, ω) be a symplectic manifold, and let L be its Lagrangian submanifold. For $\beta \in H_2(M, L; \mathbb{Z})$, we defined the moduli space $\mathcal{M}_{\ell, k}(\beta)$ as in Section 2. In Section 2, we wrote $\mathcal{M}_{\ell, k}(\beta; L; J)$ or $\mathcal{M}_{\ell}^{\text{cl}}(\alpha; J)$. Hereafter we omit J from the notation when no confusion can occur.

We include the case $k = 0$. In that case we compactify it by adding $\mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta}) \times_M L$ if $\partial\beta = 0$ as is explained in [11, Section 7.4.1]. Namely, we have an embedding

$$(3.1) \quad \text{clop} : \mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta})_{\text{ev}^{\text{int}}} \times_M L \rightarrow \mathcal{M}_{\ell, 0}(\beta).$$

Note that $\mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta})$ is a moduli space of a pseudoholomorphic map from a genus zero stable map (without boundary). Here $\tilde{\beta} \in H_2(X; \mathbb{Z})$ is a class that goes to the class β by the natural homomorphism $H_2(X; \mathbb{Z}) \rightarrow H_2(X, L; \mathbb{Z})$. We fix a compatible system of Kuranishi structures on them so that evaluation maps are submersion. And we do not change them.

We have a forgetful map

$$(3.2) \quad \text{forget} : \mathcal{M}_{\ell, 1}(\beta) \rightarrow \mathcal{M}_{\ell, 0}(\beta).$$

We also consider the embedding

$$(3.3a) \quad \text{glue} : \mathcal{M}_{\ell_1, 1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2, 1}(\beta_2) \rightarrow \partial\mathcal{M}_{\ell_1+\ell_2, 0}(\beta_1 + \beta_2)$$

in the case $(\beta_1, \ell_1) \neq (\beta_2, \ell_2)$ and

$$(3.3b) \quad \text{glue} : (\mathcal{M}_{\ell_1, 1}(\beta')_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2, 1}(\beta')) / \mathbb{Z}_2 \rightarrow \partial\mathcal{M}_{\ell_1+\ell_2, 0}(2\beta'),$$

where \mathbb{Z}_2 acts by exchanging the factors.

DEFINITION 3.1

Kuranishi structures of $\mathcal{M}_{\ell, 1}(\beta)$ and of $\mathcal{M}_{\ell, 0}(\beta)$ are said to be *forgetful-map compatible* to each other if the following holds.

Let $\tilde{\mathbf{p}} = [(\Sigma, z_0, \tilde{z}^{\text{int}}, u) \in \mathcal{M}_{\ell, 1}(\beta)$ and $\mathbf{p} = \text{forget}([(\Sigma, z_0, \tilde{z}^{\text{int}}, u) \in [(\Sigma, \tilde{z}^{\text{int}}, u) \in \mathcal{M}_{\ell, 0}(\beta)$. We first consider the case when \mathbf{p} is not in the image of (3.1).

There exist Kuranishi neighborhoods $(V_{\bar{\mathbf{p}}}, E_{\bar{\mathbf{p}}}, \Gamma_{\bar{\mathbf{p}}}, \psi_{\bar{\mathbf{p}}}, s_{\bar{\mathbf{p}}})$ and $(V_{\mathbf{p}}, E_{\mathbf{p}}, \Gamma_{\mathbf{p}}, \psi_{\mathbf{p}}, s_{\mathbf{p}})$ of them, respectively, such that

- (1) $V_{\bar{\mathbf{p}}} = V_{\mathbf{p}} \times (0, 1)$;
- (2) $E_{\bar{\mathbf{p}}} = E_{\mathbf{p}} \times (0, 1)$;
- (3) $\Gamma_{\bar{\mathbf{p}}} = \Gamma_{\mathbf{p}}$; the action of $\Gamma_{\bar{\mathbf{p}}}$ preserves identifications given in (1) and (2),

where the action to the factor $(0, 1)$ is trivial;

- (4) $s_{\bar{\mathbf{p}}}(x, t) = (s_{\mathbf{p}}(x), t)$ by the identification given in (1) and (2);
- (5) $\text{forget} \circ \psi_{\bar{\mathbf{p}}}$ coincides with the composition of $\psi_{\mathbf{p}}$ and the projection to the first factor.

We next consider the case when \mathbf{p} is in the image of (3.1). This implies that u is a constant map on the disc component Σ_0 containing z_0 and that Σ_0 has exactly one singular point and has no interior or boundary marked points other than z_0 . Let $\bar{\Sigma}$ be a closed semistable curve of genus zero without boundary, which is obtained by removing Σ_0 from Σ . Let z_0^{int} be the point $\bar{\Sigma} \cap \Sigma_0 \subset \bar{\Sigma}$. We put $\bar{z}_+^{\text{int}} = (z_0^{\text{int}}, \bar{z}^{\text{int}})$. Then $((\bar{\Sigma}, \bar{z}_+^{\text{int}}, u), u(z_0^{\text{int}}))$ is an element of $\mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta}) \times L$ such that

$$\text{clop}((\bar{\Sigma}, \bar{z}_+^{\text{int}}, u), u(z_0^{\text{int}})) = \mathbf{p}.$$

Let $(V_{\bar{\mathbf{p}}}, E_{\bar{\mathbf{p}}}, \Gamma_{\bar{\mathbf{p}}}, \psi_{\bar{\mathbf{p}}}, s_{\bar{\mathbf{p}}})$ be a Kuranishi neighborhood of $\bar{\mathbf{p}}$.

Let $\bar{\mathbf{p}} = [\bar{\Sigma}, \bar{z}_+^{\text{int}}, u]$, and let $(V_{\bar{\mathbf{p}}}, E_{\bar{\mathbf{p}}}, \Gamma_{\bar{\mathbf{p}}}, \psi_{\bar{\mathbf{p}}}, s_{\bar{\mathbf{p}}})$ be its Kuranishi neighborhood in $\mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta})$. Since $\text{ev}_0^{\text{int}} : \mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta}) \rightarrow M$ is strongly continuous and weakly submersive, it induces a submersion $\text{ev}_0^{\text{int}} : V_{\bar{\mathbf{p}}} \rightarrow M$. Let $V_{\bar{\mathbf{p}}} \cap L = (\text{ev}_0^{\text{int}})^{-1}(L) \subset V_{\bar{\mathbf{p}}}$. The fiber product Kuranishi structure on $\mathcal{M}_{1+\ell}^{\text{cl}}(\tilde{\beta})_{\text{ev}_0^{\text{int}}} \times_M L$ is $(V_{\bar{\mathbf{p}}} \cap L, E_{\bar{\mathbf{p}}} \cap L, \Gamma_{\bar{\mathbf{p}}}, \psi_{\bar{\mathbf{p}}}, s_{\bar{\mathbf{p}}})$. (Here we write $E_{\bar{\mathbf{p}}} \cap L$ for the restriction of $E_{\bar{\mathbf{p}}}$ to $V_{\bar{\mathbf{p}}} \cap L$ by abuse of notation. We also write $s_{\bar{\mathbf{p}}}$ or $\psi_{\bar{\mathbf{p}}}$ for its appropriate restriction.) We have the following:

- (1) $V_{\bar{\mathbf{p}}} = (V_{\bar{\mathbf{p}}} \cap L) \times [0, 1)$;
- (2) $E_{\bar{\mathbf{p}}} = E_{\bar{\mathbf{p}}} \cap L \times [0, 1)$;
- (3) $\Gamma_{\bar{\mathbf{p}}} = \Gamma_{\bar{\mathbf{p}}}$; the action of $\Gamma_{\bar{\mathbf{p}}}$ preserves identifications given in (1) and (2),

where the action to the factor $(0, 1)$ is trivial;

- (4) $s_{\bar{\mathbf{p}}}(x, t) = (s_{\bar{\mathbf{p}}}(x), t)$ by the identification given in (1) and (2);
- (5) $\text{forget} \circ \psi_{\bar{\mathbf{p}}}$ coincides with the composition of $\psi_{\bar{\mathbf{p}}}$ and the projection to the first factor.

The main result of this section is the following.

THEOREM 3.1

There exists a system of Kuranishi structures on $\mathcal{M}_{\ell,1}(\beta)$ and $\mathcal{M}_{\ell,0}(\beta)$, such that

- (1) *they are forgetful-map compatible in the sense of Definition 3.1;*
- (2) *$\text{ev}_0 : \mathcal{M}_{\ell,1}(\beta) \rightarrow L$ is strongly submersive;*

(3) they are compatible with (3.3). Namely, the fiber product Kuranishi structure on $\mathcal{M}_{\ell_1,1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2,1}(\beta_2)$ (which is well defined by (2)) coincides with the pullback of the $\partial\mathcal{M}_{\ell_1+\ell_2,0}(\beta_1+\beta_2)$ by **glue** (in the case $\beta_1 = \beta_2$, we divide the fiber product by \mathbb{Z}_2 -action exchanging the factors);

(4) they are componentwise in the sense of Definition 2.3.

Proof

The most essential part of the proof is the following lemma. Let $\tilde{\mathbf{p}} = [(\Sigma, z_0, \bar{z}^{\text{int}}), u] \in \mathcal{M}_{\ell,1}(\beta)$ and $\mathbf{p} = \text{forget}([\Sigma, z_0, \bar{z}^{\text{int}}), u] = [(\Sigma, \bar{z}^{\text{int}}), u] \in \mathcal{M}_{\ell,0}(\beta)$. We consider the case when \mathbf{p} is not in the image of (3.1).

We consider the linearization of the pseudoholomorphic curve equation (2.9). We consider the case $\Sigma = D^2$.

LEMMA 3.1

For any open subset U of $\text{Int } D^2$, there exists a finite-dimensional linear subspace $E(u)$ of sections of $u^*TM \otimes \Lambda^{0,1}$ such that the following holds.

- (1) Each of elements of $E(u)$ is smooth and supported in U .
- (2) We put

$$(3.4) \quad K(u) = (D_u \bar{\partial})^{-1}(E(u)).$$

Then for any $z_0 \in \partial D^2$, the map $\text{Ev}_{z_0} : K(u) \rightarrow T_{u(z_0)}L$ defined by

$$(3.5) \quad \text{Ev}_{z_0}(v) = v(z_0)$$

is surjective.

We remark that elements of $K(u)$ are smooth by elliptic regularity. Therefore (3.5) is well defined.

Proof

For each fixed z_0 we can find $E(u; z_0)$ satisfying 1 and such that for $K(u; z_0) = (D_u \bar{\partial})^{-1}(E(u; z_0))$, the map $\text{Ev}_{z_0; z_0} : K(u; z_0) \rightarrow T_{u(z_0)}L$ defined by (3.5) is surjective. (This is a consequence of unique continuation; see [11].)

Then there exists a neighborhood $W(z_0)$ of z_0 in ∂D^2 such that if $z \in W(z_0)$, then the map $\text{Ev}_{z_0; z} : K(u; z_0) \rightarrow T_{u(z)}L$ defined by $\text{Ev}_{z_0; z}(v) = v(z)$ is a submersion. We cover ∂D^2 by finitely many $W(z_0)$, say, $W(z_i)$, $i = 1, \dots, N$. Then $E(u) = \bigoplus_{i=1}^N E(u; z_i)$ has the required property. \square

We next consider the case of the boundary point corresponding to $\mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta})_{\text{ev}_1^{\text{int}}} \times_M L$. For this purpose, we need to take a Kuranishi structure on the moduli space of a pseudoholomorphic sphere.

LEMMA 3.2

There exists a system of Kuranishi structures on $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ for various $\ell \geq 0$ and $\alpha \in \pi_2(M)$ with the following properties.

(1) *The action of a permutation group of order $\ell!$ on $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ exchanging marked points is extended to an action of the Kuranishi structures.*

(2) *The evaluation map $\text{ev}^{\text{int}} : \mathcal{M}_\ell^{\text{cl}}(\alpha) \rightarrow M^\ell$ is strongly continuous and weakly submersive.*

(3) *Let $\ell_1 + \ell_2 = \ell + 2$, $\alpha_1 + \alpha_2 = \alpha$. Then by the embedding*

$$\mathcal{M}_{\ell_1}^{\text{cl}}(\alpha_1)_{\text{ev}_1^{\text{int}}} \times_{\text{ev}_1^{\text{int}}} \mathcal{M}_{\ell_2}^{\text{cl}}(\alpha_2) \subset \mathcal{M}_\ell^{\text{cl}}(\alpha),$$

the Kuranishi structure in the right-hand side restricts to the fiber product Kuranishi structure in the left-hand side. In particular, our Kuranishi structures are componentwise.

This is proved in [16].

We now consider the boundary point $(v, x) \in \mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta})_{\text{ev}_1^{\text{int}}} \times_M L$, where $v \in \mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta})$ and $x = \text{ev}_1^{\text{int}}(v) \in L$. Let (V, Γ, E, s, ψ) be a Kuranishi neighborhood of v we have taken in Lemma 3.2. We take

$$V' = (\text{ev}_1^{\text{int}})^{-1}(L).$$

This is smooth, and $\text{ev} : V' \rightarrow L$ (which is the restriction of ev^{int} to V') is a submersion. Therefore $(V', \Gamma, E|_{V'}, s|_{V'}, \psi|_{(s|_{V'})^{-1}(0)/\Gamma})$ can be regarded as the Kuranishi neighborhood of (v, x) in $\mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta})_{\text{ev}_1^{\text{int}}} \times_M L$. We put

$$\text{Conf}^o(k; (\partial D^2, 0)) = \{(z_1, \dots, z_k) \in (\partial D^2)^k \mid z_i \text{ respects cyclic order}\} / S^1$$

and let $\text{Conf}(k; (\partial D^2, 0))$ be its compactification. Then

$$\text{Conf}(k; (\partial D^2, 0)) \times V'$$

can be regarded as a stratum of a Kuranishi neighborhood of the pullback of (v, x) in $\mathcal{M}_{\ell,k}(\beta)$. We can extend it to a Kuranishi neighborhood of (v, x) . Then properties (1) and (2) of Lemma 3.1 are satisfied.

We thus described a way to obtain a space $E(u)$ satisfying properties (1) and (2) of Lemma 3.1 at each point of the compactification of $\mathcal{M}_{\ell,0}(\beta)$.

Now we are in position to complete the proof of Theorem 3.1. The proof is by induction on $\beta \cap \omega$. We assume that we have chosen already the Kuranishi structure satisfying the conclusion of Theorem 3.1 for β' with $\beta' \cap \omega < \beta \cap \omega$.

We consider $\mathcal{M}_{\ell,0}(\beta)$. A component of its boundary is

$$\mathcal{M}_{\ell_1,1}(\beta_1) \times_L \mathcal{M}_{\ell_2,1}(\beta_2) \quad (\text{resp., } (\mathcal{M}_1(\ell', \beta') \times_L \mathcal{M}_1(\ell', \beta')) / \mathbb{Z}_2)$$

with $\beta_1 + \beta_2 = \beta$, $\ell_1 + \ell_2 = \ell$ (resp., $2\beta' = \beta$, $2\ell' = \ell$). We already fixed a Kuranishi structure on each of the factors. We take the fiber product Kuranishi structure on it. We then lift it to a Kuranishi structure of a boundary component of $\mathcal{M}_{\ell,1}(\beta)$. We claim that ev_0 is weakly submersive for this Kuranishi structure. In fact, the boundary component we are studying is one of the following three cases:

$$(3.6a) \quad \mathcal{M}_{\ell_1,2}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_1(\ell_2, \beta_2),$$

where $\text{ev}_0 : \mathcal{M}_{\ell,1}(\beta) \rightarrow L$ is the map ev_1 of the first factor;

$$(3.6b) \quad \mathcal{M}_{\ell_1,1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_2(\ell_2, \beta_2),$$

where $\text{ev}_0 : \mathcal{M}_{\ell,1}(\beta) \rightarrow L$ is the map ev_1 of the second factor;

$$(3.6c) \quad \mathcal{M}_{\ell_1,1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2,1}(\beta_2),$$

where ev_0 is the map $((\Sigma_1, v_1), (\Sigma_2, v_2)) \mapsto \text{ev}_0(\Sigma_1, v_1) = \text{ev}_0(\Sigma_2, v_2)$.

We remark that (3.6c) is identified with

$$(\mathcal{M}_{\ell_1,1}(\beta_1) \times \mathcal{M}_{\ell_2,1}(\beta_2))_{(\text{ev}_0, \text{ev}_0)} \times_{(\text{ev}_1, \text{ev}_2)} \mathcal{M}_3(0).$$

Here zero in $\mathcal{M}_3(0)$ is $0 \in H_2(M, L; \mathbb{Z})$.

We first consider the case (3.6a). By the induction hypothesis,

$$(\text{ev}_1, \text{ev}_0) : \mathcal{M}_{\ell_1,2}(\beta_1) \times \mathcal{M}_{\ell_2,1}(\beta_2) \rightarrow L^2$$

is weakly submersive. It follows from Lemma 2.2 that ev_0 is submersive on (3.6a). The cases (3.6b) and (3.6c) are similar.

The case $\beta_1 = \beta_2 = \beta'$, $\ell_1 = \ell_2 = \ell'$ can be discussed in the same way.

In the case $[\partial\beta] = 0$, there is another boundary component $\mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta}) \times_M L$ of $\mathcal{M}_{\ell,0}(\beta)$. We have already explained the way to impose an obstruction bundle and then Kuranishi structure on $\mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta})$ so that the map ev_0 is weakly submersive. Therefore we can define a Kuranishi structure on a neighborhood of this boundary component of $\mathcal{M}_{\ell,0}(\beta)$ by extending fiber product Kuranishi structure. We thus constructed the required Kuranishi structure on a neighborhood of the boundary of $\mathcal{M}_{\ell,0}(\beta)$. Because of the inductive way to constructing our Kuranishi structures, they are compatible at their intersections.

Now we can use Lemma 3.1 in the same way as [11, Section 7.2] to extend it the whole $\mathcal{M}_{\ell,0}(\beta)$. We define Kuranishi structure on $\mathcal{M}_{\ell,1}(\beta)$ by taking the pullback of the Kuranishi structure of $\mathcal{M}_{\ell,0}(\beta)$ via the forgetful map. The proof of Theorem 3.1 is now complete. \square

We consider the forgetful map

$$\text{forget}_{k+1,1} : \mathcal{M}_{\ell,k+1}(\beta) \rightarrow \mathcal{M}_{\ell,1}(\beta),$$

forgetting the 2nd, \dots , $k+1$ th boundary marked points. Note that we enumerate marked points as z_0, \dots, z_k . We forget z_1, \dots, z_k .

COROLLARY 3.1

There exists a system of Kuranishi structures on $\mathcal{M}_{\ell,k+1}(\beta)$, $k \geq 0$, $\ell \geq 0$, with the following properties.

- (1) *It is compatible with $\text{forget}_{k+1,1}$.*
- (2) *It is invariant under the cyclic permutation of the boundary marked points.*
- (3) *It is invariant of the permutation of interior marked points.*
- (4) *$\text{ev}_0 : \mathcal{M}_{\ell,k+1}(\beta) \rightarrow L$ is strongly submersive.*

(5) We consider the decomposition of the boundary:

$$(3.7) \quad \begin{aligned} \partial \mathcal{M}_{\ell, k+1}(\beta) = & \bigcup_{1 \leq i \leq j+1 \leq k+1} \bigcup_{\beta_1 + \beta_2 = \beta} \bigcup_{L_1 \cup L_2 = \{1, \dots, \ell\}} \\ & \mathcal{M}_{\#L_1, j-i+1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_i} \mathcal{M}_{\#L_2, k-j+i}(\beta_2) \end{aligned}$$

(see [11, Section 7.1.1]). Then the restriction of the Kuranishi structure of $\mathcal{M}_{\ell, k+1}(\beta)$ in the left-hand side coincides with the fiber product Kuranishi structure in the right-hand side.

We first explain the statement.

(1) is similar to Definition 3.1. The only difference is that we replace $(0, 1)$ appearing there with $(0, 1)^{k-m} \times [0, 1]^m$ for some appropriate m .

The cyclic permutation of boundary marked points is defined as follows. Let (Σ, v) be a point in $\mathcal{M}_{\ell, k+1}(\beta)$. Let z_0, z_1, \dots, z_k be boundary marked points of Σ . We change them to z_1, \dots, z_k, z_0 to obtain (Σ', v) . Property (2) claims that this action extends to the Kuranishi structure (see [11, Section A1.3] for the definition of finite group action to Kuranishi structure).

The meaning of Property (3) is similar. Here we consider not only cyclic permutation but also an arbitrary permutation of the interior marked points.

We remark that Property (4) and Lemma 2.19 imply that the right-hand side of (3.7) has a Kuranishi structure. Then Property (5) claims that the boundary of the moduli space of $\mathcal{M}_{\ell, k+1}(\beta)$ as Kuranishi space decomposes as in the right-hand side of (3.7). We remark that the i th, \dots , j th boundary marked points of $\mathcal{M}_{\ell, k+1}(\beta)$ correspond to the 1st, \dots , $j - i + 1$ th marked points of the first factor and the other boundary marked points (except the 0th) of $\mathcal{M}_{\ell, k+1}(\beta)$ correspond to the boundary marked points (except i th) of the right-hand side. Also, the interior marked points correspond to each other in an obvious way. We then have a compatibility statement of evaluation maps in an obvious way. It is part of statement (5).

Note that $k + 1 \geq 1$. Therefore the extra boundary component $\mathcal{M}_{\ell+1}^{\text{cl}}(\tilde{\beta}) \times_M L$ of $\mathcal{M}_{\ell, 0}(\beta)$ does not appear in (3.7).

Proof

We defined the Kuranishi structure on $\mathcal{M}_{\ell, 1}(\beta)$ in Theorem 3.1. We define the Kuranishi structure on $\mathcal{M}_{\ell, k+1}(\beta)$ so that (1) holds. (This determines the Kuranishi structure uniquely.) Then by Theorem 3.1(1), our Kuranishi structure of $\mathcal{M}_{\ell, k+1}(\beta)$ is pulled back from one on $\mathcal{M}_{\ell, 0}(\beta)$. Property (2) follows immediately. Property (3) is a consequence of Theorem 3.1(4). Property (4) is a consequence of Theorem 3.1(2). Property (5) is a consequence of Theorem 3.1(3). □

REMARK 3.1

The Kuranishi structure we constructed is *not* compatible with the forgetful map of the *interior* marked points. The reason is rather technical. Namely, we require

the support of the obstruction bundle E_a to be disjoint from marked or singular points. This is automatic for the boundary marked points since we also assume that it is disjoint from the boundary. However, if we try to imitate the proof of Theorem 3.1 to obtain a Kuranishi structure that is compatible with the forgetful map, this causes a problem. Namely, if we construct the Kuranishi structure of $\mathcal{M}_{0,0}(\beta)$ and $\mathcal{M}_0^{\text{cl}}(\alpha)$ so that the evaluation map is submersive for the pullback Kuranishi structure on $\mathcal{M}_{1,0}(\beta)$ and $\mathcal{M}_1^{\text{cl}}(\alpha)$, then, a priori, we cannot assume the support of the obstruction bundle E_a to be disjoint from marked points. (In fact, we need to fix E_a for elements of $\mathcal{M}_0^{\text{cl}}(\alpha)$. So the interior marked point of the pullback Kuranishi structure $\mathcal{M}_1^{\text{cl}}(\alpha)$ can be arbitrary. Therefore we cannot exclude that it is in the support of E_a .)

The reason why the support of the obstruction bundle E_a is assumed to be disjoint from the singular point is as follows.

- (1) The glueing analysis is easier. If E_a hits the singular point, we need to study the case when perturbation is put on the neck region also.
- (2) We need to identify the obstruction bundle of the pieces Σ_a as a section of appropriate bundles after resolving the singularity of $\bigcup \Sigma_a$ (see (2.12)). This is easier in the case when the support of E_a is away from singular points.

We remark that we cannot distinguish marked points from singular points when we want to make the perturbation componentwise. This is the reason why it is assumed that the support of E_a is disjoint from marked points. However, the description above also shows that by working harder we might remove this restriction and then find a Kuranishi structure on $\mathcal{M}_{\ell,k}(\beta)$ which is compatible with the forgetful map of the interior marked points also. Since we do not need it in this article, we do not try to prove it here.

REMARK 3.2

For the Kuranishi structure we constructed in Corollary 3.1, the evaluation map ev_0 is weakly submersive. As a consequence of cyclic symmetry it implies that ev_i is submersive for all but fixed i . On the other hand, in [11] we used a Kuranishi structure such that $(\text{ev}_0, \dots, \text{ev}_{k-1}) : \mathcal{M}_{\ell,k}(\beta) \rightarrow L^k$ is weakly submersive. We remark that there does not exist a system of Kuranishi structures such that $(\text{ev}_0, \dots, \text{ev}_{k-1})$ is weakly submersive and is compatible with the forgetful map in the sense of Corollary 3.1.1 at the same time. In fact, if d is a dimension of the Kuranishi neighborhood $V_{\mathbf{p}}$ of a point in $\mathcal{M}_{\ell,1}(\beta)$, then the dimension of the Kuranishi neighborhood $V_{\bar{\mathbf{p}}}$ of compatible Kuranishi structure in $\mathcal{M}_{\ell,k}(\beta)$ is $k + d - 1$. (We remark that the dimension here is one of the manifold $V_{\mathbf{p}}$. It is different from the dimension as the Kuranishi space.) If k is large, then dimension of L^k , which is nk , is certainly bigger than $k + d - 1$. Therefore $(\text{ev}_0, \dots, \text{ev}_{k-1})$ cannot be weakly submersive.

A key idea of the proof of Corollary 3.1 is the observation that the submersivity of ev_0 is enough to carry out the inductive construction. (This observation was due to [12].) We remark also that for this observation to hold the assump-

tion, genus = 0 is essential. Namely, the same method does not work for its generalization to higher genus.

4. Continuous family of multisections: Review

In Section 5 we construct a perturbation of the Kuranishi map of the Kuranishi structure constructed in Section 3 so that it is compatible with the forgetful map of the boundary marked points and is cyclically symmetric (as its consequence). The author does not know how to do it using multi-(but finitely many) valued sections. So we use a continuous family of multisections. The notion of a continuous family of multisections had already been used in [14, Section 33], [7], [15], and so on. We first review them in this section.

First, we recall the notion of a good coordinate system. Let $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ be Kuranishi charts parameterized by $\alpha \in \mathfrak{A}$. We assume that the index set \mathfrak{A} has a partial order $<$, where either $\alpha_1 \leq \alpha_2$ or $\alpha_2 \leq \alpha_1$ holds for $\alpha_1, \alpha_2 \in \mathfrak{A}$ if

$$\psi_{\alpha_1}(s_{\alpha_1}^{-1}(0)/\Gamma_{\alpha_1}) \cap \psi_{\alpha_2}(s_{\alpha_2}^{-1}(0)/\Gamma_{\alpha_2}) \neq \emptyset.$$

Moreover, we assume that if $\alpha_1, \alpha_2 \in \mathfrak{A}$ and $\alpha_1 \leq \alpha_2$, then there exists a coordinate transformation from $(V_{\alpha_1}, E_{\alpha_1}, \Gamma_{\alpha_1}, \psi_{\alpha_1}, s_{\alpha_1})$ to $(V_{\alpha_2}, E_{\alpha_2}, \Gamma_{\alpha_2}, \psi_{\alpha_2}, s_{\alpha_2})$, in the sense described in Section 2. We assume compatibility between coordinate transformations in the sense of [16] and [11]. The existence of a good coordinate system is proved in [16, Lemma 6.3].

We next review multisections (see [16, Section 3]). Let $(V_\alpha, E_\alpha, \psi_\alpha, s_\alpha, \Gamma_\alpha)$ be a Kuranishi chart of \mathcal{M} . For $x \in V_\alpha$ we consider the fiber $E_{\alpha,x}$ of the bundle E_α at x . We take its l copies and consider the direct product $E_{\alpha,x}^l$. We divide it by the action of a symmetric group of order $l!$ and let $\mathcal{S}^l(E_{\alpha,x})$ be the quotient space. There exists a map $tm_m : \mathcal{S}^l(E_{\alpha,x}) \rightarrow \mathcal{S}^{lm}(E_{\alpha,x})$, which sends $[a_1, \dots, a_l]$ to $[\underbrace{a_1, \dots, a_1}_{m \text{ copies}}, \dots, \underbrace{a_l, \dots, a_l}_{m \text{ copies}}]$. A multisection s of the orbibundle $E_\alpha \rightarrow V_\alpha$ consists of an open covering $\bigcup_i U_i = V_\alpha$ and s_i which sends $x \in U_i$ to $s_i(x) \in \mathcal{S}^{l_i}(E_{\alpha,x})$. They are required to have the following properties.

- (1) U_i is Γ_α -invariant; s_i is Γ_α -equivariant. (We remark that there exists an obvious map $\gamma : \mathcal{S}^{l_i}(E_{\alpha,x}) \rightarrow \mathcal{S}^{l_i}(E_{\alpha,\gamma x})$ for each $\gamma \in \Gamma_\alpha$.)
- (2) If $x \in U_i \cap U_j$, then we have $tm_{l_j}(s_i(x)) = tm_{l_i}(s_j(x)) \in \mathcal{S}^{l_i l_j}(E_{\alpha,\gamma x})$.
- (3) s_i is *liftable and smooth* in the following sense. For each x there exists a smooth section \tilde{s}_i of $\underbrace{E_\alpha \oplus \dots \oplus E_\alpha}_{l_i \text{ times}}$ in a neighborhood of x such that

$$(4.1) \quad \tilde{s}_i(y) = (s_{i,1}(y), \dots, s_{i,l_i}(y)), \quad s_i(y) = [s_{i,1}(y), \dots, s_{i,l_i}(y)].$$

We identify two multisections $(\{U_i\}, \{s_i\}, \{l_i\}), (\{U'_i\}, \{s'_i\}, \{l'_i\})$ if

$$tm_{l_j}(s_i(x)) = tm_{l'_j}(s'_j(x)) \in \mathcal{S}^{l_i l'_j}(E_{\alpha,\gamma x})$$

on $U_i \cap U'_j$. We say that $s_{i,j}$ is a *branch* of s_i in the situation of (4.1).

We next discuss a continuous family of multisections and its transversality. Let W_α be a finite-dimensional manifold, and consider the pullback bundle

$$\pi_\alpha^* E_\alpha \rightarrow V_\alpha \times W_\alpha$$

under $\pi_\alpha : V_\alpha \times W_\alpha \rightarrow V_\alpha$. The action of Γ_α on W_α is, by definition, trivial.

DEFINITION 4.1

(1) A W_α -parameterized family \mathfrak{s}_α of multisections is by definition a multisection of $\pi_\alpha^* E_\alpha$.

(2) We fix a metric of our bundle E_α . We say that \mathfrak{s}_α is ϵ -close to s_α in C^0 -topology if the following holds. Let $(x, w) \in V_\alpha \times W_\alpha$. Then for any branch $\mathfrak{s}_{\alpha,i,j}$ of \mathfrak{s}_α , we have

$$\text{dist}(\mathfrak{s}_{\alpha,i,j}(y, w), s_\alpha(y)) < \epsilon$$

if y is in a neighborhood of x .

(3) \mathfrak{s}_α is said to be *transversal to zero* if the following holds. Let $(x, w) \in V_\alpha \times W_\alpha$. Then any branch $\mathfrak{s}_{\alpha,i,j}$ of \mathfrak{s}_α is transversal to zero.

(4) Let $f_\alpha : V_\alpha \rightarrow M$ be a Γ_α -equivariant smooth map. We assume that \mathfrak{s}_α is transversal to zero. We then say that $f_\alpha|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a *submersion* if the following holds. Let $(x, w) \in V_\alpha \times W_\alpha$. Then for any branch $\mathfrak{s}_{\alpha,i,j}$ of \mathfrak{s}_α , the restriction of $f_\alpha \circ \pi_\alpha : V_\alpha \times W_\alpha \rightarrow M$ to

$$(4.2) \quad \{(x, w) \mid \mathfrak{s}_{\alpha,i,j}(x, w) = 0\}$$

is a submersion. We remark that (4.2) is a smooth manifold by (3).

REMARK 4.1

In the case when \mathcal{M} has boundary or corner, (4.2) has boundary or corner. In this case we require that the restriction of f_α to each of the stratum of (4.2) be a submersion.

LEMMA 4.1

We assume that $f_\alpha : V_\alpha \rightarrow M$ is a submersion. Then there exists W_α such that for any ϵ there exists a W_α -parameterized family \mathfrak{s}_α of multisections which is ϵ -close to s_α , transversal to zero, and such that $f_\alpha|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a submersion.

Moreover, there exists $0 \in W_\alpha$ such that the restriction of \mathfrak{s}_α to $V_\alpha \times \{0\}$ coincides with s_α .

If \mathfrak{s}_α is already given and satisfies the required condition on a neighborhood of a compact set K_α , then we may extend it to the whole W_α without changing it on K_α .

We omit the proof (see [15]). We next describe the compatibility conditions among the W_α -parameterized families of multisections for various α . During the construction we need to shrink V_α a bit several times. We do not mention it explicitly below.

Let $\alpha_1 < \alpha_2$. We consider the normal bundle $N_{\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})}V_{\alpha_2}$ of the embedding $\phi_{\alpha_2\alpha_1} : V_{\alpha_1\alpha_2} \rightarrow V_{\alpha_2}$. We take a small neighborhood $U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2}))$ of its image and identify it with an ϵ -neighborhood $B_\epsilon N_{\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})}V_{\alpha_2}$ of the zero section $\cong V_{\alpha_1\alpha_2}$ of $N_{\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})}V_{\alpha_2}$. We then have a projection $\text{Pr} : U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})) \rightarrow V_{\alpha_1\alpha_2}$.

We pull back the bundle $E_{\alpha_1}|_{V_{\alpha_1\alpha_2}}$ by Pr to obtain $\text{Pr}^*E_{\alpha_1} \rightarrow V_{\alpha_1\alpha_2}$. We extend the bundle embedding $\hat{\phi}_{\alpha_2\alpha_1} : E_{\alpha_1}|_{V_{\alpha_1\alpha_2}} \rightarrow E_{\alpha_2}$ to a bundle embedding

$$\hat{\phi}_{\alpha_2\alpha_1} : \text{Pr}^*E_{\alpha_1} \rightarrow E_{\alpha_2}|_{U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2}))}.$$

We consider the section (Kuranishi map) $s_{\alpha_2} : U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})) \rightarrow E_{\alpha_2}$ and compose it with the projection to obtain

$$(4.3) \quad \pi \circ s_{\alpha_2} : U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})) \rightarrow \frac{E_{\alpha_2}}{\text{Pr}^*E_{\alpha_1}}.$$

We remark that (4.3) is zero on the zero section $= \phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})$ and the fiber derivative of it there induces an isomorphism.

Let $\text{Exp} : B_\epsilon N_{\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})}V_{\alpha_2} \rightarrow U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2}))$ be the isomorphism we mentioned above. By modifying it using fiberwise diffeomorphism, we may assume that

$$(4.4) \quad \pi \circ s_{\alpha_2} \circ \text{Exp} : B_\epsilon N_{\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})}V_{\alpha_2} \rightarrow \frac{E_{\alpha_2}}{\text{Pr}^*E_{\alpha_1}}$$

is a restriction of a (linear) isomorphism of vector bundles.

Now, let $U_{i,\alpha_1} \subset V_{\alpha_1}$, and let $\{\mathfrak{s}_{\alpha_1,i,j} \mid j = 1, \dots, l_i\}$ be a multisection on $U_{i,\alpha_1} \times W_{\alpha_1}$. We take W_{α_2,α_1} and put $W_{\alpha_2} = W_{\alpha_1} \times W_{\alpha_2,\alpha_1}$. We define

$$\text{Pr}^{-1}(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})) = U_{i,\alpha_2} \subset U_\epsilon(\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})).$$

DEFINITION 4.2

A W_{α_2} -parameterized family of multisections $\{\mathfrak{s}_{\alpha_2,i,j} \mid j = 1, \dots, l_i\}$ on U_{i,α_2} is said to be *compatible* with $\{\mathfrak{s}_{\alpha_1,i,j} \mid j = 1, \dots, l_i\}$ if the following holds: $\{\mathfrak{s}_{\alpha_2,i,j} \mid j = 1, \dots, l_i\}$ is a multisection of E_{α_2} on $U_{i,\alpha_2} \times W_{\alpha_2}$.

Let $y = \text{Exp}(x, \xi)$ with $x \in U_{i,\alpha_1} \cap U_{\alpha_1,\alpha_2}$, $\xi \in (N_{\phi_{\alpha_2\alpha_1}(V_{\alpha_1\alpha_2})}V_{\alpha_2})_x$ ($\|\xi\| < \epsilon$), and $w = (w_1, w_2) \in W_{\alpha_2} = W_{\alpha_1} \times W_{\alpha_2,\alpha_1}$. Then we have

$$(4.5) \quad \mathfrak{s}_{\alpha_2,i,j}(y, w) \equiv (\pi \circ s_{\alpha_2})(y) \pmod{\text{Pr}^*E_{\alpha_1}}.$$

We assume also that

$$(4.6) \quad \mathfrak{s}_{\alpha_2,i,j}(\text{Exp}(x, 0), w) = \mathfrak{s}_{\alpha_1,i,j}(x, w_1).$$

We remark that for given $\{\mathfrak{s}_{\alpha_1,i,j} \mid j = 1, \dots, l_i\}$, we can always find $\{\mathfrak{s}_{\alpha_2,i,j} \mid j = 1, \dots, l_i\}$ which is compatible to it. In fact, we can use the splitting

$$E_{\alpha_2} = \text{Pr}^*E_{\alpha_1} \oplus \frac{E_{\alpha_2}}{\text{Pr}^*E_{\alpha_1}}$$

to construct it.

Moreover, if $f = \{f_\alpha\} : \mathcal{M} \rightarrow N$ is a strongly continuous and weakly submersive map and $f_{\alpha_1}|_{\mathfrak{s}_{\alpha_1}^{-1}(0)}$ is a submersion, then $f_{\alpha_2}|_{\mathfrak{s}_{\alpha_2}^{-1}(0)}$ is a submersion, for small ϵ .

Thus we can prove the following by induction of α with respect to the order $<$ and by using Lemma 4.1.

PROPOSITION 4.1

Let \mathcal{M} be a Kuranishi space with a good coordinate system $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ ($\alpha \in \mathfrak{A}$). Let $f = \{f_\alpha\} : \mathcal{M} \rightarrow N$ be a strongly continuous and weakly submersive map. Then there exists a system of continuous families of multisections $\mathfrak{s}_\alpha = \{\mathfrak{s}_{\alpha,i,j} \mid j = 1, \dots, l_i\}$ of E_α such that

- (1) they are compatible in the sense of Definition 4.2;
- (2) they are transversal to zero in the sense of Definition 4.1(3);
- (3) $f_\alpha|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a submersion in the sense of Definition 4.1(4);
- (4) they are C^0 -close to the original Kuranishi map in the sense of Definition 4.1(2).

REMARK 4.2

We can prove a relative version of Proposition 4.1. Namely, if there exists an open set $\mathcal{U} \subset \mathcal{M}$ and a compact set $K \subset \mathcal{U}$ and \mathfrak{s}_α satisfying Proposition 4.1(1)–(4) are given on \mathcal{U} , then we can extend it with required properties without changing it on K .

We next review the way to use a family of multisections to define smooth correspondence.

We work in the following situation. Let \mathcal{M} be a Kuranishi space, let $f^s = \{f_\alpha^s\} : \mathcal{M} \rightarrow N_s$ be a strongly continuous map, and let $f^t = \{f_\alpha^t\} : \mathcal{M} \rightarrow N_t$ be a strongly continuous and weakly submersive map. Here N_s, N_t are smooth manifolds. (Here s and t stand for source and target, resp.) Let $\Lambda^d(M)$ denote the set of smooth d -forms on M . We define

$$(4.7) \quad \text{Corr}_*(\mathcal{M}; f^s, f^t) : \Lambda^d(N_s) \rightarrow \Lambda^\ell(N_t),$$

where $\ell = d + \dim N_t - \dim \mathcal{M}$.

We first take a continuous family of multisections $\mathfrak{s}_\alpha = \{\mathfrak{s}_{\alpha,i,j} \mid j = 1, \dots, l_i\}$ on \mathcal{M} such that $f_\alpha^t|_{\mathfrak{s}_\alpha^{-1}(0)}$ is a submersion. Let $\rho \in \Lambda^d(N_s)$. We consider a representative $\mathfrak{s}_{\alpha,i,j}$ of \mathfrak{s}_α . It is a section of E_α on $U_{i,\alpha} \times W_\alpha$. We take a top-dimensional smooth form ω_α on W_α of compact support such that its total mass is 1. Let χ_i be a partition of unity subordinate to the covering $\{U_{i,\alpha}\}$. We consider

$$(4.8) \quad \sum_i f_\alpha^t \frac{1}{l_i} \chi_i ((f_\alpha^s)^* \rho \wedge \omega_\alpha) \Big|_{\mathfrak{s}_{\alpha,i,j}}.$$

Here l_i is the number of branches, and $f_{\alpha^i}^t$ are integrations along fiber. This defines the U_α part of $\text{Corr}_*(\mathcal{M}; f^s, f^t)$. We use the partition of unity again to glue them for various α to obtain a map (4.7) (see [15] for details).

REMARK 4.3

The map (4.7) depends on the choice of multisection \mathfrak{s}_α and the smooth form ω_α . But it is independent of the choice of partition of unity.

The smooth correspondence we defined above has the following two properties.

PROPOSITION 4.2 (STOKES)

We have

$$(4.9) \quad d \circ \text{Corr}_*(\mathcal{M}; f^s, f^t) - \text{Corr}_*(\mathcal{M}; f^s, f^t) \circ d = \text{Corr}_*(\partial\mathcal{M}; f^s, f^t).$$

Proof

On each chart this is a consequence of Stokes’s theorem. By using partition of unity in a standard way, we obtain the proposition. \square

To state the next proposition, we need some notation. We consider oriented Kuranishi spaces $\mathcal{M}_1, \mathcal{M}_2$. Let $f^{1,s} : \mathcal{M}_1 \rightarrow N_s^1, f^{2,s} : \mathcal{M}_2 \rightarrow N_t^1 \times N_s^2$ be strongly continuous maps, and let $f^{1,t} : \mathcal{M}_1 \rightarrow N_t^1, f^{2,t} : \mathcal{M}_2 \rightarrow N_t^2$ be strongly continuous and weakly submersive maps. We put

$$(4.10) \quad \mathcal{M} = \mathcal{M}_1 \times_{N_t^1} \mathcal{M}_2,$$

where we use the second factor $\mathcal{M}_2 \rightarrow N_t^1$ of the map $f^{2,s}$ to define the above fiber product. The maps $f^{1,s}$ and $f^{2,s}$ induce a strongly continuous map $f^s : \mathcal{M} \rightarrow N_s^1 \times N_s^2$; $f^{2,t}$ induces a strongly continuous and weakly submersive map $f^t : \mathcal{M} \rightarrow N_t^2$ (see Lemma 2.1).

PROPOSITION 4.3 (COMPOSITION FORMULA)

If $\rho_i \in \Lambda(N_s^i)$ ($i = 1, 2$), then we have

$$(4.11) \quad \begin{aligned} & \text{Corr}_*(\mathcal{M}; f^s, f^t)(\rho_1 \times \rho_2) \\ &= \text{Corr}_*(\mathcal{M}_2; f^{2,s}, f^{2,t})(\text{Corr}_*(\mathcal{M}_1; f^{1,s}, f^{1,t})(\rho_1) \times \rho_2). \end{aligned}$$

Proof

On each chart this is obvious. So we can use partition of unity in a standard way to prove the propositions. \square

REMARK 4.4

We need to take and fix appropriate orientation in Propositions 4.2 and 4.3. We do it later but only in the case we use (see the end of Section 7).

We also note the next lemma. Let $(\mathcal{M}; f^s, f^t)$ be as above. Here $f^t : \mathcal{M} \rightarrow N_t$ is weakly submersive. We consider $(\mathcal{M}; f^s \times f^t, \text{const})$ where $\text{const} : \mathcal{M} \rightarrow \{\text{point}\}$ is a constant map to a point.

LEMMA 4.2

Let $\rho_s \in \Lambda(N_s)$ and $\rho_t \in \Lambda(N_t)$. Then

$$(4.12) \quad \int_{N_t} \text{Corr}(\mathcal{M}; f^s, f^t)(\rho_s) \wedge \rho_t = \text{Corr}(\mathcal{M}; f^s \times f^t, \text{const})(\rho_s \wedge \rho_t).$$

Note that the right-hand side is an element of $\Lambda(\{\text{point}\}) = \mathbb{R}$.

5. Forgetful-map-compatible continuous family of multisections

In this section, we define a system of continuous families of multisections on the Kuranishi space produced in Section 3. We construct the system so that it is compatible with forgetful maps.

We first define this compatibility precisely. We consider the moduli spaces $\mathcal{M}_{\ell,1}(\beta)$, $\mathcal{M}_{\ell,0}(\beta)$ and consider their Kuranishi structures, which are compatible in the sense of Definition 3.1. We take their good coordinate systems, which are compatible in a similar sense. More precise definition of this compatibility is in order. We have Kuranishi charts $(V_\alpha, E_\alpha, \psi_\alpha, s_\alpha, \Gamma_\alpha)$ ($\alpha \in \mathfrak{A}$) of $\mathcal{M}_{\ell,0}(\beta)$ and $(V_{\tilde{\alpha}}, E_{\tilde{\alpha}}, \psi_{\tilde{\alpha}}, s_{\tilde{\alpha}}, \Gamma_{\tilde{\alpha}})$ ($\tilde{\alpha} \in \tilde{\mathfrak{A}}$) of $\mathcal{M}_{\ell,1}(\beta)$. Here \mathfrak{A} and $\tilde{\mathfrak{A}}$ are partially ordered sets. We require that there exist an order-preserving map $\tilde{\mathfrak{A}} \rightarrow \mathfrak{A}$, $\tilde{\alpha} \mapsto \alpha$ such that

- (1) $V_{\tilde{\alpha}} = V_\alpha \times (0, 1)$;
- (2) $E_{\tilde{\alpha}} = E_\alpha \times (0, 1)$;
- (3) $\Gamma_{\tilde{\alpha}} = \Gamma_\alpha$; The action of $\Gamma_{\tilde{\alpha}}$ preserves identifications given in (1) and (2),

where the action to the factor $(0, 1)$ is trivial;

- (4) $s_{\tilde{\alpha}}(x, t) = (s_\alpha(x), t)$ by the identifications given in (1) and (2);
- (5) **forget** $\circ \psi_{\tilde{\alpha}}$ coincides with the composition of ψ_α and the projection to the first factor.

DEFINITION 5.1

Let $(U_{\alpha,i}, W_\alpha, \{\mathfrak{s}_{\alpha,i,j}\})$ define a compatible system of families of multisections on $\mathcal{M}_{\ell,0}(\beta)$, and let $(U_{\tilde{\alpha},i}, W_{\tilde{\alpha}}, \{\mathfrak{s}_{\tilde{\alpha},i,j}\})$ define a compatible system of family of multisections on $\mathcal{M}_{\ell,1}(\beta)$.

We say that they are *compatible* if the following conditions are satisfied:

- (1) $U_{\tilde{\alpha},i} = U_{\alpha,i} \times (0, 1)$;
- (2) $W_{\tilde{\alpha}} = W_\alpha$;
- (3) $\mathfrak{s}_{\tilde{\alpha},i,j}(w, x, t) = \mathfrak{s}_{\alpha,i,j}(w, (x, t))$.

The main result of this section is as follows.

THEOREM 5.1

For each $E_0 > 0$, $\ell_0 \in \mathbb{Z}_{\geq 0}$, and $\epsilon > 0$, there exists a compatible system of families of multisections $(U_{\tilde{\alpha},i}, W_{\tilde{\alpha}}, \{\mathfrak{s}_{\tilde{\alpha},i,j}\})$, $(U_{\alpha,i}, W_{\alpha}, \{\mathfrak{s}_{\alpha,i,j}\})$ on $\mathcal{M}_{\ell,1}(\beta)$, $\mathcal{M}_{\ell,0}(\beta)$ for $\beta \cap \omega \leq E_0$ and $\ell \leq \ell_0$, with the following properties.

- (1) They are ϵ -close to the Kuranishi map.
- (2) They are compatible in the sense of Definition 5.1.
- (3) They are transversal to zero in the sense of Definition 4.1(3).
- (4) $\text{ev}_0 : \mathcal{M}_{\ell,1}(\beta) \rightarrow L$ induces submersions $(\text{ev}_0)_{\tilde{\alpha}}|_{\mathfrak{s}_{\tilde{\alpha}}^{-1}(0)} : \mathfrak{s}_{\tilde{\alpha}}^{-1}(0) \rightarrow L$.
- (5) They are compatible with (3.3) in the sense we describe.

We describe condition (5) precisely below. We consider $U_{\alpha_1,i_1}, \mathfrak{s}_{\tilde{\alpha}_1,i_1,j_1}$ of $\mathcal{M}_{\ell_1,1}(\beta_1)$ and $U_{\alpha_2,i_2}, \mathfrak{s}_{\tilde{\alpha}_2,i_2,j_2}$ of $\mathcal{M}_{\ell_2,1}(\beta_2)$. We put $\alpha = (\alpha_1, \alpha_2)$. The fiber product

$$(5.1) \quad V_{\alpha} := V_{\alpha_1 \text{ev}_0} \times_{\text{ev}_0} V_{\alpha_2}$$

is a Kuranishi neighborhood of the Kuranishi space

$$\mathcal{M}_{\ell_1,1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2,1}(\beta_2);$$

(5.1) contains a fiber product $U_{\alpha,(i_1,i_2)} = U_{\alpha_1,i_1 \text{ev}_0} \times_{\text{ev}_0} U_{\alpha_2,i_2}$. An obstruction bundle on (5.1) is a restriction of $E_{\alpha_1} \times E_{\alpha_2}$. We put $W_{\alpha} = W_{\alpha_1} \times W_{\alpha_2}$. Now we put

$$(5.2) \quad \mathfrak{s}_{\alpha,(i_1,i_2),(j_1,j_2)} = (\mathfrak{s}_{\alpha_1,i_1,j_1}, \mathfrak{s}_{\alpha_2,i_2,j_2})|_{U_{\alpha,(i_1,i_2)}}.$$

It defines a compatible system of continuous families of multisections on $\mathcal{M}_{\ell_1,1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2,1}(\beta_2)$. Condition (5) requires that this system coincide with the restriction of the one in $\mathcal{M}_{\ell,0}(\beta)$. Here $\beta = \beta_1 + \beta_2$, $\ell + 2 = \ell_1 + \ell_2$.

In the cases $\beta_1 = \beta_2 = \beta'$, $\ell_1 = \ell_2 = \ell'$, and $\beta = 2\beta'$, $2\ell' = \ell + 2$, we put two copies of $\mathcal{M}_{\ell,1}(\beta')$. Then the parameter space is given as $W_{\alpha} = W_{\alpha_1} \times W_{\alpha_2}$ in the same way. (In the case when W_{α_1} coincides with W_{α_2} , we take the square of it.) Then we obtain a continuous family of multisections on

$$\mathcal{M}_{\ell',1}(\beta')_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell',1}(\beta').$$

Note that the Kuranishi structure is invariant of \mathbb{Z}_2 -action exchanging the factors. It is easy to see that any (local) multisection with m branches induces a (local) multisection with $2m$ branches to the quotient of the \mathbb{Z}_2 -action. Therefore we obtain a continuous family of multisections on

$$(\mathcal{M}_{\ell',1}(\beta')_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell',1}(\beta'))/\mathbb{Z}_2.$$

We require compatibility with this multisection as in (5.2).

Proof

The strategy of the proof is similar to the proof of Theorem 3.1. We first consider the situation of Lemma 3.1 and use the notation there.

LEMMA 5.1

Let $V_0 \subset \bar{V}_0 \subset V_1 \subset \bar{V}_1 \subset V$ be neighborhoods of u in V_α , where V_α is a Kuranishi neighborhood of u .

Then there exists a finite-dimensional vector space $W(u)$ and a section $s_u : V \times W(u) \rightarrow E(u) \times W(u)$ with the following properties:

- (1) $s_u(v, 0) = D_v \bar{\partial}$;
- (2) $s_u(v, w) = D_v \bar{\partial}$ if $v \notin V_1$;
- (3) if $D_v \bar{\partial} = 0$ and $v \in V_0$, then the map

$$D_{(v,0)}s_u : T_v V \times T_0 W(u) \rightarrow E(u)$$

is surjective;

- (4) if v is as in (3) and if $z_0 \in \partial D^2$, then

$$D_{(v,0)}s_u \oplus d_{v,0} \text{Ev}_{z_0} : T_v V \times T_0 W(u) \rightarrow E(u) \oplus T_{v(z_0)} L$$

is surjective.

Proof

Let $\chi : V \rightarrow [0, 1]$ be a smooth function such that $\chi \equiv 1$ on V_0 and $\chi \equiv 0$ on the complement of V_1 . We put $W(u) = E(u)$ and

$$s_u(v, w) = D_v \bar{\partial} + \chi(v)w.$$

The lemma then follows easily from (3.5). □

We consider a Kuranishi chart V_α of $\mathcal{M}_0(\beta)$. For simplicity of notation, we assume that each element of V_α is represented by a map from a disc (without singular point). Let $u \in V_\alpha$. We apply Lemma 5.1 and obtain V_0, V_1, V , which we denote by $V_0(u), V_1(u), V(u)$. We also have $E(u)$ and obtain $s_u, W(u)$. (We remark that $E(u)$ here is the restriction of E_α , the obstruction bundle of our Kuranishi chart.) Let $\Gamma(u) \subset \Gamma_\alpha$ be the isotropy group of u . We may assume that $U(u) \cap \gamma V(u) \neq \emptyset, \gamma \in \Gamma_\alpha$, implies that $\gamma \in \Gamma(u)$ and that $V_0(u), V_1(u), V_2(u)$ are $\Gamma(u)$ -invariant. (We do not, and cannot, require s_u to be $\Gamma(u)$ -equivariant.)

We take a Γ_α -invariant relatively compact subset $V'_\alpha \subset V_\alpha$ such that V'_α/Γ_α still cover $\mathcal{M}_0(\beta)$. V'_α/Γ_α is covered by finitely many $V_0(u)/\Gamma(u)$'s, which we write as $V_0(u_c)/\Gamma(u_c), c = 1, \dots, N$. We put

$$W_\alpha = W(u_1) \times \dots \times W(u_N).$$

For each c , the map s_{u_c} determines a multisection on V_α/Γ_α which coincides with s_α outside $V_1(u_c)/\Gamma(u_c)$ (see Lemma 5.1(2)). We denote it as s_{u_c} by abuse of notation. Let $\chi_c : V_\alpha/\Gamma_\alpha \rightarrow [0, 1]$ be a partition of unity subordinate to the covering $\{V_1(u_c)/\Gamma(u_c) \mid c = 1, \dots, N\}$. We now put

$$(5.3) \quad \mathfrak{s}_\alpha(v; w_1, \dots, w_I) = s_\alpha(v) + \sum_i \chi_c(v)(s_c(v, w) - s_\alpha(v))$$

(see [16, Definition 3.4] for the sum of multisections, which appear in the right-hand side of (5.3)). Now we have the following.

LEMMA 5.2

There exists $\epsilon > 0$ such that the following hold.

- (1) \mathfrak{s}_α is transversal to zero.
- (2) We take $U_i \subset V'_\alpha/\Gamma_\alpha$ such that \mathfrak{s}_α has a representative $(\mathfrak{s}_{\alpha,i,j})_{j=1}^l$ on $U_i \times W$. We put

$$\mathfrak{s}_{\alpha,i,j}^{-1}(0) = \{(v, w) \in U_i \times W \mid \mathfrak{s}_{\alpha,i,j}(v, w) = 0, \|w\| \leq \epsilon\}.$$

This is a manifold by (1). Then for each $z_0 \in \partial D^2$, the map

$$\text{Ev}_{z_0} : \mathfrak{s}_{\alpha,i,j}^{-1}(0) \rightarrow L$$

defined by $(v, w) \mapsto \text{Ev}_{z_0}(v, w) = v(z_0)$ is a submersion.

- (3) We have $\mathfrak{s}_{\alpha,i,j}(v, 0) = s_\alpha(v)$.

Proof

Statements (1), (2), and (3) follow from Lemma 5.1(3), (4), (1), respectively. \square

Lemma 5.2 enables us to construct the family of multisections required in Theorem 5.1 locally. We can prove the relative version of Lemma 5.2 also in the same way.

REMARK 5.1

We remark that we can use Lemma 5.2(3) to prove property (1) in Theorem 5.1. Namely, we can shrink W to its small neighborhood of zero. Then Lemma 5.2(3) implies that $\mathfrak{s}_{\alpha,i,j}$ is C^0 -close to the Kuranishi map s_α .

We next state the analog of Lemma 3.2.

LEMMA 5.3

For each E_0 and ℓ_0 , there exists a system of continuous families of multisections on various $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ for various $\ell \geq 0$ and $\alpha \in \pi_2(M)$ with $\alpha \cap \omega \leq E_0$ and $\ell \leq \ell_0$. It has the following properties.

- (1) The action of permutation groups of order $\ell!$ on $\mathcal{M}_\ell^{\text{cl}}(\alpha)$ exchanging marked points preserves the family of multisections.
- (2) It is transversal to zero.
- (3) The evaluation map $\text{ev}^{\text{int}} : \mathcal{M}_\ell^{\text{cl}}(\alpha) \rightarrow M^\ell$ is a submersion on the zero set.
- (4) Let $\ell_1 + \ell_2 = \ell + 2$, $\alpha_1 + \alpha_2 = \alpha$. Then the embedding

$$\mathcal{M}_{\ell_1}^{\text{cl}}(\alpha_1)_{\text{ev}_1^{\text{int}}} \times_{\text{ev}_1^{\text{int}}} \mathcal{M}_{\ell_2}^{\text{cl}}(\alpha_2) \subset \mathcal{M}_\ell^{\text{cl}}(\alpha)$$

is compatible with our system of continuous families of multisections.

Proof

The proof is by induction on (α, ℓ) . We can organize the order of the induction in the same way as [11, Section 7.2]. Property (4) determines the family of

multisections in the singular locus of each of $\mathcal{M}_\ell^{\text{cl}}(\alpha)$. By (2) and (3) of the fiber product factors, we can prove that (2) and (3) are satisfied on the singular locus. We can then use the argument of Section 4 to extend it to $\mathcal{M}_\ell^{\text{cl}}(\alpha)$. \square

Note that we do not require compatibility with forgetful map here. So the proof of Lemma 5.3 is easier than that of Theorem 5.1.

Now we are in a position to complete the proof of Theorem 5.1. The proof is by induction on (β, ℓ) . We assume that we have constructed the required family of multisections on $\mathcal{M}_{\ell',0}(\beta')$ for $\beta' \cap \omega < \beta \cap \omega$ and $\ell' \leq \ell$. We study the case of $\mathcal{M}_{\ell,0}(\beta)$. We consider the boundary $\partial\mathcal{M}_{\ell,0}(\beta)$. One of its boundary components is

$$(5.4) \quad \mathcal{M}_{\ell_1,1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_0} \mathcal{M}_{\ell_2,1}(\beta_2),$$

where $\ell_1 + \ell_2 = \ell$, $\alpha_1 + \alpha_2 = \alpha$. (In the case $\beta_1 = \beta_2$, $\ell_1 = \ell_2$, we need to divide (5.4) by \mathbb{Z}_2 -action.) A continuous family of multisections on the factors of (5.4) is already given by the induction hypothesis. Moreover, properties (3) and (4) of the factors of (5.4) imply the same properties for the fiber product family of multisections.

For the other type of boundary component,

$$(5.5) \quad \mathcal{M}_\ell^{\text{cl}}(\tilde{\beta})_{\text{ev}^{\text{int}}} \times_M L,$$

we can apply Lemma 5.3 to obtain the required family of multisections. We thus obtain a family of multisections which has required properties.

Those families of the multisections on the components of the boundary $\partial\mathcal{M}_{\ell,0}(\beta)$ are consistent at the overlapped part (see [11, Lemma 7.2.55]).

We can then extend it to a neighborhood of the boundary. It is easy to see that this extended one still has the required transversality properties. Therefore we can use a relative version of Lemma 5.2 to discuss in the same way as in Section 4 to extend this family of multisections to $\mathcal{M}_{\ell,0}(\beta)$. Since we have only finitely many steps to work out, we can choose our family so that Theorem 5.1(1) is satisfied. The proof of Theorem 5.1 is now complete. \square

REMARK 5.2

See [11, Section 7.2.3] for the reason why we need to fix E_0, ℓ_0 and stop the construction of the multisections after $\beta \cap \omega > E_0$ or $\ell > \ell_0$.

In the same way as in Section 3, Theorem 5.1 has the following corollary.

COROLLARY 5.1

For each ϵ, E_0 , and ℓ_0 , there exists a system of continuous families of multisections on $\mathcal{M}_{\ell,k+1}(\beta)$, $k \geq 0, \ell_0 \geq \ell \geq 0, \beta \cap \omega \leq E_0$, with the following properties.

- (1) *It is ϵ -close to the Kuranishi map in the C^0 -sense.*
- (2) *It is compatible with $\text{forget}_{k+1,1}$.*

- (3) It is invariant under the cyclic permutation of the boundary marked points.
- (4) It is invariant by the permutation of interior marked points.
- (5) $\text{ev}_0 : \mathcal{M}_{\ell, k+1}(\beta) \rightarrow L$ induces a submersion on its zero set.
- (6) We consider the decomposition of the boundary:

$$(5.6) \quad \begin{aligned} \partial \mathcal{M}_{\ell, k+1}(\beta) = & \bigcup_{1 \leq i \leq j+1 \leq k+1} \bigcup_{\beta_1 + \beta_2 = \beta} \bigcup_{L_1 \cup L_2 = \{1, \dots, \ell\}} \\ & \mathcal{M}_{\#L_1, j-i+1}(\beta_1)_{\text{ev}_0} \times_{\text{ev}_i} \mathcal{M}_{\#L_2, k-j+i}(\beta_2) \end{aligned}$$

(see [11, Section 7.1.1]). Then the restriction of our family of multisections of $\mathcal{M}_{\ell, k+1}(\beta)$ in the left-hand side coincides with the fiber product family of multisections in the right-hand side.

6. Cyclic filtered A_∞ -algebra and cyclic filtered A_∞ -algebra modulo T^E

Let \overline{C} be a graded vector space over \mathbb{R} , and let n be a positive integer. We consider an \mathbb{R} -bilinear map

$$(6.1) \quad \langle \cdot \rangle : \overline{C}^k \otimes \overline{C}^{n-k} \rightarrow \mathbb{R}$$

such that

$$(6.2) \quad \langle x, y \rangle = (-1)^{1 + \text{deg}' x \text{deg}' y} \langle y, x \rangle.$$

Here and hereafter,

$$\text{deg}' x = \text{deg } x - 1.$$

We consider a sequence of operators

$$(6.3) \quad \overline{m}_k : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

of degree 1 for $k = 1, 2, \dots$ (Here $B_k(\overline{C}[1])$ is the tensor product of k copies of $\overline{C}[1]$.)

DEFINITION 6.1

We say that $(\overline{C}, \langle \cdot \rangle, \{\overline{m}_k\}_{k=1}^\infty)$ is a *cyclic A_∞ -algebra* of dimension n if

- (1) $\{\overline{m}_k\}_{k=1}^\infty$ satisfies the A_∞ -relation

$$(6.4) \quad \sum_{k_1+k_2=k+1} \sum_{i=1}^{k-k_2+1} (-1)^* \overline{m}_{k_1}(x_1, \dots, \overline{m}_{k_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k) = 0,$$

where $*$ = $\text{deg}' x_1 + \dots + \text{deg}' x_{i-1}$;

- (2) we have

$$(6.5) \quad \langle \overline{m}_k(\mathbf{x}_1, \dots, \mathbf{x}_k), \mathbf{x}_0 \rangle = (-1)^* \langle \overline{m}_k(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}), \mathbf{x}_k \rangle,$$

where $*$ = $\text{deg}' x_0(\text{deg}' x_1 + \dots + \text{deg}' x_k)$;

- (3) $\langle \cdot \rangle$ is nondegenerate and induces a perfect pairing on $H(\overline{C}) = \text{Ker } \overline{m}_1 / \text{Im } \overline{m}_1$.

We remark that $\overline{m}_1 \circ \overline{m}_1 = 0$ by 1. Therefore $H(\overline{C})$ is well defined. Definition 6.1(2) implies $\langle \overline{m}_1(x), y \rangle = \pm \langle \overline{m}_1(y), x \rangle$. Therefore $\langle \cdot \rangle$ induces one on $H(\overline{C})$ which satisfies (6.2).

EXAMPLE 6.1

Let M be an n -dimensional oriented closed manifold. Let $\overline{C} = \Lambda(M)$ be the de Rham complex. We put

$$\overline{m}_1(u) = (-1)^{\deg u} du, \quad \overline{m}_2(u, v) = (-1)^{\deg u \deg v + \deg u} u \wedge v,$$

$\overline{m}_k = 0$ for $k \neq 1, 2$, and

$$\langle u, v \rangle = (-1)^{\deg u \deg v + \deg u} \int_M u \wedge v.$$

DEFINITION 6.2

We say that a subset $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ is a *discrete submonoid* if the following holds. We denote by $E : G \rightarrow \mathbb{R}$ and $\mu : G \rightarrow 2\mathbb{Z}$ the projections to each of the components.

- (1) If $\beta_1, \beta_2 \in G$, then $\beta_1 + \beta_2 \in G$, $(0, 0) \in G$.
- (2) The image $E(G) \subset \mathbb{R}_{\geq 0}$ is discrete.
- (3) For each $\lambda \in \mathbb{R}_{\geq 0}$, the inverse image $G \cap E^{-1}(\lambda)$ is a finite set.

We remark that (1), (2), and (3) imply that $E^{-1}(0) \cap G = \{(0, 0)\}$.

DEFINITION 6.3

We put

$$\Lambda_0^G = \left\{ \sum a_\beta T^{E(\beta)} e^{\mu(\beta)/2} \mid \beta \in G, a_\beta \in \mathbb{R} \right\},$$

where the sum may be either finite or infinite. For each $E_0 \in \mathbb{R}_{\geq 0}$, we have a filtration

$$F^{E_0} \Lambda_0^G = \left\{ \sum a_\beta T^{E(\beta)} e^{\mu(\beta)/2} \in \Lambda_0^G \mid E(\beta) \geq E_0 \right\}.$$

It determines a topology on Λ_0^G with which Λ_0^G is complete.

We define a grading on Λ_0^G by $\deg T = 0$, $\deg e = 2$.

We remark that Λ_0^G contains a (semi)group ring of the monoid G that is nothing but the set of elements of Λ_0^G with only finitely many nonzero a_β . Λ_0^G is its completion.

The universal Novikov ring $\Lambda_{0, \text{nov}}$ which was introduced in [13] is the union (inductive limit) of all Λ_0^G for various discrete submonoids G .

DEFINITION 6.4

A G -gapped cyclic filtered A_∞ -algebra structure on \overline{C} is a sequence of operators

$$(6.6) \quad m_{k, \beta} : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

for each $\beta \in G$ and $k \in \mathbb{Z}_{\geq 0}$ of degree $1 - \mu(\beta)$ and $\langle \cdot \rangle$ with the following properties:

- (1) $\mathfrak{m}_{0,\beta} = 0$ for $\beta = (0, 0)$;
- (2)

$$(6.7) \quad \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} (-1)^* \mathfrak{m}_{k_1,\beta_1}(x_1, \dots, \mathfrak{m}_{k_2,\beta_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k) = 0,$$

holds for any $\beta \in G$ and k ; Here the sign is as in (6.1);

- (3) $\langle \cdot \rangle$ satisfies (6.1), (6.2), and

$$(6.8) \quad \langle \mathfrak{m}_{k,\beta}(\mathbf{x}_1, \dots, \mathbf{x}_k), \mathbf{x}_0 \rangle = (-1)^* \langle \mathfrak{m}_{k,\beta}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}), \mathbf{x}_k \rangle$$

holds for any β and k . Here the sign is as in (6.5). Definition 6.1(3) also holds.

DEFINITION 6.5

A G -gapped cyclic filtered A_∞ -algebra structure modulo T^{E_0} is a sequence of operators (6.6) for $E(\beta) < E_0$ which satisfies the same properties, except that (6.7) and (6.8) are assumed only for $E(\beta) < E_0$.

DEFINITION 6.6

An element $\mathbf{e} \in \overline{C}$ of degree zero is said to be a *(strict) unit of an A_∞ -algebra* if

$$(6.9) \quad \begin{aligned} \mathfrak{m}_2(\mathbf{e}, x) &= (-1)^{\deg x} \mathfrak{m}_2(x, \mathbf{e}) = x, \\ \mathfrak{m}_k(\dots, \mathbf{e}, \dots) &= 0 \quad \text{for } k \neq 2. \end{aligned}$$

An element \mathbf{e} is said to be a *unit of a filtered A_∞ -algebra* if

$$(6.10) \quad \begin{aligned} \mathfrak{m}_{2,(0,0)}(\mathbf{e}, x) &= (-1)^{\deg x} \mathfrak{m}_{2,(0,0)}(x, \mathbf{e}) = x, \\ \mathfrak{m}_{k,\beta}(\dots, \mathbf{e}, \dots) &= 0 \quad \text{for } (k, \beta) \neq (2, (0, 0)). \end{aligned}$$

A unit for a filtered A_∞ -algebra modulo T^E is defined in the same way. A (filtered) A_∞ -algebra with unit is said to be *unital*.

We put

$$\mathfrak{m}_k = \sum_{\beta} T^{E(\beta)} e^{\mu(\beta)/2} \mathfrak{m}_{k,\beta} : B_k(C[1]) \rightarrow C[1].$$

This satisfies (6.4) and (6.5).

REMARK 6.1

The sign convention of this article is slightly different from the one in [11]. We explain the difference below. In [11] the inner product

$$(6.11) \quad (u, v) = (-1)^{\deg u \deg v} \int_L u \wedge v = (-1)^{\deg u} \langle u, v \rangle$$

is used (see [11, Remark 8.4.7(1)]). Then the cyclic symmetry of \mathfrak{m}_k takes the form

$$(6.12) \quad (x_0, \mathfrak{m}_{k,\beta}(x_1, \dots, x_k)) = (-1)^* (x_1, \mathfrak{m}_{k,\beta}(x_2, \dots, x_k, x_0)),$$

where $*$ = $\deg' x_0(\deg' x_1 + \dots + \deg' x_k)$ (see [11, Proposition 8.4.8]).

LEMMA 6.1

If (\cdot) and $\langle \cdot \rangle$ are related by (6.11), then (6.8) is equivalent to (6.12).

Proof

We assume (6.11) and (6.12). Then we calculate

$$\begin{aligned} & \langle \mathfrak{m}_{k,\beta}(x_1, \dots, x_k), x_0 \rangle \\ &= (-1)^{\sum_{i=1}^k \deg' x_i} (\mathfrak{m}_{k,\beta}(x_1, \dots, x_k), x_0) \\ &= (-1)^{(\sum_{i=1}^k \deg' x_i)(\deg' x_0)} (x_0, \mathfrak{m}_{k,\beta}(x_1, \dots, x_k)) = (x_1, \mathfrak{m}_{k,\beta}(x_2, \dots, x_k, x_0)) \\ &= (-1)^{*1} (\mathfrak{m}_{k,\beta}(x_2, \dots, x_k, x_0), x_1) = (-1)^{*2} \langle \mathfrak{m}_{k,\beta}(x_2, \dots, x_k, x_0), x_1 \rangle. \end{aligned}$$

Here

$$*1 = \left(\sum_{i \neq 1} \deg' x_i \right) (\deg' x_1 + 1)$$

and

$$*2 = *1 + \sum_{i \neq 1} \deg' x_i = \left(\sum_{i \neq 1} \deg' x_i \right) \deg' x_1.$$

We thus have (6.8). The proof of the converse is similar. □

By Lemma 6.1 we can apply the discussion of orientation and sign in [11, Section 8.4.2] for the purpose of this article.

7. Cyclic filtered A_∞ -structure modulo T^E on the de Rham complex

The main result of this section is as follows.

THEOREM 7.1

For any relatively spin Lagrangian submanifold L of (M, ω) , we can assign G such that for each $E_0 > 0$ the de Rham complex $\Lambda(L)$ has the structure of a G -gapped cyclic unital filtered A_∞ -algebra modulo T^{E_0} .

Proof

This theorem follows from Corollary 5.1 as follows. We fix a tame almost complex structure J and let G be the submonoid of $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ generated by the set

$$\{(\beta \cap \omega, \mu(\beta)) \mid \beta \in H_2(X, L; \mathbb{Z}), \mathcal{M}_0(\beta) \neq \emptyset\}.$$

Gromov compactness implies that G satisfies the conditions in Definition 6.2.

We apply Corollary 5.1 by putting E_0 as above and $\ell = 0$. Then for $\rho_1, \dots, \rho_k \in \Lambda(L)$, we define

$$(7.1) \quad \begin{aligned} & \mathfrak{m}_{k,\beta}(\rho_1, \dots, \rho_k) \\ & = \text{Corr}(\mathcal{M}_{k+1}(\beta); (\text{ev}_1, \dots, \text{ev}_k), \text{ev}_0)(\rho_1 \times \dots \times \rho_k) \in \Lambda(L) \end{aligned}$$

for $\beta \neq (0, 0)$. Since ev_0 is weakly submersive by Corollary 5.1.5, the right-hand side is well defined and is a smooth form. We define \mathfrak{m}_{k,β_0} for $\beta_0 = (0, 0)$ and $\langle \cdot \rangle$ as in (6.1).

The cyclic symmetry follows from Corollary 5.1.3 (up to sign) as follows. By Lemma 4.2 and the definitions we have

$$(7.2) \quad \begin{aligned} & \langle \mathfrak{m}_{k,\beta}(\rho_1, \dots, \rho_k), \rho_0 \rangle \\ & = \text{Corr}(\mathcal{M}_{k+1}(\beta), (\text{ev}_1, \dots, \text{ev}_k, \text{ev}_0), \text{const})(\rho_1 \times \dots \times \rho_k \times \rho_0). \end{aligned}$$

By Corollary 5.1.3, the right-hand side of (7.2) is cyclically symmetric.

The filtered A_∞ -relation (6.7) is a consequence of Corollary 5.1.6 (up to sign) and Propositions 4.2 and 4.3 (see [11, Section 3.5]).

The (strict) unitality (6.9) follows from Corollary 5.1.2 as follows. We consider $\beta \neq (0, 0)$ and $k \geq 1$ and studies

$$(7.3) \quad \mathfrak{m}_{k,\beta}(\rho_1, \dots, \rho_{i-1}, 1, \rho_{i+1}, \dots, \rho_k).$$

Here $1 \in \Lambda^0(L)$ is the zero form $\equiv 1$. We consider the forgetful map

$$\mathbf{forget}_i : \mathcal{M}_k(\beta) \rightarrow \mathcal{M}_{k-1}(\beta)$$

which forgets i th marked point. We have chosen the Kuranishi structure, and the (family of) multisections are invariant of forgetful map. We consider the expression (4.8):

$$(7.4) \quad \sum_i f_\alpha^t \frac{1}{l_i} \chi_i((f_\alpha^s)^* \rho \wedge \omega_\alpha) \Big|_{\mathfrak{s}_{\alpha,i,j}}.$$

In our case, $\rho = \rho_1 \times \dots \times \rho_{i-1} \times 1 \times \rho_{i+1} \times \dots \times \rho_k$. Let X be the vector field tangent to the fiber of \mathbf{forget}_i . Then we have

$$i_X((f_\alpha^s)^* \rho \wedge \omega_\alpha) = 0.$$

On the other hand, $f^t = \text{ev}_0$ factors through \mathbf{forget}_i . Therefore (7.4) is zero in our case. Since (7.3) is obtained from (7.4) by summing up using the partition of unity, it follows that (7.3) is zero also. We thus proved the strict unitality.

We finally remark that we can handle the sign in the same way as in [11, Section 8.10.3] and [7, Section 12]. Namely, we can reduce the sign in the de Rham version to one in singular homology version. □

REMARK 7.1

We remark that the evaluation map ev_0 from our perturbed moduli space is submersive. This is enough to work with de Rham theory since we can pull back differential forms by $\text{ev}_1, \dots, \text{ev}_k$ without assuming its submersivity.

In the case when we work with (singular) chains, we need to take the fiber product of singular chains P_i of L with our perturbed moduli space by evaluation maps ev_1, \dots, ev_k . This requires the submersivity of ev_1, \dots, ev_k . As we explained in Remark 3.2, it is impossible to do so while keeping compatibility with the forgetful map.

8. Homological algebra of a cyclic filtered A_∞ -algebra: Statement

In Sections 8–10 we study the homological algebra of a cyclic filtered A_∞ -algebra. We follow [18] in various places (see also [3], [20]). However, our discussion is different from [18], not only because we study filtered the case but also in several other points. In [18] $\langle u, v \rangle \neq 0$ only when $\deg u + \deg v$ is odd. Thus our situation is included in [18] when $\dim L$ is odd. In that case, according to [18, Remark 2.1.2], the sign convention of cyclic symmetry in [18] is similar to (6.12) (i.e., the convention of [11]). The notion of pseudoisotopy of a cyclic filtered A_∞ -algebra which we introduce in this section is also new.

We first review the definition of filtered A_∞ -homomorphisms from [11, Chapter 4]. Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a discrete submonoid, and let $(C, \{\mathbf{m}_{k,\beta}\})$, $(C', \{\mathbf{m}'_{k,\beta}\})$ be G -gapped filtered A_∞ -algebras.

DEFINITION 8.1

A sequence of \mathbb{R} -linear maps

$$f_{k,\beta} : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

of degree $-\mu(\beta)$ for $k = 0, 1, 2, \dots$, $\beta \in G$ is said to be a G -gapped filtered A_∞ -homomorphism if

- (1) $f_{0,(0,0)} = 0$;
- (2) For each β , $k = 0, 1, 2, \dots$, with $(k, \beta) \neq (0, (0, 0))$ and $x_1, \dots, x_k \in \overline{C}[1]$,

we have

$$(8.1) \quad \sum \mathbf{m}_{\ell,\beta_0}(f_{k_1,\beta_1}(x_1, \dots), \dots, f_{k_\ell,\beta_\ell}(\dots, x_k)) \\ = \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} (-1)^* f_{\beta_1,k_1}(x_1, \dots, \mathbf{m}_{k_2,\beta_2}(x_i, \dots), \dots, x_k).$$

Here the sum in the left-hand side is taken over ℓ , β_i , k_i such that $\beta_0 + \beta_1 + \dots + \beta_\ell = \beta$ and $k_1 + \dots + k_\ell = k$, and the sign in the right-hand side is $*$ = $\deg' x_1 + \dots + \deg' x_{i-1}$.

When $f_{k,\beta}$ is defined only for $E(\beta) < E_0$ and (8.1) holds only for $E(\beta) < E_0$, we call it a *filtered A_∞ -homomorphism modulo T^{E_0}* . This is defined if $(C, \{\mathbf{m}_{k,\beta}\})$, $(C', \{\mathbf{m}'_{k,\beta}\})$ are filtered A_∞ -algebras modulo T^{E_0} .

DEFINITION 8.2

- (1) A filtered A_∞ -homomorphism is said to be *strict* if $f_{0,\beta} = 0$ for any β .

(2) Suppose that $(C, \{\mathbf{m}_{k,\beta}\})$, $(C', \{\mathbf{m}'_{k,\beta}\})$ are unital. We say a filtered A_∞ -homomorphism is *unital* if

$$(8.2) \quad \begin{aligned} f_{1,\beta}(\mathbf{e}) &= \begin{cases} \mathbf{e} & \text{if } \beta = (0, 0), \\ 0 & \text{if } \beta \neq (0, 0), \end{cases} \\ f_{k,\beta}(\dots, \mathbf{e}, \dots) &= 0, \quad k > 1. \end{aligned}$$

(3) We say that a filtered algebra or homomorphism is *gapped* when it is G -gapped for some G , which we do not specify.

DEFINITION 8.3 ([18] DEFINITION 2.13)

Let $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}\})$, $(C', \langle \cdot, \cdot \rangle, \{\mathbf{m}'_{k,\beta}\})$ be G -gapped cyclic filtered A_∞ -algebras. A G -gapped filtered A_∞ -homomorphism $f = \{f_{k,\beta}\} : C \rightarrow C'$ is said to be *cyclic* if the following holds:

$$(8.3) \quad \langle f_{1,(0,0)}(x), f_{1,(0,0)}(y) \rangle = \langle x, y \rangle$$

for any x, y ;

$$(8.4) \quad \sum_{\beta_1 + \beta_2 = \beta} \sum_{k_1 + k_2 = k} \langle f_{k_1,\beta_1}(x_1, \dots, x_{k_1}), f_{k_2,\beta_2}(x_{k_1+1}, \dots, x_k) \rangle = 0$$

holds for $(k, \beta) \neq (2, (0, 0))$ and x_1, \dots, x_k .

When (8.4) holds only for $E(\beta) < E_0$, we say that it is *cyclic modulo T^{E_0}* .

We define the composition of a filtered A_∞ -homomorphism as in [11, Definition 3.2.31]. It is easy to see that compositions of cyclic filtered A_∞ -homomorphisms are cyclic.

REMARK 8.1

(1) We say that a (cyclic) filtered A_∞ -homomorphism $\{f_{k,\beta}\}$ is a *weak homotopy equivalence* if $f_{1,(0,0)}$ induces an isomorphism on $\mathfrak{m}_{1,(0,0)}$ -cohomology.

(2) We say that a filtered A_∞ -homomorphism $\{f_{k,\beta}\}$ is an *isomorphism* if it has an inverse, that is, a filtered A_∞ -homomorphism $\{g_{k,\beta}\}$ such that the compositions of them are identity. (An identity morphism $\{h_{k,\beta}\}$ is defined by $h_{k,\beta} = 0$ for $(k, \beta) \neq (1, (0, 0))$ and $h_{1,(0,0)} = \text{identity}$.) It is easy to see that an inverse of a cyclic filtered A_∞ -homomorphism is automatically cyclic.

We next define and study the properties of pseudoisotopy of cyclic filtered A_∞ -algebras. Let \overline{C} be a graded \mathbb{R} vector space. We take a basis \mathbf{e}_i of \overline{C}^k and define $C^\infty([0, 1], \overline{C}^k)$ to be a finite sum

$$\sum_i a_i(t) \mathbf{e}_i$$

such that $a_i : [0, 1] \rightarrow \mathbb{R}$ are smooth functions.

REMARK 8.2

In the case $\overline{C} = \Lambda(L)$, the de Rham complex, we need to take into account the

Fréchet topology of $\Lambda(L)$ and define $C^\infty([0, 1], \overline{C})$ in a different way (see Section 11).

We consider the set of formal sums

$$(8.5) \quad a(t) + dt \wedge b(t),$$

where $a(t) \in C^\infty([0, 1], \overline{C}^k)$, $b(t) \in C^\infty([0, 1], \overline{C}^{k-1})$. We write the totality of such (8.5) as $C^\infty([0, 1] \times \overline{C})^k$. We consider a filtered A_∞ -structure on it. More precisely, we proceed as follows.

We assume that, for each $t \in [0, 1]$, we have operations

$$(8.6) \quad \mathbf{m}_{k,\beta}^t : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

of degree $-\mu(\beta) + 1$ and

$$(8.7) \quad \mathbf{c}_{k,\beta}^t : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

of degree $-\mu(\beta)$.

DEFINITION 8.4

We say $\mathbf{m}_{k,\beta}^t$ is *smooth* if for each x_1, \dots, x_k ,

$$t \mapsto \mathbf{m}_{k,\beta}^t(x_1, \dots, x_k)$$

is an element of $C^\infty([0, 1], \overline{C})$.

The smoothness of $\mathbf{c}_{k,\beta}^t$ is defined in the same way.

DEFINITION 8.5

We say that $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ is a *pseudoisotopy* of G -gapped cyclic filtered A_∞ -algebras if the following hold.

- (1) $\mathbf{m}_{k,\beta}^t$ and $\mathbf{c}_{k,\beta}^t$ are smooth.
- (2) For each (but fixed) t , the triple $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\})$ defines a cyclic filtered A_∞ -algebra.
- (3) For each (but fixed) t , and $x_i \in \overline{C}[1]$, we have

$$(8.8) \quad \langle \mathbf{c}_{k,\beta}^t(x_1, \dots, x_k), x_0 \rangle = (-1)^* \langle \mathbf{c}_{k,\beta}^t(x_0, x_1, \dots, x_{k-1}), x_k \rangle,$$

$* = (\deg \mathbf{x}_0 + 1)(\deg \mathbf{x}_1 + \dots + \deg \mathbf{x}_k + k)$.

- (4) For each $x_i \in \overline{C}[1]$,

$$(8.9) \quad \begin{aligned} & \frac{d}{dt} \mathbf{m}_{k,\beta}^t(x_1, \dots, x_k) \\ & + \sum_{k_1+k_2=k} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} (-1)^* \mathbf{c}_{k_1,\beta_1}^t(x_1, \dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots, x_k) \\ & - \sum_{k_1+k_2=k} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} \mathbf{m}_{k_1,\beta_1}^t(x_1, \dots, \mathbf{c}_{k_2,\beta_2}^t(x_i, \dots), \dots, x_k) \\ & = 0. \end{aligned}$$

Here $*$ = $\text{deg}' x_1 + \cdots + \text{deg}' x_{i-1}$.

(5) $\mathbf{m}_{k,(0,0)}^t$ is independent of t . $\mathbf{c}_{k,(0,0)}^t = 0$.

REMARK 8.3

Condition (5) above may be a bit more restrictive than the optional definition. We assume it here since it suffices for the purpose of this article.

We consider $x_i(t) + dt \wedge y_i(t) = \mathbf{x}_i \in C^\infty([0, 1], \overline{C})$. We define

$$\hat{\mathbf{m}}_{k,\beta}(\mathbf{x}_1, \dots, \mathbf{x}_k) = x(t) + dt \wedge y(t),$$

where

$$(8.10a) \quad x(t) = \mathbf{m}_{k,\beta}^t(x_1(t), \dots, x_k(t)),$$

$$(8.10b) \quad y(t) = \mathbf{c}_{k,\beta}^t(x_1(t), \dots, x_k(t)) - \sum_{i=1}^k (-1)^{*i} \mathbf{m}_{k,\beta}^t(x_1(t), \dots, x_{i-1}(t), y_i(t), x_{i+1}(t), \dots, x_k(t))$$

if $(k, \beta) \neq (1, (0, 0))$ and

$$(8.10c) \quad y(t) = \frac{d}{dt}x_1(t) + \mathbf{m}_{1,(0,0)}^t(y_1(t))$$

if $(k, \beta) = (1, (0, 0))$. Here $*i$ in (8.10b) is $*i = \text{deg}' x_1 + \cdots + \text{deg}' x_{i-1}$.

LEMMA 8.1

Equation (8.9) is equivalent to the filtered A_∞ -relation of $\hat{\mathbf{m}}_{k,\beta}$ defined by (8.10).

The proof is straightforward and is omitted (see [11, Lemma 4.2.13]). We define $\langle \cdot \rangle_{t_0}$ on $C^\infty([0, 1], \overline{C})$ by

$$\langle x_1(t) + dt \wedge y_1(t), x_2(t) + dt \wedge y_2(t) \rangle_{t_0} = \langle x_1(t_0), x_2(t_0) \rangle.$$

Then $(C^\infty([0, 1], \overline{C}), \langle \cdot \rangle_{t_0}, \{\hat{\mathbf{m}}_{k,\beta}\})$ is a cyclic filtered A_∞ -algebra for any t_0 .

DEFINITION 8.6

A pseudoisotopy $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ is said to be *unital* if there exists $\mathbf{e} \in \overline{C}^0$ such that \mathbf{e} is a unit of $(C, \{\mathbf{m}_{k,\beta}^t\})$ for each t and if

$$\mathbf{c}_{k,\beta}^t(\dots, \mathbf{e}, \dots) = 0$$

for each k, β , and t .

DEFINITION 8.7

Let $(\overline{C}, \langle \cdot \rangle, \{\mathbf{m}_{k,(0,0)}\})$ be a (unfiltered) cyclic A_∞ -algebra. We consider two G -gapped filtered cyclic A_∞ -algebras $(\overline{C}, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^i\})$ ($i = 0, 1$) such that $\mathbf{m}_{k,(0,0)}^i = \mathbf{m}_{k,(0,0)}$ for $i = 0, 1$.

We say that $(\overline{C}, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^0\})$ is *pseudoisotopic* to $(\overline{C}, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^1\})$ if there exists a pseudoisotopy $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ with given boundary value at $t = 0, 1$.

The modulo T^{E_0} and/or unital version is defined in a similar way.

LEMMA 8.2

Pseudoisotopy of G -gapped filtered cyclic A_∞ -algebras is an equivalence relation.

The modulo T^{E_0} version also holds.

Proof

Let $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ be a pseudoisotopy. First, we show that we can modify them so that $\mathbf{m}_{k,\beta}^t$ is locally constant of t and that $\mathbf{c}_{k,\beta}^t = 0$, in a neighborhood of $t \in \partial[0, 1]$, as follows. Let $t = t(s)$ be a smooth map $[0, 1] \rightarrow [0, 1]$ which is constant in a neighborhood of $0, 1$, and let $t(1) = 1, t(0) = 0$. We put

$$(8.11) \quad \mathbf{m}_{k,\beta}^s = \mathbf{m}_{k,\beta}^{t(s)}, \quad \mathbf{c}_{k,\beta}^s = \frac{dt}{ds}(s) \cdot \mathbf{c}_{k,\beta}^{t(s)}.$$

It is easy to see that $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^s\}, \{\mathbf{c}_{k,\beta}^s\})$ is a required pseudoisotopy.

Now we can easily join two pseudoisotopies satisfying the above additional condition. It implies that the pseudoisotopy relation is transitive. The other properties are easier to check. □

THEOREM 8.1

Let $E_0 < E_1$ and $(\overline{C}, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^i\})$ ($i = 0, 1$) be G -gapped cyclic filtered A_∞ -algebras modulo T^{E_i} . Let $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ be a pseudoisotopy modulo T^{E_0} between them. Then

- (1) *we can extend $(\overline{C}, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^0\})$ to a G -gapped cyclic filtered A_∞ -algebra modulo T^{E_1} ;*
- (2) *we can extend $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ to a pseudoisotopy modulo T^{E_1} between them.*

The unital version also holds.

The proof is given in Section 9.

THEOREM 8.2

If $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ is a pseudoisotopy, then there exists a filtered A_∞ -homomorphism from $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^0\})$ to $(C, \langle \cdot \rangle, \{\mathbf{m}_{k,\beta}^1\})$ which is cyclic and has an inverse.

The modulo T^E and/or unital version also holds.

The proof is given in Section 9.

REMARK 8.4

It is possible to prove that a gapped cyclic filtered A_∞ -homomorphism which is homotopy equivalence (as a gapped filtered A_∞ -homomorphism) has a homo-

topy inverse that is cyclic (see [18, Theorem 5.17]). The explicit construction of homotopy inverse given in [1] proves it also.

One reason why we built our story without using this theorem but used pseudoisotopy more is that if a filtered A_∞ -algebra C is pseudoisotopic to C' , then C is homotopy equivalent to C' . But the converse may not hold. (An invariant of a kind of “Reidemeister torsion” may distinguish them.) So for future application (especially the one in [8]) to keep track of pseudoisotopy type rather than homotopy type seems essential.

This does not seem to be the case when we do not include cyclic symmetry and inner product in the story.

Let $(C, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}\})$ be a G -gapped cyclic filtered A_∞ -algebra. We have $\mathfrak{m}_{1,(0,0)} \circ \mathfrak{m}_{1,(0,0)} = 0 : \overline{C} \rightarrow \overline{C}$. We put

$$\overline{H} = \frac{\text{Ker } \mathfrak{m}_{1,(0,0)}}{\text{Im } \mathfrak{m}_{1,(0,0)}}.$$

In [11, Theorem 5.4.2’], a G -gapped filtered A_∞ -structure $\{\mathfrak{m}_{k,\beta}^{\text{can}}\}$ on H is defined. Moreover, a G -gapped filtered A_∞ -homomorphism $\mathfrak{f} : H \rightarrow C$ (which is a homotopy equivalence) is defined. By (6.7), the inner product $\langle \cdot, \cdot \rangle$ on \overline{C} induces one on \overline{H} , which we denote also by $\langle \cdot, \cdot \rangle$.

THEOREM 8.3

We assume that \overline{C} is either finite-dimensional or is a de Rham complex. Then $(H, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{\text{can}}\})$ is cyclic. Moreover, $\mathfrak{f} : H \rightarrow C$ is cyclic.

The modulo T^{E_0} and/or unital version is also true.

We prove Theorem 8.3 in Section 10.

DEFINITION 8.8

We call $(H, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{\text{can}}\})$ the *canonical model* of the cyclic filtered A_∞ -algebra $(C, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}\})$.

Weak homotopy equivalence between (cyclic) canonical filtered A_∞ -algebras is an isomorphism (see [11, Proposition 5.4.5]).

THEOREM 8.4

If $(C, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^0\})$ is pseudoisotopic to $(C, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^1\})$, then their canonical models are also pseudoisotopic to each other.

We prove Theorem 8.4 in Section 10.

9. Pseudoisotopy of cyclic filtered A_∞ -algebras

In this section we prove Theorems 8.1 and 8.2. We begin with the proof of Theorem 8.2. We construct the required isomorphism by taking an appropriate sum over trees with some additional data, which we describe below.

A *ribbon tree* is a tree T together with isotopy type of an embedding $T \rightarrow \mathbb{R}^2$. (This is equivalent to fixing the cyclic order of the set of edges containing a given vertex.) A *rooted ribbon tree* is a pair (T, v_0) of a ribbon tree T and its vertex v_0 such that v_0 has exactly one edge. Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a discrete submonoid. We consider the triple $\Gamma = (T, v_0, \beta(\cdot))$ together with some other data that has the following properties.

- (1) (T, v_0) is a rooted ribbon tree.
- (2) The set of vertices $C_0(T)$ is divided into the disjoint union $C_0^{\text{int}}(T) \cup C_0^{\text{ext}}(T)$, $v_0 \in C_0^{\text{ext}}(T)$. Each of $v \in C_0^{\text{ext}}(T)$ has exactly one edge.
- (3) $\beta(\cdot) : C_0^{\text{int}}(T) \rightarrow G$ is a map.
- (4) If $\beta(v) = (0, 0)$, then v has at least three edges.

DEFINITION 9.1

We write $\text{Gr}(\beta, k)$ as the set of all $(T, v_0, \beta(\cdot))$ as above such that

- (1) $\sum_{v \in C_0^{\text{int}}(T)} \beta(v) = \beta$;
- (2) $\#C_0^{\text{ext}}(T) = k + 1$.

We call an element of $C_0^{\text{ext}}(T)$ an *exterior vertex* and an element of $C_0^{\text{int}}(T)$ an *interior vertex*. An edge is said to be *exterior* if it contains an exterior edge. It is called *interior* otherwise. The set of exterior edges and interior edges are denoted by $C_1^{\text{ext}}(T)$ and $C_1^{\text{int}}(T)$, respectively.

We call v_0 the *root* of (T, v_0) .

DEFINITION 9.2 (SEE [11, DEFINITION 4.6.6])

For a rooted ribbon tree (T, v_0) , we define a partial order $<$ on $C_0(T)$ as follows.

We have $v < v'$ if all the paths joining v with v_0 contain v' .

DEFINITION 9.3 (SEE [11, DEFINITION 7.1.53])

The *time allocation* of an element $(T, v_0, \beta(\cdot)) \in \text{Gr}(\beta, k)$ is a map $\tau : C_0^{\text{int}}(T) \rightarrow [0, 1]$ such that if $v < v'$, then $\tau(v) \leq \tau(v')$.

Let $0 \leq \tau_b \leq \tau_a \leq 1$. We denote by $\mathfrak{M}(T, v_0, \beta(\cdot); \tau_a, \tau_b)$ the set of all time allocations τ such that $\tau(v) \in [\tau_b, \tau_a]$ for all v . We write $\mathfrak{M}(T, v_0, \beta(\cdot)) = \mathfrak{M}(T, v_0, \beta(\cdot); 1, 0)$.

We may regard

$$(9.1) \quad \mathfrak{M}(T, v_0, \beta(\cdot)) \subseteq [0, 1]^{\#C_0^{\text{int}}(T)}.$$

For $(T, v_0, \beta(\cdot)) \in \text{Gr}(\beta, k)$ and $\tau \in \mathfrak{M}(T, v_0, \beta(\cdot))$, we associate an \mathbb{R} -linear map

$$\mathfrak{c}(T, v_0, \beta(\cdot), \tau) : B_k(\overline{\mathcal{C}}[1]) \rightarrow \overline{\mathcal{C}}[1]$$

of degree $-\mu(\beta)$ by induction on $\#C_0^{\text{int}}(T)$ as follows.

Suppose $\#C_0^{\text{int}}(T) = 0$. Then T has only one edge and two (exterior) vertices. So $\beta(\cdot)$ is void. We put

$$(9.2) \quad \mathbf{c}(T, v_0, \beta(\cdot), \tau) = \text{identity}$$

in this case.

Suppose $\#C_0^{\text{int}}(T) = 1$. Let v be the unique interior vertex, and let $\beta = \beta(v)$. The vertex v has exactly $k + 1$ edges. We put

$$(9.3) \quad \mathbf{c}(T, v_0, \beta(\cdot), \tau)(x_1, \dots, x_k) = -\mathbf{c}_{k, \beta}^{\tau(v)}(x_1, \dots, x_k).$$

(Note that $\tau(v) \in [0, 1]$ and that $\mathbf{c}^t(\dots)$ for $t \in [0, 1]$ is defined as in (8.7).)

Let $\#C_0^{\text{int}}(T) > 1$. We take the unique edge e_0 containing v_0 . Let v'_0 be the vertex of e_0 other than v_0 ; v'_0 is necessarily interior. We remove v_0, e_0 , and v'_0 from T and then obtain ℓ components T_1, \dots, T_ℓ . Here $\ell + 1$ is the number of edges of v'_0 . We number them so that v'_0, T_1, \dots, T_ℓ , respects the counterclockwise cyclic order induced by the canonical orientation of \mathbb{R}^2 . We take the closures of T_i and denote it by the same symbol by an abuse of notation. Together with the other data that is induced in an obvious way from one of $(T, v_0, \beta(\cdot), \tau)$, we obtain $(T_i, v'_0, \beta_i(\cdot), \tau_i)$ for $i = 1, \dots, \ell$. We now put

$$(9.4) \quad \begin{aligned} &\mathbf{c}(T, v_0, \beta(\cdot), \tau) \\ &= -\mathbf{c}_{\ell, \beta(v'_0)}^{\tau(v'_0)} \circ (\mathbf{c}(T_1, v'_0, \beta_1(\cdot), \tau_1) \otimes \dots \otimes \mathbf{c}(T_\ell, v'_0, \beta_\ell(\cdot), \tau_\ell)). \end{aligned}$$

Note that the right-hand side is already defined by induction hypothesis.

Now we integrate on $\mathfrak{M}(T, v_0, \beta(\cdot))$ and define

$$(9.5) \quad \mathbf{c}(T, v_0, \beta(\cdot)) = \int_{\tau \in \mathfrak{M}(T, v_0, \beta(\cdot))} \mathbf{c}(T, v_0, \beta(\cdot), \tau) d\tau.$$

Here we regard $\mathfrak{M}(T, v_0, \beta(\cdot)) \subset [0, 1]^{\#C_0^{\text{int}}(T)}$ and use standard the measure $d\tau$ to integrate. We define $\mathbf{c}(T, v_0, \beta(\cdot); \tau_a, \tau_b)$ in the same way by integrating on $\mathfrak{M}(T, v_0, \beta(\cdot); \tau_a, \tau_b)$.

DEFINITION 9.4

We have

$$(9.6) \quad \mathbf{c}(k, \beta) = \sum_{(T, v_0, \beta(\cdot)) \in \text{Gr}(\beta, k)} \mathbf{c}(T, v_0, \beta(\cdot)).$$

We define $\mathbf{c}(k, \beta; \tau_a, \tau_b)$ in a similar way.

PROPOSITION 9.1

The system of maps $\{\mathbf{c}(k, \beta; \tau_a, \tau_b)\}$ defines a G -gapped filtered A_∞ -homomorphism from $(C, \{\mathbf{m}_{k, \beta}^{\tau_b}\})$ to $(C, \{\mathbf{m}_{k, \beta}^{\tau_a}\})$.

Proof

Let $E(G) = \{0, E_1, \dots, E_k, \dots\}$ with $E_i < E_{i+1}$. We prove that $\{\mathbf{c}(k, \beta; \tau_a, \tau_b)\}$ is a filtered A_∞ -homomorphism modulo E_j by induction on j .

We remark that

$$(9.7) \quad \mathbf{c}(k, (0, 0)) = \begin{cases} \text{identity} & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

by Definition 8.5.5. The case $j = 0 + 1 = 1$ follows immediately.

We assume that $\{\mathbf{c}(k, \beta; \tau, \tau_b)\}$ is a filtered A_∞ -homomorphism modulo E_j . Let $E(\beta) = E_j$. We study the two maps (9.8) and (9.9):

$$(9.8) \quad \sum \mathbf{m}_{\ell, \beta_0}^\tau \circ (\mathbf{c}(k_1, \beta_1; \tau, \tau_b) \otimes \dots \otimes \mathbf{c}(k_\ell, \beta_\ell; \tau, \tau_b)),$$

where the sum is taken over all ℓ, k_i, β_i with $\beta_0 + \beta_1 + \dots + \beta_\ell = \beta, k_1 + \dots + k_\ell = k$;

$$(9.9) \quad x_1 \otimes \dots \otimes x_k \mapsto \sum (-1)^* \mathbf{c}(\beta_1, k_1; \tau, \tau_b)(x_1, \dots, \mathbf{m}_{k_2, \beta_2}^{\tau_b}(x_i, \dots), \dots, x_k),$$

where the sum is taken over $k_1, k_2, \beta_1, \beta_2, i$ with $k_1 + k_2 = k + 1, \beta_1 + \beta_2 = \beta$, and $i = 1, \dots, k - k_2 + 1$ and $* = \text{deg}' x_1 + \dots + \text{deg}' x_{i-1}$.

We denote (9.8) by $\mathfrak{P}(k, \beta; \tau, \tau_b)$ and (9.9) by $\mathfrak{Q}(k, \beta; \tau, \tau_b)$. To prove Proposition 9.1 it suffices to show that $\mathfrak{P}(k, \beta; \tau, \tau_b) = \mathfrak{Q}(k, \beta; \tau, \tau_b)$.

We calculate

$$(9.10) \quad \begin{aligned} & \frac{d}{d\tau} \mathfrak{P}(k, \beta; \tau, \tau_b) \\ &= \sum \left(\frac{d}{d\tau} \mathbf{m}_{\ell, \beta_b}^\tau \right) \circ (\mathbf{c}(k_1, \beta_1; \tau, \tau_b) \otimes \dots \otimes \mathbf{c}(k_\ell, \beta_\ell; \tau, \tau_b)) \\ & \quad + \mathbf{m}_{\ell, \beta_0}^\tau \circ \frac{d}{d\tau} (\mathbf{c}(k_1, \beta_1; \tau, \tau_b) \otimes \dots \otimes \mathbf{c}(k_\ell, \beta_\ell; \tau, \tau_b)). \end{aligned}$$

By using (8.9), the first term of (9.10) becomes the sum of the following two formulas.

(1) We have the sum of the composition of

$$(9.11) \quad \mathbf{c}(k_1, \beta_1; \tau, \tau_b) \otimes \dots \otimes \mathbf{c}(k_\ell, \beta_\ell; \tau, \tau_b)$$

and

$$(9.12) \quad x_1 \otimes \dots \otimes x_\ell \mapsto (-1)^* \mathbf{c}_{\ell_1, \beta_0}^\tau (\dots, \mathbf{m}_{\ell_2, \beta'_0}^\tau(x_i, \dots), \dots).$$

Here the sum is taken over all $\beta_0, \beta'_0, \beta_1, \dots, \beta_\ell, \ell_1, \ell_2, k_1, \dots, k_\ell$ such that $\beta = \beta_0 + \beta'_0 + \beta_1 + \dots + \beta_\ell, \ell_1 + \ell_2 = \ell + 1, k_1 + \dots + k_\ell = k$. The sign is $* = \text{deg}' x_1 + \dots + \text{deg}' x_{i-1}$.

(2) We have the composition of (9.11) and

$$(9.13) \quad x_1 \otimes \dots \otimes x_\ell \mapsto -\mathbf{m}_{\ell_1, \beta_0}^\tau (\dots, \mathbf{c}_{\ell_2, \beta'_0}^\tau(x_i, \dots), \dots).$$

We remark that the minus sign in (9.13) is induced by the minus sign in the third line of (8.9).

Using the induction hypothesis, we can show that (1) above is equal to (9.9). (Note that the minus sign in (9.3), (9.4) is essential here.)

On the other hand, by definition we can show that (2) above cancels with the second term of (9.10). (We again use the minus sign in (9.3), (9.4) here.)

The proof of Proposition 9.1 is now complete. □

PROPOSITION 9.2

The filtered A_∞ -homomorphism $\{\mathbf{c}(k, \beta)\}$ is cyclic.

Proof

Let $(k, \beta) \neq (2, (0, 0))$. We prove

$$(9.14) \quad \sum_{\substack{k_1+k_2=k \\ \beta_1+\beta_2=\beta}} \langle \mathbf{c}(k_1, \beta_1)(x_1, \dots, x_{k_1}), \mathbf{c}(k_2, \beta_2)(x_{k_1+1}, \dots, x_k) \rangle = 0.$$

A term of (9.14) is written as

$$(9.15) \quad \int_{\tau \in \mathfrak{M}(\Gamma_1)} \int_{\tau' \in \mathfrak{M}(\Gamma_2)} \langle \mathbf{c}(\Gamma_1; \tau)(x_1, \dots, x_{k_1}), \mathbf{c}(\Gamma_2; \tau')(x_{k_1+1}, \dots, x_k) \rangle d\tau d\tau'.$$

Here $\Gamma_i = (T_i, v_0^i, \beta_i(\dots)) \in \text{Gr}(k_i, \beta_i)$ with $k_1 + k_2 = k$, $\beta_1 + \beta_2 = \beta$.

We put

$$\tau_{\max} = \max\{\tau(v) \mid v \in C_0^{\text{int}}(T_1)\} = \tau(v_0^{1'}).$$

Here $v_0^{1'}$ is the unique interior vertex that is joined with v_0^1 . We define

$$\tau'_{\max} = \max\{\tau'(v) \mid v \in C_0^{\text{int}}(T_2)\} = \tau'(v_0^{2'})$$

in the same way. We divide the domain of integration (9.15) into two:

- (1) $\tau_{\max} \geq \tau'_{\max}$;
- (2) $\tau_{\max} \leq \tau'_{\max}$.

Integration on the domain (1) is the sum of the terms

$$(9.16) \quad - \int_0^1 \langle (\mathbf{c}_{\ell, \beta_{(0)}}^t \circ (\mathbf{c}(\Gamma(1); t, \tau_b) \otimes \dots \otimes \mathbf{c}(\Gamma(\ell); t, \tau_b))) (x_1, \dots, x_{k_1}), \mathbf{c}(\Gamma(0); t, \tau_b)(x_{k_1+1}, \dots, x_k) \rangle dt.$$

Here $\Gamma(i) = (T_i, v_0^i, \beta^i(\dots)) \in \text{Gr}(k(i), \beta^i)$ such that $\sum_{i=1}^\ell k(i) = k_1$, $k(0) = k_2$, $k_1 + k_2 = k$, $\sum_{i=0}^\ell \beta^i + \beta_{(0)} = \beta$.

In a similar way, integration on the domain (2) is the sum of the terms

$$(9.17) \quad - \int_0^1 \langle \mathbf{c}(\Gamma(0); t, \tau_b)(x_1, \dots, x_{k_2}), (\mathbf{c}_{\ell, \beta_{(0)}}^t \circ (\mathbf{c}(\Gamma(1); t, \tau_b) \otimes \dots \otimes \mathbf{c}(\Gamma(\ell); t, \tau_b))) (x_{k_2+1}, \dots, x_k) \rangle dt.$$

Therefore (9.14) follows from the next lemma.

LEMMA 9.1

We have

$$(9.18) \quad \langle \mathbf{c}_{\ell,\beta}^t(x_1, \dots, x_\ell), x_0 \rangle + \langle x_1, \mathbf{c}_{\ell,\beta}^t(x_2, \dots, x_\ell, x_0) \rangle = 0.$$

Proof

We have

$$\begin{aligned} \langle \mathbf{c}_{\ell,\beta}^t(x_1, \dots, x_\ell), x_0 \rangle &= (-1)^{(\deg' x_1)(\sum_{i \neq 1} \deg' x_i)} \langle \mathbf{c}_{\ell,\beta}^t(x_2, \dots, x_\ell, x_0), x_1 \rangle \\ &= -\langle x_1, \mathbf{c}_{\ell,\beta}^t(x_2, \dots, x_\ell, x_0) \rangle. \end{aligned}$$

Here we use the cyclic symmetry of $\mathbf{c}_{\ell,\beta}^t$ in the first equality and (6.2) in the second equality. □

The proof of Proposition 9.2 is complete. □

Proof of Theorem 8.2

By Propositions 9.1 and 9.2, we obtain a cyclic filtered A_∞ -homomorphism $\{\mathbf{c}_{k,\beta}\}$. We remark that $\mathbf{c}_{1,(0,0)}$ is identity and $\mathbf{c}_{k,(0,0)} = 0$ for $k \neq 1$ by definition. We can use this fact to show that $\{\mathbf{c}_{k,\beta}\}$ has an inverse by induction on energy filtration. The proof of Theorem 8.2 is complete. □

Proof of Theorem 8.1

We may assume that $E(G) \cap [E_0, E_1] = \{E_0, E_1\}$. (In the general case we can divide the interval $[E_0, E_1]$ into the pieces so that the above assumption holds.)

We use the modulo T^{E_0} version of Theorem 8.2 we proved above and obtain a cyclic filtered A_∞ -homomorphism $\{\mathbf{c}_{k,\beta}\}$ modulo T^{E_0} .

Let $E(\beta) > E_0$. We put $\mathbf{c}_{k,\beta}^t = 0$. We then define $\mathbf{m}_{k,\beta}^t$ by solving (8.9). Namely, we put

$$\begin{aligned} (9.19) \quad & \mathbf{m}_{k,\beta}^T(x_1, \dots, x_k) \\ &= \mathbf{m}_{k,\beta}^1(x_1, \dots, x_k) \\ & - \sum_{\substack{k_1+k_2=k \\ \beta_1+\beta_2=\beta}} \sum_{i=1}^{k-k_2+1} (-1)^{*i} \int_{\tau}^1 \mathbf{c}_{k_1,\beta_1}^t(\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots) dt \\ & + \sum_{\substack{k_1+k_2=k \\ \beta_1+\beta_2=\beta}} \sum_{i=1}^{k-k_2+1} \int_{\tau}^1 \mathbf{m}_{k_1,\beta_1}^t(\dots, \mathbf{c}_{k_2,\beta_2}^t(x_i, \dots), \dots) dt. \end{aligned}$$

Here $*i = \deg' x_1 + \dots + \deg' x_{i-1}$.

We remark that if $\mathbf{c}_{k,\beta}^t \neq 0$, then $E(\beta) > 0$. Therefore the right-hand side of (9.19) is already defined by the induction hypothesis.

Definitions 8.5(1), (3), (4), (5) are obvious.

LEMMA 9.2

The operators $\mathbf{m}_{k,\beta}^\tau$ in (9.19) satisfies the filtered A_∞ -relation (6.7).

Proof

We remark that $\mathbf{m}_{k,\beta}^1$ satisfies (6.7) by assumption. We prove (6.7) by induction on $E(\beta)$. Since $\mathbf{m}_{k,(0,0)}^\tau$ is independent of τ , (6.7) holds for $E(\beta) = 0$. We assume that it is satisfied for β' with $E(\beta') < E(\beta)$ and consider the case of β . We calculate

$$\begin{aligned} & \frac{d}{dt} \left(\sum (-1)^{*i} \mathbf{m}_{k_1,\beta_1}^t (\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots) \right) \\ &= \sum (-1)^{*i,j} \mathbf{c}_{k_1,\beta_1}^t (\dots, \mathbf{m}_{k_2,\beta_2}^t(x_j, \dots), \dots, \mathbf{m}_{k_3,\beta_3}^t(x_i, \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{c}_{k_1,\beta_1}^t (\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots, \mathbf{m}_{k_3,\beta_3}^t(x_j, \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{c}_{k_1,\beta_1}^t (\dots, \mathbf{m}_{k_2,\beta_2}^t(x_j, \dots, \mathbf{m}_{k_3,\beta_3}^t(x_i, \dots), \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{m}_{k_1,\beta_1}^t (\dots, \mathbf{c}_{k_2,\beta_2}^t(x_j, \dots), \dots, \mathbf{m}_{k_3,\beta_3}^t(x_i, \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{m}_{k_1,\beta_1}^t (\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots, \mathbf{c}_{k_3,\beta_3}^t(x_j, \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{m}_{k_1,\beta_1}^t (\dots, \mathbf{c}_{k_2,\beta_2}^t(x_j, \dots, \mathbf{m}_{k_3,\beta_3}^t(x_i, \dots), \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{m}_{k_1,\beta_1}^t (\dots, \mathbf{c}_{k_2,\beta_2}^t(x_i, \dots, \mathbf{m}_{k_3,\beta_3}^t(x_j, \dots), \dots), \dots) \\ & \quad + \sum (-1)^{*i,j} \mathbf{m}_{k_1,\beta_1}^t (\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots, \mathbf{c}_{k_3,\beta_3}^t(x_j, \dots), \dots), \dots). \end{aligned}$$

Here the first six terms are obtained by differentiating $\mathbf{m}_{k_1,\beta_1}^t$ and the last two terms are obtained by differentiating $\mathbf{m}_{k_2,\beta_2}^t$. The signs are given by

$$\begin{aligned} *^1_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + \text{deg}' x_1 + \dots + \text{deg}' x_{j-1}, \\ *^2_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + \text{deg}' x_1 + \dots + \text{deg}' x_{j-1} + 1, \\ *^3_{i,j} &= \text{deg}' x_j + \dots + \text{deg}' x_{i-1}, \\ *^4_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + 1, \\ *^5_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + 1, \\ *^6_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + 1, \\ *^7_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + \text{deg}' x_1 + \dots + \text{deg}' x_{j-1}, \\ *^8_{i,j} &= \text{deg}' x_1 + \dots + \text{deg}' x_{i-1} + 1. \end{aligned}$$

Now the first and second terms cancel. The third term is zero by the induction hypothesis (A_∞ -relation for \mathbf{m} ; we remark that $\mathbf{c}_{k,\beta} \neq 0$ only if $E(\beta) > 0$). The sum of the fourth, fifth, and eighth terms are zero by the induction hypothesis also. The sixth and seventh terms cancel. The proof of Lemma 9.2 is now complete. □

LEMMA 9.3

The operators $\mathbf{m}_{k,\beta}^{\tau}$ is cyclically symmetric.

Proof

We consider the following formulas:

$$(9.20) \quad \sum_{\substack{k_1+k_2=k \\ \beta_1+\beta_2=\beta}} \sum_{i=1}^{k-k_2+1} (-1)^{*i+1} \langle \mathbf{c}_{k_1,\beta_1}^t(\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots), x_0 \rangle,$$

where $*i = \deg' \rho_1 + \dots + \deg' \rho_{i-1}$, and

$$(9.21) \quad \sum_{\substack{k_1+k_2=k \\ \beta_1+\beta_2=\beta}} \sum_{i=1}^{k-k_2+1} \langle \mathbf{m}_{k_1,\beta_1}^t(\dots, \mathbf{c}_{k_2,\beta_2}^t(x_i, \dots), \dots), x_0 \rangle.$$

We denote (9.20) as $\mathfrak{P}(x_1, \dots, x_k, x_0)$ and (9.21) as $\mathfrak{Q}(x_1, \dots, x_k, x_0)$.

We prove

$$(9.22) \quad \begin{aligned} & \mathfrak{P}(x_0, x_1, \dots, x_k) + \mathfrak{Q}(x_0, x_1, \dots, x_k) \\ &= (-1)^{(\deg' x_0)(\deg' x_1 + \dots + \deg' x_k)} (\mathfrak{P}(x_1, \dots, x_k, x_0) + \mathfrak{Q}(x_1, \dots, x_k, x_0)). \end{aligned}$$

We have

$$(9.23) \quad \begin{aligned} & \mathfrak{P}(x_0, x_1, \dots, x_k) \\ &= - \sum \langle \mathbf{c}_{k_1,\beta_1}^t(\mathbf{m}_{k_2,\beta_2}^t(x_0, \dots), \dots), x_k \rangle \\ & \quad + \sum (-1)^{\deg' x_0 + *i+1} \langle \mathbf{c}_{k_1,\beta_1}^t(x_0, \dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots), x_k \rangle \end{aligned}$$

and

$$(9.24) \quad \begin{aligned} & \mathfrak{Q}(x_0, x_1, \dots, x_k) \\ &= \sum \langle \mathbf{m}_{k_1,\beta_1}^t(\mathbf{c}_{k_2,\beta_2}^t(x_0, \dots), \dots), x_k \rangle \\ & \quad + \sum \langle \mathbf{m}_{k_1,\beta_1}^t(x_0, \dots, \mathbf{c}_{k_2,\beta_2}^t(\dots), \dots), x_k \rangle. \end{aligned}$$

Moreover,

$$(9.25) \quad \begin{aligned} & (-1)^{(\deg' x_0)(\deg' x_1 + \dots + \deg' x_k)} \mathfrak{P}(x_1, \dots, x_k, x_0) \\ &= \sum (-1)^{*+1+*i} \langle \mathbf{c}_{k_1,\beta_1}^t(\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots), \dots, x_k), x_0 \rangle \\ & \quad + \sum (-1)^{*+*i+1} \langle \mathbf{c}_{k_1,\beta_1}^t(\dots, \mathbf{m}_{k_2,\beta_2}^t(x_i, \dots, x_k)), x_0 \rangle \end{aligned}$$

with $* = (\deg' x_0)(\deg' x_1 + \dots + \deg' x_k)$ and

$$(9.26) \quad \begin{aligned} & (-1)^{(\deg' x_0)(\deg' x_1 + \dots + \deg' x_k)} \mathfrak{Q}(x_1, \dots, x_k, x_0) \\ &= \sum (-1)^* \langle \mathbf{m}_{k_1,\beta_1}^t(\dots, \mathbf{c}_{k_2,\beta_2}^t(x_i, \dots), \dots, x_k), x_0 \rangle \\ & \quad + \sum (-1)^* \langle \mathbf{m}_{k_1,\beta_1}^t(\dots, \mathbf{c}_{k_2,\beta_2}^t(x_i, \dots, x_k)), x_0 \rangle. \end{aligned}$$

The second term of (9.23) coincides with first term of (9.25) by the cyclic symmetry of \mathfrak{c}^t . The second term of (9.24) coincides with first term of (9.26) by the cyclic symmetry of \mathfrak{m}^t .

The first term of (9.23) coincides with the second term of (9.26). In fact,

$$\begin{aligned} & -\langle \mathfrak{c}_{k_1, \beta_1}^t(\mathfrak{m}_{k_2, \beta_2}^t(x_0, \dots, x_{i-1}), \dots), x_k \rangle \\ &= (-1)^{1+(\ast_i+1)(\deg' x_i + \dots + \deg' x_k)} \langle \mathfrak{c}_{k_1, \beta_1}^t(x_i, \dots, x_k), \mathfrak{m}_{k_2, \beta_2}^t(x_0, \dots, x_{i-1}) \rangle \\ &= \langle \mathfrak{m}_{k_2, \beta_2}^t(x_0, \dots, x_{i-1}), \mathfrak{c}_{k_1, \beta_1}^t(x_i, \dots, x_k) \rangle, \end{aligned}$$

which is equal to the second term of (9.26).

In the same way, the first term of (9.24) coincides with the second term of (9.25). We thus proved (9.22).

Lemma 9.3 now follows from (9.22) and (9.19). □

Theorem 8.1 follows from Lemmas 9.2 and 9.3. □

10. Canonical model of cyclic filtered A_∞ -algebras

In this section we prove Theorems 8.3 and 8.4. We first review the construction of the filtered A_∞ -structure $\mathfrak{m}_{k, \beta}^{\text{can}}$ on H and the filtered A_∞ -homomorphism $\{f_{k, \beta}\} : H \rightarrow C$ from [11, Section 5.4.4].

We consider the chain complex $(\overline{C}, \mathfrak{m}_{1, (0,0)})$ together with its inner product. We take an \mathbb{R} -linear subspace $\overline{H} \subset \overline{C}$ such that $\mathfrak{m}_{1, (0,0)} = 0$ on \overline{H} and \overline{H} is identified with the $\mathfrak{m}_{1, (0,0)}$ -cohomology by an obvious map. In the case when \overline{C} is the de Rham complex of L , we take a Riemannian metric on L and let \overline{H} be the space of harmonic forms.

LEMMA 10.1 (CF. [18, SECTION 5.1])

We assume that \overline{C} either is finite-dimensional or is a de Rham complex. There exists a map $\Pi : \overline{C} \rightarrow \overline{C}$ of degree zero and $G : \overline{C} \rightarrow \overline{C}$ of degree +1 with the following properties:

- (1) $\Pi \circ \Pi = \Pi$: *The image of Π is a \overline{H} ;*
- (2)
- (10.1) $\text{identity} - \Pi = -(\mathfrak{m}_{1, (0,0)} \circ G + G \circ \mathfrak{m}_{1, (0,0)})$;
- (3) $G \circ G = 0$;
- (4) $\langle \Pi x, y \rangle = \langle x, \Pi y \rangle$;
- (5)
- (10.2) $\langle x, Gy \rangle = (-1)^{\deg x \deg y} \langle y, Gx \rangle$.

Proof

We first assume that \overline{C} is finite-dimensional. We remark that

$$(10.3) \quad \langle \mathfrak{m}_{1, (0,0)} x, y \rangle = (-1)^{\deg' x \deg' y + 1} \langle \mathfrak{m}_{1, (0,0)} y, x \rangle.$$

We put $\overline{B} = \text{Im } \mathfrak{m}_{1,(0,0)}$. Equation (10.3) implies $\langle \overline{B}, \overline{H} \rangle = 0$, $\langle \overline{B}, \overline{B} \rangle = 0$. We put

$$\overline{C}' = \{x \in \overline{C} \mid \langle x, \overline{H} \rangle = 0\}.$$

Since $\langle \cdot \rangle$ is nondegenerate on \overline{H} , it follows that it is nondegenerate also on \overline{C}' . By an easy linear algebra, we can find $\overline{D} \subset \overline{C}'$ such that $\overline{B} \oplus \overline{D} = \overline{C}'$ and $\langle \overline{D}, \overline{D} \rangle = 0$. We thus have a decomposition

$$(10.4) \quad \overline{C} = \overline{B} \oplus \overline{D} \oplus \overline{H}.$$

We use this decomposition to define projections Π_B, Π_D, Π to $\overline{B}, \overline{D}, \overline{H}$, respectively. We have

$$(10.5) \quad \langle \Pi x, y \rangle = \langle x, \Pi y \rangle, \quad \langle \Pi_B x, y \rangle = \langle x, \Pi_D y \rangle, \quad \langle \Pi_D x, y \rangle = \langle x, \Pi_B y \rangle.$$

Thus we have (1) and (4).

By construction, the restriction of $\mathfrak{m}_{1,(0,0)}$ to \overline{D} induces an isomorphism $:\overline{D} \rightarrow \overline{B}$. Let \mathfrak{n} be its inverse. We put

$$(10.6) \quad G = -\mathfrak{n} \circ \Pi_B = -\Pi_D \circ \mathfrak{n} \circ \Pi_B.$$

It is easy to check (2) and (3). We prove (5). We first show that

$$(10.7) \quad \langle x, \mathfrak{n}(y) \rangle = (-1)^{\text{deg } x \text{ deg } y} \langle y, \mathfrak{n}(x) \rangle.$$

To prove (10.7) we may assume $x, y \in \overline{B}$. We put $x = \mathfrak{m}_{1,(0,0)} a$, $y = \mathfrak{m}_{1,(0,0)} b$. Then

$$\langle x, \mathfrak{n}(y) \rangle = \langle \mathfrak{m}_{1,(0,0)} a, b \rangle = (-1)^{\text{deg}' a \text{ deg}' b} \langle \mathfrak{m}_{1,(0,0)} b, a \rangle = (-1)^{\text{deg } x \text{ deg } y} \langle y, \mathfrak{n}(x) \rangle,$$

as required. Now we have

$$\begin{aligned} \langle x, G(y) \rangle &= -\langle x, \Pi_D \circ \mathfrak{n} \circ \Pi_B(y) \rangle \\ &= -\langle \Pi_B(x), \mathfrak{n}(\Pi_B(y)) \rangle \\ &= -(-1)^{\text{deg } x \text{ deg } y} \langle \Pi_B(y), \mathfrak{n}(\Pi_B(x)) \rangle \\ &= (-1)^{\text{deg } x \text{ deg } y} \langle y, G(x) \rangle. \end{aligned}$$

The proof of Lemma 10.1 is complete in the case when \overline{C} is finite-dimensional. In the case of the de Rham complex, we take δ the L^2 -conjugate to $\mathfrak{m}_{1,(0,0)}$ and let \overline{D} be the image of it. Equation (10.4) is nothing but the Hodge-Kodaira decomposition; (10.5) is well known. The rest of the proof is the same. \square

Let $\text{Gr}(\beta, k)$ be as in Section 9. For each $\Gamma = (T, v_0, \beta(\cdot)) \in \text{Gr}(\beta, k)$, we define

$$(10.8) \quad \mathfrak{f}_\Gamma : B_k(\overline{H}[1]) \rightarrow \overline{C}[1]$$

of degree $-\mu(\beta)$ and

$$(10.9) \quad \mathfrak{m}_\Gamma : B_k(\overline{H}[1]) \rightarrow \overline{H}[1]$$

of degree $1 - \mu(\beta)$ by induction on $\#C_0^{\text{int}}(T)$.

Suppose $\#C_0^{\text{int}}(T) = 0$. Then $k = 1$. We put $f_\Gamma = \text{identity}$ and $m_\Gamma = m_{1,(0,0)}$.
 Suppose $\#C_0^{\text{int}}(T) = 1$. Then $v \in C_0^{\text{int}}(T)$ has $k + 1$ edges. We put

$$\begin{cases} f_\Gamma(x_1, \dots, x_k) = G(m_{k,\beta(v)}(x_1, \dots, x_k)), \\ m_\Gamma(x_1, \dots, x_k) = \Pi(m_{k,\beta(v)}(x_1, \dots, x_k)). \end{cases}$$

We next assume $\#C_0^{\text{int}}(T) \geq 2$. Let e be the edge containing v_0 , and let v be another vertex of e ; v is necessarily interior. We remove v_0, v, e from T and obtain T_1, \dots, T_ℓ , where v has $\ell + 1$ edges. We number T_i so that e, T_1, \dots, T_ℓ respects the counterclockwise cyclic order induced by the orientation of \mathbb{R}^2 . The tree T_i , together with the data induced from Γ in an obvious way, determines $\Gamma_i \in \text{Gr}(k_i, \beta_i)$. We have $\beta = \beta(v) + \sum \beta_i$, $k = \sum k_i$. We put

$$\begin{cases} f_\Gamma = G \circ m_{k,\beta(v)} \circ (f_{\Gamma_1} \otimes \dots \otimes f_{\Gamma_\ell}), \\ m_\Gamma = \Pi \circ m_{k,\beta(v)} \circ (f_{\Gamma_1} \otimes \dots \otimes f_{\Gamma_\ell}). \end{cases}$$

We now define

$$(10.10) \quad f_{k,\beta} = \sum_{\Gamma \in \text{Gr}(\beta,k)} f_\Gamma, \quad m_{k,\beta}^{\text{can}} = \sum_{\Gamma \in \text{Gr}(\beta,k)} m_\Gamma.$$

LEMMA 10.2

The system of operators $\{m_{k,\beta}^{\text{can}}\}$ defines a structure of a filtered A_∞ -algebra on H .
 The system of operators $\{f_{k,\beta}^{\text{can}}\}$ defines a filtered A_∞ -homomorphism: $H \rightarrow C$.
 Unital and/or $\text{mod } T^{E_0}$ versions also hold.

We omit the proof and refer to [11, Section 5.4.4].

We next prove the cyclicity of $m_{k,\beta}^{\text{can}}$. We need some more notation. Let $\Gamma = (T, v_0, \beta(\cdot)) \in \text{Gr}(\beta, k)$. A *flag* of Γ is a pair (v, e) , where v is an interior vertex of T and e is an edge containing v . For each (Γ, v, e) we define

$$(10.11) \quad m(\Gamma, v, e) : B_{k+1}(\overline{C}[1]) \rightarrow \mathbb{R}$$

as follows. We remove v from T . Let T_0, T_1, \dots, T_ℓ be the components of the complement. We assume $e \in T_0$, and T_0, T_1, \dots, T_ℓ respects the counterclockwise cyclic order induced by the standard orientation of \mathbb{R}^2 . Together with data induced from Γ , the ribbon tree T_i determines $\Gamma_i = (T_i, v, \beta_i(\cdot)) \in \text{Gr}(k_i, \beta_i)$ such that $\beta(v) + \sum \beta_i = \beta$ and $\sum k_i = k$. (We remark that the root of Γ_i is always v by convention.)

We enumerate the exterior vertices of Γ as v_0, v_1, \dots, v_k so that it respects the counterclockwise cyclic order. We take j_i such that $v_{j_i}, \dots, v_{j_i+k_i-1}$ are vertices of T_i . (In the case $j_i + k_i - 1 > k$, we identify $v_{j_i+k_i-1}$ with $v_{j_i+k_i-1-k}$.)

DEFINITION 10.1

We have

$$(10.12) \quad \begin{aligned} & m(\Gamma, v, e)(x_1, \dots, x_k, x_0) \\ &= (-1)^* \langle m_{\ell,\beta(v)}(f_{\Gamma_1}(x_{j_1}, \dots), \dots, f_{\Gamma_\ell}(x_{j_\ell}, \dots)), f_{\Gamma_0}(x_{j_0}, \dots, x_{j_1-1}) \rangle. \end{aligned}$$

Here

$$* = (\deg' x_{j_1} + \deg' x_{j_1+1} + \cdots + \deg' x_0)(\deg' x_1 + \cdots + \deg' x_{j_1-1}).$$

PROPOSITION 10.1

The element $\mathfrak{m}(\Gamma, v, e)(x_1, \dots, x_k, x_0)$ is independent of the flag (v, e) but depends only on $\Gamma, x_1, \dots, x_k, x_0$.

Proof

Independence of e is a consequence of the cyclic symmetry of $\mathfrak{m}_{k,\beta}$. We prove the independence of v . Let v and v' be interior vertices. We assume that there exists an edge e joining v and v' .

We first consider the flag (v, e) . We then obtain $\Gamma_0, \dots, \Gamma_\ell$ as above. We next consider the flag (v', e) . We then obtain $\Gamma'_0, \dots, \Gamma'_{\ell'}$ as above. It is easy to see the following:

$$(10.13) \quad \begin{cases} \Gamma_1 \cup \cdots \cup \Gamma_\ell \cup e \cup \Gamma'_1 \cup \cdots \cup \Gamma'_{\ell'} = \Gamma, \\ e \cup \Gamma'_1 \cup \cdots \cup \Gamma'_{\ell'} = \Gamma_0, \\ e \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell = \Gamma'_0. \end{cases}$$

Let $v_{j_i}, \dots, v_{j_i+k_i-1}$ be the vertices of Γ_i , and let $v'_{j'_i}, \dots, v'_{j'_i+k'_i-1}$ be the vertices of Γ'_i . We put

$$y_i = \mathfrak{f}_{\Gamma_i}(x_{j_i}, \dots, x_{j_i+k_i-1}), \quad z_i = \mathfrak{f}_{\Gamma'_i}(x'_{j'_i}, \dots, x'_{j'_i+k'_i-1}).$$

Now by the definitions and (10.13), we have

$$(10.14) \quad \begin{aligned} \mathfrak{m}(\Gamma, v, e)(x_1, \dots, x_k, x_0) \\ = (-1)^{*1} \langle \mathfrak{m}_{\ell,\beta(v)}(y_1, \dots, y_\ell), G(\mathfrak{m}_{\ell',\beta(v')}(z_1, \dots, z_{\ell'})) \rangle, \end{aligned}$$

where

$$*1 = (\deg' x_{j_1} + \deg' x_{j_1+1} + \cdots + \deg' x_0)(\deg' x_1 + \cdots + \deg' x_{j_1-1}).$$

On the other hand, we have

$$(10.15) \quad \begin{aligned} \mathfrak{m}(\Gamma, v', e)(x_1, \dots, x_k, x_0) \\ = (-1)^{*2} \langle \mathfrak{m}_{\ell',\beta(v')}(z_1, \dots, z_{\ell'}), G(\mathfrak{m}_{\ell,\beta(v)}(y_1, \dots, y_\ell)) \rangle, \end{aligned}$$

where

$$*2 = (\deg' x'_{j'_1} + \deg' x'_{j'_1+1} + \cdots + \deg' x_0)(\deg' x_1 + \cdots + \deg' x_{j'_1-1}).$$

Equation (10.14) coincides with (10.15) by (10.2). (We use the fact that the degree of \mathfrak{m} is $+1$ here.) □

LEMMA 10.3

The operator $\mathfrak{m}_{k,\beta}^{\text{can}}$ is cyclically symmetric.

Proof

We consider

$$\begin{aligned}
 \langle \mathbf{m}_{k,\beta}^{\text{can}}(x_1, \dots, x_k), x_0 \rangle &= \sum_{\Gamma \in \text{Gr}(\beta, k)} \langle \mathbf{m}_{\Gamma}^{\text{can}}(x_1, \dots, x_k), x_0 \rangle \\
 (10.16) \qquad \qquad \qquad &= \sum_{\Gamma \in \text{Gr}(\beta, k)} \mathbf{m}(\Gamma, (v'_0, e_0))(x_1, \dots, x_k, x_0).
 \end{aligned}$$

Let e_0 be the edge containing v_0 (the root of Γ), and let v'_0 be the other edge of e_0 .

We number the exterior edges of Γ as v_0, \dots, v_k so that it respects counter-clockwise cyclic order. Let e_i be the edge containing v_i , and let v'_i be the other vertex of e_i . Proposition 10.1 implies that (10.16) is equal to

$$(-1)^* \sum_{\Gamma \in \text{Gr}(\beta, k)} \mathbf{m}(\Gamma, (v'_i, e_i))(x_1, \dots, x_k, x_0).$$

Here $*$ = $(\text{deg}' x_{i+1} + \dots + \text{deg}' x_0)(\text{deg}' x_1 + \dots + \text{deg}' x_i)$. Clearly this is equal to

$$(-1)^* \langle \mathbf{m}_{k,\beta}^{\text{can}}(x_{i+1}, \dots, x_{i-1}), x_i \rangle.$$

The proof of Lemma 10.3 is now complete. □

LEMMA 10.4

The filtered A_∞ -homomorphism $\{f_{k,\beta}\}$ is cyclic.

Proof

We consider

$$(10.17) \qquad \qquad \langle f(\Gamma_1)(x_1, \dots, x_{k_1}), f(\Gamma_2)(x_{k_1+1}, \dots, x_k) \rangle$$

where $\Gamma_i \in \text{Gr}(k_i, \beta_i)$, $k_1 + k_2 = k + 1$, $\beta_1 + \beta_2 = \beta$.

We remark that the image of $f(\Gamma_i)$ is in $\overline{D} = \text{Im } G$ if $(k_i, \beta_i) \neq (1, (0, 0))$. If $(k_i, \beta_i) = (1, (0, 0))$ then the image of $f(\Gamma_i)$ is in \overline{H} . Moreover $\langle \overline{D}, \overline{D} \rangle = \langle \overline{D}, \overline{H} \rangle = 0$. Therefore (10.17) is 0 unless $(k, \beta) = (2, (0, 0))$. In the case $(k, \beta) = (2, (0, 0))$, (10.17) is $\langle x_1, x_2 \rangle$. The lemma follows. □

Proof of Theorem 8.3

Lemmas 10.2, 10.3, and 10.4 imply Theorem 8.3. □

Proof of Theorem 8.4

Let $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^t\}, \{\mathbf{c}_{k,\beta}^t\})$ be a pseudoisotopy. We take G, Π as in Lemma 10.1. Since $\mathbf{m}_{1,(0,0)}^t$ is independent of t , we can choose G, Π to be independent of t . We take the canonical model $(H, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{\text{can}}\})$ for each fixed t . It is easy to see from the construction that $\mathbf{m}_{k,\beta}^{\text{can}}$ is smooth with respect to t . We next define $\mathbf{c}_{k,\beta}^{\text{can}}$.

We consider a pair (Γ, v_s) of $\Gamma \in \text{Gr}(\beta, k)$ and an interior vertex v_s of Γ . We denote by $\text{Gr}^+(k, \beta)$ the set of all such pairs. We define

$$(10.18) \qquad \qquad \mathbf{c}^t(\Gamma, v_s) : B_k(\overline{H}[1]) \rightarrow \overline{H}[1]$$

of degree $-\mu(\beta)$ and

$$(10.19) \quad \mathfrak{h}^t(\Gamma, v_s) : B_k(\overline{H}[1]) \rightarrow \overline{c}[1]$$

of degree $-1 - \mu(\beta)$, by induction on $\#C_0^{\text{int}}(\Gamma)$.

Since $v_s \in C_0^{\text{int}}(\Gamma)$, we have $\#C_0^{\text{int}}(\Gamma) \geq 1$.

If $\#C_0^{\text{int}}(\Gamma) = 1$, we put

$$\begin{aligned} \mathfrak{h}^t(\Gamma, v_s)(x_1, \dots, x_k) &= G(\mathfrak{c}_{k, \beta(v_s)}^t(x_1, \dots, x_k)), \\ \mathfrak{c}^t(\Gamma, v_s)(x_1, \dots, x_k) &= \Pi(\mathfrak{c}_{k, \beta(v_s)}^t(x_1, \dots, x_k)). \end{aligned}$$

Suppose that $\#C_0^{\text{int}}(\Gamma) > 1$. Let e_1 be the edge containing v_0 , and let v be the other vertex of e_1 .

Case 1: $v = v_s$.

We remove e, v, v_0 from Γ and obtain $\Gamma_1, \dots, \Gamma_\ell$. (We assume that $e, \Gamma_1, \dots, \Gamma_\ell$ respects counterclockwise cyclic order.) We put

$$\begin{aligned} \mathfrak{h}^t(\Gamma, v_s) &= G \circ \mathfrak{c}_{\ell, \beta(v_s)}^t \circ (\mathfrak{f}^t(\Gamma_1) \otimes \dots \otimes \mathfrak{f}^t(\Gamma_\ell)), \\ \mathfrak{c}^t(\Gamma, v_s) &= \Pi \circ \mathfrak{c}_{\ell, \beta(v_s)}^t \circ (\mathfrak{f}^t(\Gamma_1) \otimes \dots \otimes \mathfrak{f}^t(\Gamma_\ell)). \end{aligned}$$

Here $\mathfrak{f}^t(\Gamma)$ is the operator that appeared in the definition of $\mathfrak{f}_{k, \beta}^t$.

Case 2: $v \neq v_s$.

We remove e, v, v_0 from Γ and obtain $\Gamma_1, \dots, \Gamma_\ell$. (We assume that $e, \Gamma_1, \dots, \Gamma_\ell$ respects counterclockwise cyclic order.) We assume that $v_s \in \Gamma_i$. We now put

$$\begin{aligned} \mathfrak{h}^t(\Gamma, v_s) &= G \circ \mathfrak{m}_{\ell, \beta(v_s)}^t \circ (\mathfrak{f}^t(\Gamma_1) \otimes \dots \otimes \mathfrak{h}^t(\Gamma_i) \otimes \dots \otimes \mathfrak{f}^t(\Gamma_\ell)), \\ \mathfrak{c}^t(\Gamma, v_s) &= -\Pi \circ \mathfrak{m}_{\ell, \beta(v_s)}^t \circ (\mathfrak{f}^t(\Gamma_1) \otimes \dots \otimes \mathfrak{h}^t(\Gamma_i) \otimes \dots \otimes \mathfrak{f}^t(\Gamma_\ell)). \end{aligned}$$

Note that $\mathfrak{f}^t(\Gamma)$ is of even degree and $\mathfrak{h}^t(\Gamma)$ is of odd degree. We take their tensor product as follows:

$$(10.20) \quad \begin{aligned} &(\mathfrak{f}^t(\Gamma_1) \otimes \dots \otimes \mathfrak{h}^t(\Gamma_i) \otimes \dots \otimes \mathfrak{f}^t(\Gamma_\ell))(x_1, \dots, x_k) \\ &= (-1)^{*i} \mathfrak{f}^t(\Gamma_1)(x_1, \dots, x_{k_1}) \otimes \dots \otimes \mathfrak{h}^t(\Gamma_i)(x_j, \dots, x_{j+k_i-1}) \\ &\quad \otimes \dots \otimes \mathfrak{f}^t(\Gamma_\ell)(x_{k-k_\ell+1}, \dots, x_k). \end{aligned}$$

Here $j = k_1 + k_2 + \dots + k_{i-1} + 1$, $*i = \text{deg}' x_1 + \dots + \text{deg}' x_{j-1}$.

We remark that the minus sign in the definition of $\mathfrak{c}^t(\Gamma, v_s)$ appears since we change the order of $\mathfrak{m}_{\ell, \beta(v_s)}^T$ and dt .

We put

$$\mathfrak{c}_{k, \beta}^{\text{tcan}} = \sum_{(\Gamma, v_s) \in \text{Gr}^+(k, \beta)} \mathfrak{c}^t(\Gamma, v_s).$$

It is easy to see that $(H, \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k, \beta}^{\text{tcan}}\}, \{\mathfrak{c}_{k, \beta}^{\text{tcan}}\})$ satisfies Definition 8.5(1), (2), (5).

We next prove (4) by using Lemma 8.1. We consider $C^\infty([0, 1], \overline{H})$ and define $\hat{\mathfrak{m}}_{k, \beta}^{\text{can}}$ by (8.10). We next define

$$\mathfrak{F}_{k, \beta} : B_k(C^\infty([0, 1], \overline{H})[1]) \rightarrow C^\infty([0, 1], \overline{H})[1]$$

as follows. Let $x_i(t) + dt \wedge y_i(t) = \mathbf{x}_i \in C^\infty([0, 1], \overline{H})$. We put

$$\mathfrak{F}_{k,\beta}(\mathbf{x}_1, \dots, \mathbf{x}_k) = x(t) + dt \wedge y(t),$$

where

$$(10.21a) \quad x(t) = \mathfrak{f}_{k,\beta}^t(x_1(t), \dots, x_k(t)),$$

$$(10.21b) \quad y(t) = \mathfrak{h}_{k,\beta}^t(x_1(t), \dots, x_k(t))$$

$$- \sum_{i=1}^k (-1)^{*i} \mathfrak{m}_{k,\beta}^t(x_1(t), \dots, x_{i-1}(t), y_i(t), x_{i+1}(t), \dots, x_k(t))$$

if $(k, \beta) \neq (1, (0, 0))$ and

$$(10.21c) \quad y(t) = \frac{d}{dt} x_1(t)$$

if $(k, \beta) = (1, (0, 0))$. Here $*i$ in (10.21b) is $*i = \deg' x_1 + \dots + \deg' x_{i-1}$. (We remark that $\mathfrak{f}_{1,(0,0)}^t = \text{identity}$ and $\mathfrak{h}_{1,(0,0)}^t = 0$.)

LEMMA 10.5

The system of maps $\{\mathfrak{F}_{k,\beta}\}$ is a filtered A_∞ -homomorphism.

Proof

We regard G and Π as homomorphisms

$$C^\infty([0, 1] \times \overline{C}) \rightarrow C^\infty([0, 1] \times \overline{C}).$$

Then we can apply (the proof of) Lemma 10.2. It implies Lemma 10.5. \square

COROLLARY 10.1

The operators $\mathfrak{c}^{t\text{can}}$ and $\mathfrak{m}^{t\text{can}}$ satisfies (8.9).

Proof

Since $\mathfrak{F}_{1,(0,0)}$ is injective, Lemma 10.5 implies that $\hat{\mathfrak{m}}^{t\text{can}}$ satisfies A_∞ -relation. The Corollary then follows from Lemma 8.1. \square

LEMMA 10.6

The operator $\mathfrak{c}^{t\text{can}}$ is cyclically symmetric.

The proof is similar to the proof of Lemma 9.3 and so is omitted. The proof of Theorem 8.4 is now complete. (The proof of unitality is easy.) \square

11. Geometric realization of pseudoisotopy of cyclic filtered A_∞ -algebras

The main result of this section is Theorem 11.1. Let (M, ω) be a symplectic manifold, and let L be a relatively spin Lagrangian submanifold. We take two almost complex structures J_0, J_1 tamed by ω . Let $E_0 \leq E_1$. We apply Theorem 7.1 for L, J_0, E_0 and L, J_1, E_1 . Then we obtain $G(J_i)$ -gapped cyclic filtered

A_∞ -structures $(\Lambda(L), \{\mathbf{m}_{k,\beta}^{(i)}\}, \langle \cdot, \cdot \rangle)$ modulo T^{E_i} on the de Rham complex. (Note that the discrete submonoid $G(J)$ depends on the almost complex structure J ; see the beginning of the proof of Theorem 7.1.)

THEOREM 11.1

There exists $G \supset G(J_0), G(J_1)$ such that $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(0)}\})$ is pseudoisotopic to $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(1)}\})$ as G -gapped cyclic unital filtered A_∞ -algebras modulo T^{E_0} .

Before proving Theorem 11.1, we clarify the definition of pseudoisotopy in the (infinite-dimensional) case of the de Rham complex. We consider $\overline{C} = \Lambda(L)$. We put

$$(11.1) \quad C^\infty([0, 1] \times \overline{C}) = \Lambda([0, 1] \times L).$$

Note that an element of $C^\infty([0, 1] \times \overline{C})$ is uniquely written as

$$x(t) + dt \wedge y(t),$$

where $x(t), y(t) \in \Lambda(L)$. So this notation is consistent with one in Section 8. However, the assumption on the smoothness here on the map $t \mapsto x(t), t \mapsto y(t)$ is different from the finite-dimensional case.

We consider the operations $\mathbf{m}_{k,\beta}^t : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$ of degree $-\mu(\beta) + 1$ and $\mathbf{c}_{k,\beta}^t : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$ of degree $-\mu(\beta)$.

DEFINITION 11.1

We say that $\mathbf{m}_{k,\beta}^t$ is *smooth* if for each x_1, \dots, x_k ,

$$t \mapsto \mathbf{m}_{k,\beta}^t(x_1, \dots, x_k)$$

is an element of $C^\infty([0, 1] \times \overline{C})$.

The smoothness of $\mathbf{c}_{k,\beta}^t$ is defined in the same way.

This is the same as Definition 11.1, except that we use (11.1) for $C^\infty([0, 1] \times \overline{C})$. We use this definition of the smoothness and define pseudoisotopy on the de Rham complex in the same way as Definition 8.5. Theorems 8.1 and 8.2 can be proved in the same way.

Proof of Theorem 11.1

We take a path $\mathcal{J} = \{J_t\}_{t \in [0,1]}$ of tame almost complex structures joining J_0 to J_1 . We consider the moduli spaces $\mathcal{M}_k(\beta; J_t)$ of J_t -holomorphic discs of homology class $\beta \in H_2(M, L; \mathbb{Z})$. We put

$$(11.2) \quad \mathcal{M}_k(\beta; \mathcal{J}) = \bigcup_{t \in [0,1]} \{t\} \times \mathcal{M}_k(\beta; J_t).$$

We have evaluation maps

$$\text{ev} = (\text{ev}_0, \dots, \text{ev}_{k-1}) : \mathcal{M}_k(\beta; \mathcal{J}) \rightarrow L^k$$

together with $\text{ev}_t : \mathcal{M}_k(\beta; \mathcal{J}) \rightarrow [0, 1]$, where $\text{ev}_t(\{t\} \times \mathcal{M}_k(\beta; J_t)) = \{t\}$.

LEMMA 11.1

There exists a system of Kuranishi structures on $\mathcal{M}_1(\beta; \mathcal{J})$ and $\mathcal{M}_0(\beta; \mathcal{J})$ with the following properties.

- (1) ev_t extends to a strongly continuous and weakly submersive map. So it induces a Kuranishi structure on $\mathcal{M}_k(\beta; J_t)$ for each of $t \in [0, 1]$, $k = 0, 1$.
- (2) The induced Kuranishi structure on $\mathcal{M}_k(\beta; J_i)$ for $i = 0, 1$, $k = 0, 1$ coincides with one produced in Theorem 3.1.
- (3) For any $t \in [0, 1]$, $k = 0, 1$, the induced Kuranishi structure on $\mathcal{M}_k(\beta; J_t)$ satisfies the conclusions of Theorem 3.1.

Proof

The proof is the same as the proof of Theorem 3.1. Namely, we define the obstruction bundle, that is, a subspace of $C^\infty(\Sigma, u^*TM \otimes \Lambda^{0,1})$, large enough so that the submersivity of ev_0 and ev_t holds. We can do it inductively so that the parts already defined are untouched. \square

LEMMA 11.2

There exists a system of Kuranishi structures on $\mathcal{M}_{\ell, k+1}(\beta; \mathcal{J})$, $k \geq 0$, with the following properties.

- (1) ev_t extends to a strongly continuous and weakly submersive map. So it induces a Kuranishi structure on $\mathcal{M}_k(\beta; J_t)$ for each of $t \in [0, 1]$.
- (2) The induced Kuranishi structure on $\mathcal{M}_k(\beta; J_i)$ for $i = 0, 1$ coincides with one produced in Corollary 3.1.
- (3) For any $t \in [0, 1]$, the induced Kuranishi structure on $\mathcal{M}_k(\beta; J_t)$ satisfies the conclusion of Corollary 3.1.

Proof

The proof is the same as the proof of Corollary 3.1. \square

LEMMA 11.3

For each $\epsilon > 0$, there exists a compatible systems of families of multisections $(U_{\bar{\alpha}, i}, W_{\bar{\alpha}}, \{\mathfrak{s}_{\bar{\alpha}, i, j}\})$, $(U_{\alpha, i}, W_\alpha, \{\mathfrak{s}_{\alpha, i, j}\})$ on $\mathcal{M}_1(\beta; \mathcal{J})$, $\mathcal{M}_0(\beta; \mathcal{J})$ for $\beta \cap \omega \leq E_0$, with the following properties.

- (1) At $t = 0, 1$, they coincide with the family of multisections produced in Theorem 5.1.
- (2) They are ϵ -close to the Kuranishi map.
- (3) They are transversal to zero in the sense of Definition 4.1(3).
- (4) $(ev_0, ev_t) : \mathcal{M}_1(\beta; \mathcal{J}) \rightarrow L \times [0, 1]$ induces submersions $(ev_0)_{\bar{\alpha}}|_{\mathfrak{s}_{\bar{\alpha}}^{-1}(0)} : \mathfrak{s}_{\bar{\alpha}}^{-1}(0) \rightarrow L \times [0, 1]$.
- (5) They are compatible with (3.3) in the same sense as Theorem 5.1.

Proof

The proof is the same as the proof of Theorem 5.1. □

REMARK 11.1

We remark that Lemma 11.3(4) implies submersivity of $\text{ev}_0 : \mathcal{M}_1(\beta; J_t) \rightarrow L$ for any t . Since there exist uncountably many t 's, we cannot do it when we are working with multi-, but finitely many, valued sections. Since we are working with a continuous family of multisections, this becomes possible. In fact, we can take the dimension of our parameter space W as large as we want.

LEMMA 11.4

For each ϵ and E_0 , there exists a system of continuous families of multisections on $\mathcal{M}_{k+1}(\beta; \mathcal{J})$, $k \geq 0$, $\beta \cap \omega \leq E_0$, with the following properties.

- (1) At $t = 0, 1$ they coincide with the family of multisections produced in Corollary 5.1.
- (2) It is ϵ -close to the Kuranishi map.
- (3) It is compatible with $\text{forget}_{k+1,1}$.
- (4) It is invariant under the cyclic permutation of the boundary marked points.
- (5) It is invariant by the permutation of interior marked points.
- (6) $(\text{ev}_0, \text{ev}_t) : \mathcal{M}_{k+1}(\beta; \mathcal{J}) \rightarrow L \times [0, 1]$ induces a submersion on its zero set.
- (7) We consider the decomposition of the boundary

$$(11.3) \quad \partial \mathcal{M}_{k+1}(\beta; \mathcal{J}) \supset \bigcup_{1 \leq i \leq j+1 \leq k+1} \bigcup_{\beta_1 + \beta_2 = \beta} \mathcal{M}_{j-i+1}(\beta_1; \mathcal{J})_{(\text{ev}_0, \text{ev}_t)} \times_{(\text{ev}_i, \text{ev}_t)} \mathcal{M}_{k-j+i}(\beta_2; \mathcal{J}).$$

Then the restriction of our family of multisections of $\mathcal{M}_{k+1}(\beta; \mathcal{J})$ in the left-hand side coincides with the fiber product family of multisections in the right-hand side.

Proof

The proof is the same as the proof of Corollary 5.1. □

Now we are in a position to complete the proof of Theorem 11.1. Let $\rho_1, \dots, \rho_k \in \Lambda(L)$. We put

$$(11.4) \quad \text{Corr}_*(\mathcal{M}_{k+1}(\beta; \mathcal{J}); (\text{ev}_1, \dots, \text{ev}_k), \text{ev}_0)(\rho_1 \times \dots \times \rho_k) = \rho(t) + dt \wedge \sigma(t).$$

Here we use the continuous family of multisections produced in Lemma 11.4 to define the left-hand side. We define

$$(11.5) \quad \mathfrak{m}_{k,\beta}^t(\rho_1, \dots, \rho_k) = \rho(t), \quad \mathfrak{c}_{k,\beta}^t(\rho_1, \dots, \rho_k) = \sigma(t).$$

Using Lemma 11.4, we can prove that they satisfy the required properties in the same way as the proof of Theorem 7.1. □

12. Cyclic filtered A_∞ -structures on the de Rham complex and on de Rham cohomology

In this section, we use Theorems 7.1 and 11.1 to produce a gapped, cyclic, and unital filtered A_∞ -structure on de Rham cohomology. We first construct a cyclic filtered A_∞ -structure on a de Rham complex.

THEOREM 12.1

For any relatively spin Lagrangian submanifold L , we can associate a gapped, cyclic, unital filtered A_∞ -algebra, $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}\})$ on its de Rham complex.

It is independent of the choices up to pseudoisotopy as cyclic unital filtered A_∞ -algebra modulo T^E for any E .

Proof

We fix J and take $E_1 < E_2 < \dots$. For each E_i we apply Theorem 7.1 to obtain a gapped, cyclic, unital filtered A_∞ -algebra modulo T^{E_i} , $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i)}\})$ on its de Rham complex. By Theorem 11.1, there exists a pseudoisotopy $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{t,(i)}\}, \{\mathfrak{c}_{k,\beta}^{t,(i)}\})$ of gaped, cyclic, unital filtered A_∞ -algebra modulo T^{E_i} between $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i)}\})$ and $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i+1)}\})$. Now we use Theorem 8.2 to extend unital pseudoisotopy modulo T^{E_0} , $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{t,(i)}\}, \{\mathfrak{c}_{k,\beta}^{t,(i)}\})$ and a unital and cyclic filtered A_∞ -algebra modulo T^{E_0} , and $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i)}\})$ to a unital pseudoisotopy and unital and cyclic filtered A_∞ -algebra (see [11, Section 7.2.8]).

Let us take two choices J_j ($j = 0, 1$) of J , perturbation, and so on, and obtain a cyclic, unital filtered A_∞ -structure extending the one on $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i_0,j)}\})$. We take i such that $E_i > E$. Then, by construction, $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i_0,j)}\})$ is pseudoisotopic to $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i,j)}\})$ as a cyclic, unital filtered A_∞ -algebra. (Here modulo T^E is superfluous.) On the other hand, by Theorem 11.1, $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i,0)}\})$ is pseudoisotopic to $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(i,1)}\})$ as a cyclic, unital filtered A_∞ -algebra modulo T^E . The uniqueness part of Theorem 12.1 follows. \square

Theorems 12.1, 8.3, and 8.4 immediately imply the following.

COROLLARY 12.1

For any relatively spin Lagrangian submanifold L , we can associate a gapped, cyclic, unital filtered A_∞ -algebra, $(H(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{\text{can}}\})$ on its de Rham cohomology.

It is independent of the choices up to homotopy equivalence as a cyclic, unital filtered A_∞ -algebra modulo T^E for any E .

REMARK 12.1

In Corollary 12.1 we proved that the cyclic filtered A_∞ -structure on the de Rham cohomology is well defined up to homotopy equivalence modulo T^E but *not* up to pseudoisotopy modulo T^E . Theorem 8.4 implies that it is well defined up to pseudoisotopy modulo T^E once we fix operators G and Π satisfying the conclusion

of Lemma 10.1. It does not seem to be so immediate to prove its independence of G and Π up to pseudoisotopy (modulo T^E). The proof up to homotopy equivalence follows from the fact that f in Theorem 8.3 is homotopy equivalence.

So Theorem 12.1 gives a stronger conclusion than Corollary 12.1.

REMARK 12.2

The difference between pseudoisotopy modulo T^E for arbitrary E and pseudoisotopy is not important for most of the applications. To improve the statement of Theorem 12.1 up to pseudoisotopy, we need to work out the story of pseudoisotopy of pseudoisotopies. We present the detail of this construction in Section 14 for completeness.

REMARK 12.3

Here we are using the bifurcation method rather than the cobordism method (see [11, Section 7.2.14] for the comparison between these two methods). In [11] we used the cobordism method mainly. In [1] the bifurcation method is used. The reason why we use the bifurcation method here is related to Remark 12.1. Namely, pseudoisotopy seems stronger than homotopy equivalence.

By carefully looking at the proof of Theorem 8.2, we find that they finally give the same homotopy equivalence. In fact, time ordered product, which was used in [11], appears during the proof of Theorem 8.2.

13. Adic convergence of filtered A_∞ -structures

Proof of Theorem 1.2

We begin with the proof of properties (1) and (2) of Theorem 1.2. We define the convergence used here first. Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a discrete submonoid. Let \overline{C} be a finite-dimensional \mathbb{R} -vector space, and let $C_G = \overline{C} \otimes_{\mathbb{R}} \Lambda_0^G$, $C = \overline{C} \otimes_{\mathbb{R}} \Lambda_{0, \text{nov}}$.

DEFINITION 13.1

A sequence of elements $v_i \in C_G$ is said to *converge to* v if

$$v_i = \sum_{\beta \in G} v_{i,\beta} T^{E(\beta)} e^{\mu(\beta)/2},$$

$$v = \sum_{\beta \in G} v_\beta T^{E(\beta)} e^{\mu(\beta)/2}$$

and if search of $v_{i,\beta}$ converges to v_β in the topology of \overline{C} (induced by the ordinary topology of \mathbb{R}).

A sequence $v_i \in C$ is said to *converge to* v if there exists G independent of i such that $v_i, v \in C_G$ and v_i converges to v .

Let L be a relatively spin Lagrangian submanifold of M . We take an almost complex structure J on M tamed by ω . We consider the cyclic, unital filtered A_∞ -algebra $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}\})$ in Theorem 12.1 and $(H(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{\text{can}}\})$ in Corol-

lary 12.1. Let $\mathbf{e}_1, \dots, \mathbf{e}_{b_1}$ be a basis of $H^1(L; \mathbb{Z})$. We take its representative as a closed one-form and denote it by the same symbol. We put

$$(13.1) \quad \mathbf{b} = \sum_{i=1}^{b_1} x_i \mathbf{e}_i$$

with $x_i \in \Lambda_G^{(0)}$. (Namely, x_i does not contain e , the grading parameter of $\Lambda_{0, \text{nov}}$, and $\mathbf{x}_i \in \Lambda(L)$ ($i = 1, \dots, k$).

LEMMA 13.1

We have

$$(13.2) \quad \begin{aligned} & \sum_{m_0 + \dots + m_k = m} \mathbf{m}_{k+m, \beta}(\mathbf{b}^{\otimes m_0}, \mathbf{x}_1, \mathbf{b}^{\otimes m_1}, \dots, \mathbf{b}^{\otimes m_{k-1}}, \mathbf{x}_k, \mathbf{b}^{\otimes m_k}) \\ &= \frac{1}{m!} \left(\sum_{i=1}^{b_1} (\partial\beta \cap \mathbf{e}_i) x_i \right)^m \mathbf{m}_k(\mathbf{x}_1, \dots, \mathbf{x}_k). \end{aligned}$$

Proof

We consider the set $(m_0, \dots, m_k) \in \mathbb{Z}_{\geq 0}$ with $m_0 + \dots + m_k = m$ and denote it by $A(m)$. For each $\vec{m} = (m_0, \dots, m_k) \in A(m)$, we take a copy of $\mathcal{M}_{k+m}(\beta)$ and denote it by $\mathcal{M}_{\vec{m}}(\beta)$. We then consider the forgetful map

$$(13.3) \quad \text{forget}_{\vec{m}} : \mathcal{M}_{\vec{m}}(\beta) \rightarrow \mathcal{M}_k(\beta),$$

which forgets the first, \dots , m_0 th, $m_0 + \text{second}$, $m_0 + m_1 + \text{first}$, $m_0 + m_1 + \text{third}$, and, \dots , $m_0 + \dots + m_i + \text{first}$, $m_0 + \dots + m_i + i + \text{third}$, \dots marked points. In other words, we forget the marked points where \mathbf{b} are assigned in the left-hand side of (13.2).

We consider

$$(13.4) \quad \text{ev}_{\vec{m}} : \mathcal{M}_{\vec{m}}(\beta) \rightarrow L^m,$$

the evaluation map at the marked points which we forget in (13.3). We take the (continuous family of) perturbations as in Corollary 5.1. We write its zero-set as $\mathcal{M}_{\vec{m}}(\beta)^{\mathfrak{s}}$, $\mathcal{M}_k(\beta)^{\mathfrak{s}}$. Since the perturbation is compatible with the forgetful map, there exists a map

$$(13.5) \quad \text{forget}_{\vec{m}}^{\mathfrak{s}} : \mathcal{M}_{\vec{m}}(\beta)^{\mathfrak{s}} \rightarrow \mathcal{M}_k(\beta)^{\mathfrak{s}}.$$

For each $\mathbf{p} \in \mathcal{M}_k(\beta)^{\mathfrak{s}}$, the fiber $(\text{forget}_{\vec{m}}^{\mathfrak{s}})^{-1}(\mathbf{p})$ is m -dimensional. Moreover, we have the following. We represent \mathbf{p} by (Σ, u) . Then the cycle

$$\sum_{\vec{m}} (\text{ev}_{\vec{m}})_* ((\text{forget}_{\vec{m}}^{\mathfrak{s}})^{-1}(\mathbf{p}))$$

is equal to

$$\{ (u(t_1), \dots, u(t_m)) \mid t_1, \dots, t_m \in [0, 1], t_1 \leq \dots \leq t_m \}$$

as currents. Here we identify $\partial\Sigma = [0, 1]$ so that zero corresponds to the zeroth boundary marked point. In fact, by counting the dimension of the support it

suffices to consider the case when $k + m + 1$ boundary marked points are all distinct. In that case, there are various possibilities in which marked points, among $k + m$ marked points, become k marked points that remain after applying forgetful maps. Those possibilities correspond to the choice of \vec{m} .

Therefore we have

$$\sum_{\vec{m}} \int_{(\text{forget } t_{\vec{m}}^s)^{-1}(\mathbf{p})} \text{ev}_{\vec{m}}^*(\mathbf{b} \times \cdots \times \mathbf{b}) = \frac{1}{m!} \left(\sum_{i=1}^{b_1} (\partial\beta \cap \mathbf{e}_i) x_i \right)^m.$$

We remark that the sign is independent of \vec{m} in the left-hand side. This is because the shifted degree of \mathbf{b} is even (see [11, Lemma 8.4.3]). The same argument appears also in the proof of [12, Lemma 11.8].

Equation (13.2) follows ($1/m!$ is the volume of the domain $\{(t_1, \dots, t_m) \mid t_1, \dots, t_m \in [0, 1], t_1 \leq \dots \leq t_m\}$). \square

LEMMA 13.2

We have

$$\begin{aligned} (13.6) \quad & \sum_{m_0 + \cdots + m_k = m} \mathfrak{m}_{k+m, \beta}^{\text{can}}(\mathbf{b}^{\otimes m_0}, \mathbf{x}_1, \mathbf{b}^{\otimes m_1}, \dots, \mathbf{b}^{\otimes m_{k-1}}, \mathbf{x}_k, \mathbf{b}^{\otimes m_k}) \\ &= \frac{1}{m!} \left(\sum_{i=1}^{b_1} (\partial\beta \cap \mathbf{e}_i) x_i \right)^m \mathfrak{m}_k^{\text{can}}(\mathbf{x}_1, \dots, \mathbf{x}_k). \end{aligned}$$

Proof

We have

$$\begin{aligned} (13.7) \quad & \sum_{m_0 + \cdots + m_k = m} \mathfrak{f}_{k+m, \beta}(\mathbf{b}^{\otimes m_0}, \mathbf{x}_1, \mathbf{b}^{\otimes m_1}, \dots, \mathbf{b}^{\otimes m_{k-1}}, \mathbf{x}_k, \mathbf{b}^{\otimes m_k}) \\ &= \frac{1}{m!} \left(\sum_{i=1}^{b_1} (\partial\beta \cap \mathbf{e}_i) x_i \right)^m \mathfrak{f}_k(\mathbf{x}_1, \dots, \mathbf{x}_k), \end{aligned}$$

by its inductive construction and (13.2). Equation (13.6) then follows from the definition, (13.2), and (13.7). \square

COROLLARY 13.1

We have

$$\begin{aligned} (13.8) \quad & \lim_{N \rightarrow \infty} \sum_{m_0 + \cdots + m_k = m \leq N} \mathfrak{m}_{k+m, \beta}^{\text{can}}(\mathbf{b}^{\otimes m_0}, \mathbf{x}_1, \mathbf{b}^{\otimes m_1}, \dots, \mathbf{b}^{\otimes m_{k-1}}, \mathbf{x}_k, \mathbf{b}^{\otimes m_k}) \\ &= \exp \left(\sum_{i=1}^{b_1} (\partial\beta \cap \mathbf{e}_i) x_i \right) \mathfrak{m}_k^{\text{can}}(\mathbf{x}_1, \dots, \mathbf{x}_k). \end{aligned}$$

Namely, the left-hand side converges to the right-hand side in the usual topology of $H(L; \mathbb{R})$, that is, the topology induced by the usual topology of \mathbb{R} .

The right-hand side depends only on $y_i = e^{x_i}$ and \mathbf{x}_i . Namely, it is independent of the change $x_i \mapsto x_i + 2\pi\sqrt{-1}a_i$ for $a_i \in H(L; \Lambda_{0,\text{nov}}^{\mathbb{Z}})$.

This is immediate from Lemma 13.2.

So far we consider the bounding cochain \mathbf{b} consisting of a cohomology class of degree 1. The degree-zero class does not appear in the bounding cochain. We next consider the class of degree > 1 . We put

$$(13.9) \quad \mathbf{b}_{\text{high}} = \sum_{i > b_1} x_i \mathbf{e}_i.$$

Here $\mathbf{e}_i, i = b_1 + 1, \dots$, is a basis of $\bigoplus_{d \geq 1} H^{2d+1}(L; \mathbb{Z})$.

LEMMA 13.3

There exists $E(m)$ such that $\lim_{m \rightarrow \infty} E(m) = \infty$ and that

$$(13.10) \quad T^{E(\beta)} \mathbf{m}_{k+m, \beta}^{\text{can}} (\mathbf{b}_{\text{high}}^{\otimes m_0}, \mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{b}_{\text{high}}^{\otimes m_k}) \equiv 0 \pmod{T^{E(m)}}$$

if $m = m_0 + \dots + m_k$. $E(m)$ is independent of β .

Proof

Since the degree of each term of \mathbf{b}_{high} is strictly larger than 1, we have

$$\mu(\beta) > md + C,$$

where C depends only on $\mathbf{x}_1, \dots, \mathbf{x}_k$ and L . By Gromov compactness (see Definition 6.2(2), (3)), it implies that $E(\beta) \rightarrow \infty$ as $m \rightarrow \infty$. □

We put

$$(13.11) \quad \mathbf{m}_k = \sum_{\beta \in G} T^{E(\beta)} e^{\mu/2} \mathbf{m}_{k, \beta}.$$

Then Theorem 1.2(1), (2) follow from Corollary 13.1 and Lemma 13.3.

We turn to the proof of Theorem 1.2(3), (4). We take a Weinstein neighborhood U of L . Namely, U is symplectomorphic to a neighborhood U' of zero section in T^*L . We choose δ_1 such that for $c = (c_1, \dots, c_b) \in [-\delta_1, +\delta_1]^b$, the graph of the closed one-form $\sum_{i=1}^{b_1} c_i \mathbf{e}_i$ is contained in U' . We send it by the symplectomorphism to U and denote it by $L(c)$. We may take $\delta_2 < \delta_1$ such that if $c = (c_1, \dots, c_b) \in [-\delta_2, +\delta_2]^b$, then there exists a diffeomorphism $F_c : M \rightarrow M$ such that

$$(13.12) \quad F_c(L) = L(c);$$

$$(13.13) \quad (F_c)_* J \text{ is tamed by } \omega.$$

REMARK 13.1

It is essential here to consider tame almost complex structures rather than compatible almost complex structures. In fact, the compatibility is used to prove Gromov compactness, which was actually proved in [17] for the tame almost

complex structure. In fact, we cannot take F_c to be a symplectomorphism in general. So in general, $(F_c)_*J$ is not compatible with ω . However, it is tamed by ω if c is sufficiently small. This is because the condition for an almost complex structure to be tame is an open condition.

We consider the cyclic filtered A_∞ -algebra $(\Lambda(L(c)), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{(F_c)_*J}\})$. We compare it with $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathfrak{m}_{k,\beta}^J\})$. (Here we include $(F_c)_*J$ and J in the notation to specify the complex structure we use.) The closed one-form \mathbf{e}_i representing the basis $H^1(L; \mathbb{Z})$ is transformed to a closed one-form $\mathbf{e}_i(c)$ on $L(c)$ by the diffeomorphism F_c . For \mathbf{b} in (13.1) we put

$$\mathbf{b}_c(x_1, \dots, x_{b_1}) = \mathbf{b} = \sum_{i=1}^{b_1} (x_i \cdot (F_c)_*(\mathbf{e}_i)).$$

We define

$$\begin{aligned} \mathbf{b}_{\text{high},c}(x_{b_1+1}, \dots, x_b) \\ = \mathbf{b}_{\text{high},c} = \sum_{i>b_1} (x_i \cdot (F_c)_*(\mathbf{e}_i)) \end{aligned}$$

and $\mathbf{b}_{c+} = \mathbf{b}_c + \mathbf{b}_{\text{high},c}$, $\mathbf{b}_+ = \mathbf{b} + \mathbf{b}_{\text{high}}$.

LEMMA 13.4

We may choose perturbation of the choices entering in the definition of $\mathfrak{m}_{k,\beta}^{(F_c)_*J}$, $\mathfrak{m}_{k,\beta}^J$ such that the following holds:

$$\begin{aligned} (13.14) \quad \mathfrak{m}_{k+m, (F_c^{-1})^*\beta}^{(F_c)_*J}(\mathbf{b}_{c+}^{\otimes m_0}, (F_c^{-1})^*\mathbf{x}_1, \mathbf{b}_{c+}^{\otimes m_1}, \dots, \mathbf{b}_{c+}^{\otimes m_{k-1}}, (F_c^{-1})^*\mathbf{x}_k, \mathbf{b}_{c+}^{\otimes m_k}) \\ = \mathfrak{m}_{k+m, \beta}^J(\mathbf{b}_+^{\otimes m_0}, \mathbf{x}_1, \mathbf{b}_+^{\otimes m_1}, \dots, \mathbf{b}_+^{\otimes m_{k-1}}, \mathbf{x}_k, \mathbf{b}_+^{\otimes m_k}). \end{aligned}$$

Proof

We remark that F_c gives an isomorphism

$$(13.15) \quad (F_c)_* : \mathcal{M}_k(\beta; J) \rightarrow \mathcal{M}_k((F_c)_*\beta; (F_c)_*J).$$

We can extend this isomorphism to one of the Kuranishi structures. Therefore we can take the continuous family of perturbations in Corollary 5.1 so that it is preserved by (13.15). The lemma follows immediately. \square

LEMMA 13.5

If $\partial\beta \cap \mathbf{e}_i = g_i$, then

$$(F_c)_*(\beta) \cap (F_c^{-1})^*(\mathbf{e}_i) = \beta \cap \mathbf{e}_i + \sum_{j=1}^{b_1} c_j g_i.$$

Proof

This follows from the fact that $F_c(L) = L(c)$ is the graph of $\sum c_i \mathbf{e}_i$. \square

Let $\mathbf{b}_+ = \sum_i x_i \mathbf{e}_i$. We put $y_i = e^{x_i}$, $i = 1, \dots, b_1$, and $\vec{x} = (y_1, \dots, y_{b_1}, x_{b_1+1}, \dots, x_b)$. We define

$$(13.16) \quad \begin{aligned} & \mathfrak{m}_{k,\beta}^{(F_c)_* J, \vec{x}}((F_c^{-1})^* \mathbf{x}_1, \dots, (F_c^{-1})^* \mathbf{x}_k) \\ &= \sum_m \sum_{m=m_0+\dots+m_k} T^{(F_c)_*(\beta) \cap \omega} e^{\mu(\beta)/2} \\ & \quad \times \mathfrak{m}_{k+m, (F_c^{-1})^* \beta}^{(F_c)_* J}(\mathbf{b}_{c_+}^{\otimes m_0}, (F_c^{-1})^* \mathbf{x}_1, \dots, (F_c^{-1})^* \mathbf{x}_k, \mathbf{b}_{c_+}^{\otimes m_k}). \end{aligned}$$

We write

$$(13.17) \quad \begin{aligned} & \mathfrak{m}_{k,\beta}^{J, \vec{x}}(\mathbf{x}_1, \dots, \mathbf{x}_k) \\ &= \sum_m \sum_{m=m_0+\dots+m_k=m} T^{\beta \cap \omega} e^{\mu(\beta)/2} \mathfrak{m}_{k+m, \beta}^J(\mathbf{b}_+^{\otimes m_0}, \mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{b}_+^{\otimes m_k}). \end{aligned}$$

We put

$$\vec{x}(c) = (T^{c_1} y_1, \dots, T^{c_{b_1}} y_{b_1}, x_{b_1+1}, \dots, x_b).$$

Then Lemmas 13.4 and 13.5 imply

$$(13.18) \quad \mathfrak{m}_{k,\beta}^{(F_c)_* J, \vec{x}}((F_c^{-1})^* \mathbf{x}_1, \dots, (F_c^{-1})^* \mathbf{x}_k) = \mathfrak{m}_{k,\beta}^{J, \vec{x}(c)}(\mathbf{x}_1, \dots, \mathbf{x}_k).$$

We apply Corollary 12.1 to $L(c)$ and $(F_c)_* J$. Then the sum of the left-hand side of (13.18) over β converges for $y_i \in 1 + \Lambda_{0, \text{nov}}^+$ and $v(x_i) \geq 0$ ($i = b_1 + 1, \dots$). Therefore the sum of the right-hand side of (13.18) over β also converges there. Hence, by taking various c_i , we obtain Theorem 1.2(3), (4). \square

REMARK 13.2

By the construction of this section, we can prove a similar convergence result for the pseudoisotopy we constructed in Sections 11, 12, and 14. Therefore the family of unital, cyclic filtered A_∞ -algebras in Theorem 1.2.4 is well defined up to the homotopy equivalence of unital, cyclic filtered A_∞ -algebras.

14. Pseudoisotopy of pseudoisotopies

Let $\overline{\mathcal{C}}$ be a finite-dimensional \mathbb{R} -vector space or a de Rham complex $\Lambda(L)$. The vector space $C^\infty([0, 1]^2, \overline{\mathcal{C}})$ is the set of all smooth maps $[0, 1]^2 \rightarrow \overline{\mathcal{C}}$. Let $C^\infty([0, 1]^2 \times \overline{\mathcal{C}})$ be the set of the formal expression

$$(14.1) \quad x(t, s) + dt \wedge y(t, s) + ds \wedge z(t, s) + dt \wedge ds \wedge w(t, s),$$

where $x(t, s), y(t, s), z(t, s), w(t, s) \in C^\infty([0, 1]^2 \times \overline{\mathcal{C}})$. We define the degree by putting $\deg dt = \deg ds = 1$.

In the case of the de Rham complex $\overline{\mathcal{C}} = \Lambda(L)$, we put

$$C^\infty([0, 1]^2 \times \overline{\mathcal{C}}) = \Lambda([0, 1]^2 \times L).$$

(In this case also, an element of $\Lambda([0, 1]^1 \times L)$ can be uniquely written as (14.1).)

We assume that, for each $(t, s) \in [0, 1]^2$, we have operations

$$(14.2) \quad \mathfrak{m}_{k,\beta}^{t,s} : B_k(\overline{\mathcal{C}}[1]) \rightarrow \overline{\mathcal{C}}[1]$$

of degree $-\mu(\beta) + 1$,

$$(14.3) \quad \mathfrak{c}_{k,\beta}^{t,s} : B_k(\overline{C}[1]) \rightarrow \overline{C}[1], \quad \mathfrak{d}_{k,\beta}^{t,s} : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

of degree $-\mu(\beta)$, and

$$(14.4) \quad \mathfrak{e}_{k,\beta}^t : B_k(\overline{C}[1]) \rightarrow \overline{C}[1]$$

of degree $-\mu(\beta) - 1$. We assume that they are smooth in the following sense:

$$(s, t) \mapsto \mathfrak{m}_{k,\beta}^{t,s}(x_1, \dots, x_k) \in C^\infty([0, 1], \overline{C}),$$

and similarly for $\mathfrak{c}_{k,\beta}^{t,s}, \mathfrak{d}_{k,\beta}^{t,s}, \mathfrak{e}_{k,\beta}^{t,s}$. We use them to define

$$(14.5) \quad \mathfrak{M}_{k,\beta} : B_k(C^\infty([0, 1]^2 \times \overline{C})[1]) \rightarrow C^\infty([0, 1]^2 \times \overline{C})$$

as follows. Let

$$\mathbf{x}_i = x_i(t, s) + dt \wedge y_i(t, s) + ds \wedge z_i(t, s) + dt \wedge ds \wedge w_i(t, s).$$

We put

$$\mathfrak{M}_{k,\beta}(\mathbf{x}_1, \dots, \mathbf{x}_k) = x(t, s) + dt \wedge y(t, s) + ds \wedge z(t, s) + dt \wedge ds \wedge w(t, s),$$

where $x(t, s), y(t, s), z(t, s), w(t, s)$ are defined as follows:

$$(14.6a) \quad x(t, s) = \mathfrak{m}_{k,\beta}^{t,s}(x_1(t, s), \dots, x_k(t, s)),$$

$$(14.6b) \quad \begin{aligned} y(s, t) &= \mathfrak{c}_{k,\beta}^{t,s}(x_1(t, s), \dots, x_k(t, s)) \\ &+ \sum (-1)^{*i} \mathfrak{m}_{k,\beta}^{t,s}(x_1(t, s), \dots, y_i(t, s), \dots, x_k(t, s)) \end{aligned}$$

where $*i = \deg' x_1 + \dots + \deg' x_{i-1} + 1$,

$$(14.6c) \quad \begin{aligned} z(s, t) &= \mathfrak{d}_{k,\beta}^{t,s}(x_1(t, s), \dots, x_k(t, s)) \\ &+ \sum (-1)^{*i} \mathfrak{m}_{k,\beta}^{t,s}(x_1(t, s), \dots, z_i(t, s), \dots, x_k(t, s)), \end{aligned}$$

$$(14.6d) \quad \begin{aligned} w(s, t) &= \mathfrak{e}_{k,\beta}^{t,s}(x_1(t, s), \dots, x_k(t, s)) \\ &+ \sum (-1)^{*i} \mathfrak{c}_{k,\beta}^{t,s}(x_1(t, s), \dots, z_i(t, s), \dots, x_k(t, s)) \\ &+ \sum (-1)^{*i} \mathfrak{d}_{k,\beta}^{t,s}(x_1(t, s), \dots, y_i(t, s), \dots, x_k(t, s)) \\ &+ \sum_{i < j} (-1)^{*ij} \mathfrak{m}_{k,\beta}^{t,s}(x_1(t, s), \dots, y_i(t, s), \dots, z_j(t, s), \dots, x_k(t, s)) \\ &+ \sum_{i > j} (-1)^{*ij} \mathfrak{m}_{k,\beta}^{t,s}(x_1(t, s), \dots, z_j(t, s), \dots, y_i(t, s), \dots, x_k(t, s)). \end{aligned}$$

Here

$$\begin{aligned} *i^1 &= \deg' x_1 + \dots + \deg' x_{i-1}, \\ *i^2 &= \deg' x_1 + \dots + \deg' x_{i-1}, \\ *ij^3 &= \deg' y_i + \deg' x_{i+1} + \dots + \deg' x_{j-1}, \end{aligned}$$

$$*_{ij}^4 = 1 + \deg' z_i + \deg' x_{i+1} + \cdots + \deg' x_{j-1}.$$

In the case $(k, \beta) = (1, (0, 0))$, we put

$$\begin{aligned} x(t, s) &= \mathbf{m}_{1,(0,0)}(x_1(t, s)), \\ y(t, s) &= \frac{d}{dt}x_1(t, s) - \mathbf{m}_{1,(0,0)}(y_1(t, s)), \\ z(t, s) &= \frac{d}{ds}x_1(t, s) - \mathbf{m}_{1,(0,0)}(z_1(t, s)), \\ w(s, t) &= \frac{d}{ds}y_1(t, s) - \frac{d}{dt}z_1(t, s) + \mathbf{m}_{1,(0,0)}(w_1(t, s)). \end{aligned}$$

DEFINITION 14.1

We say that $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathfrak{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ is a *pseudoisotopy of pseudoisotopies* if the following hold:

- (1) $\mathfrak{M}_{k,\beta}$ satisfies filtered A_∞ -formula (6.7);
- (2) $\mathbf{m}_{k,\beta}^{t,s}, \mathbf{c}_{k,\beta}^{t,s}, \mathfrak{d}_{k,\beta}^{t,s}, \mathbf{e}_{k,\beta}^{t,s}$ are all cyclically symmetric;
- (3) $\mathbf{m}_{k,(0,0)}^{t,s}$ is independent of t, s . Moreover, $\mathbf{c}_{k,(0,0)}^{t,s}, \mathfrak{d}_{k,(0,0)}^{t,s}, \mathbf{e}_{k,(0,0)}^{t,s}$ are all zero.

The unital and/or mod T^{E_0} -version is defined in the same way.

If $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathfrak{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ is a pseudoisotopy of pseudoisotopies, then for each $s_0 \in [0, 1]$, $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s_0}\}, \{\mathbf{c}_{k,\beta}^{t,s_0}\})$ is a pseudoisotopy, and for each $t_0 \in [0, 1]$, $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t_0,s}\}, \{\mathfrak{d}_{k,\beta}^{t_0,s}\})$ is also a pseudoisotopy. We call them the *restrictions*.

THEOREM 14.1

Let $E_0 < E_1$. Let $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t_0,s_0}\})$ be cyclic filtered A_∞ -algebras modulo T^{E_1} for $s_0, t_0 \in \{0, 1\}$. For $s_0 = 0, 1$, let $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s_0}\}, \{\mathbf{c}_{k,\beta}^{t,s_0}\})$ be a pseudoisotopy modulo T^{E_1} . For $t_0 = 1$, let $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t_0,s}\}, \{\mathfrak{d}_{k,\beta}^{t_0,s}\})$ be a pseudoisotopy modulo T^{E_1} .

Let $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathfrak{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ be a pseudoisotopy of pseudoisotopies modulo T^{E_0} .

We assume that the restriction of a pseudoisotopy of pseudoisotopies to $s_0 = 0, 1$ or to $t_0 = 1$ coincides with the above pseudoisotopy as pseudoisotopies modulo T^{E_0} .

We assume Assumption 14.1.

Then $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathfrak{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ extends to a pseudoisotopy of pseudoisotopies modulo T^{E_1} so that its restriction to $s_0 = 0, 1$ or to $t_0 = 1$ coincides with the above pseudoisotopy as pseudoisotopies modulo T^{E_1} .

The unital version also holds.

REMARK 14.1

This theorem is a cyclic and de Rham version of [11, Theorem 7.2.212].

ASSUMPTION 14.1

(1) If $E(\beta) < E_0$, then $\mathbf{c}_{k,\beta}^{t,s}, \mathfrak{d}_{k,\beta}^{t,s}, \mathbf{e}_{k,\beta}^{t,s}$ are zero in a neighborhood of $\{0, 1\}^2$ and $\mathbf{m}_{k,\beta}^{t,s}$ is locally constant there.

(2) If $E(\beta) < E_1$, then $\mathbf{c}_{k,\beta}^{t,s_0}$ ($s_0 = 0, 1$) is zero if t is in a neighborhood of $\{0, 1\}$. Moreover, $\mathfrak{d}_{k,\beta}^{0,s}$ is zero if s is in a neighborhood of $\{0, 1\}$. Furthermore, $\mathbf{m}_{k,\beta}^{t,s_0}, \mathbf{m}_{k,\beta}^{0,s}$ are locally constant there.

We remark that by the method of the proof of Lemma 8.2, we may change $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathfrak{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ so that it satisfies Assumption 14.1.

Proof of Theorem 14.1

We take a map

$$h = (h_t(u, v), h_s(u, v)) : [0, 1]_u \times [0, 1]_v \rightarrow [0, 1]_t \times [0, 1]_s$$

with the following properties (we write $[0, 1]_s$, etc., to show that the parameter of this factor is s):

- (1) $h(u, 0) = (1, 0), h(u, 1) = (1, 1);$
- (2) $h(1, v) = (1, v);$
- (3) $h(0, 1/3) = (0, 0), h(0, 2/3) = (0, 1);$
- (4) the restriction of h determines a homeomorphism

$$[0, 1]^2 \setminus ([0, 1] \times \{0\}) \rightarrow [0, 1]^2 \setminus (([0, 1] \times \{0, 1\}) \cup \{0\} \times [0, 1]);$$

(5) it is a diffeomorphism outside $\{(0, 1/3), (0, 2/3)\}$ of the domain and $\{(0, 0), (0, 1)\}$ of the target.

In particular, h determines a homeomorphism

$$(14.7) \quad \{0\} \times [0, 1]_v \cong (\{0\} \times [0, 1]_s) \cup ([0, 1]_t \times \{0, 1\}).$$

$(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{0,s}\}, \{\mathfrak{d}_{k,\beta}^{0,s}\})$ and $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s_0}\}, \{\mathbf{c}_{k,\beta}^{t,s_0}\})$ define pseudoisotopies parameterized by the target of (14.7). We pull it back by (14.7) and obtain a pseudoisotopy $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{0,v}\}, \{\mathbf{c}_{k,\beta}^{0,v}\})$ modulo T^{E_1} parameterized by $[0, 1]_v$. (The pullback is defined by a formula similar to that in (8.11). We use Assumption 14.1 to show the smoothness of the pullback at $v = 1/3, 2/3$.)

We next pull back $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathfrak{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ by h as follows. We consider

$$dh_t = \frac{dh_t}{du} du + \frac{dh_t}{dv} dv, \quad dh_s = \frac{dh_s}{du} du + \frac{dh_s}{dv} dv,$$

$$dh_t \wedge dh_s = \left(\frac{dh_t}{du} \frac{dh_s}{dv} - \frac{dh_t}{dv} \frac{dh_s}{du} \right) du \wedge dv$$

and put

$$(14.8) \quad \mathbf{c}_{k,\beta}^{u,v} = \frac{dh_t}{du} \mathbf{c}_{k,\beta}^{h(u,v)} + \frac{dh_s}{du} \mathfrak{d}_{k,\beta}^{h(u,v)},$$

$$\mathfrak{d}_{k,\beta}^{u,v} = \frac{dh_t}{dv} \mathbf{c}_{k,\beta}^{h(u,v)} + \frac{dh_s}{dv} \mathfrak{d}_{k,\beta}^{h(u,v)},$$

$$\mathbf{e}_{k,\beta}^{u,v} = \left(\frac{dh_t}{du} \frac{dh_s}{dv} - \frac{dh_t}{dv} \frac{dh_s}{du} \right) \mathbf{e}_{k,\beta}^{h(u,v)}.$$

(We use Assumption 14.1 to show that the pullback is smooth.) It may be regarded as a pseudoisotopy modulo T^{E_0} from the pullback of $(C^\infty([0,1] \times \overline{C}), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{0,v}\}, \{\mathbf{c}_{k,\beta}^{0,v}\})$ to the restriction of $(C, \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s}\}, \{\mathbf{c}_{k,\beta}^{t,s}\}, \{\mathbf{d}_{k,\beta}^{t,s}\}, \{\mathbf{e}_{k,\beta}^{t,s}\})$ to the line $s = 1$. We now apply Theorem 8.2 and extend $\mathbf{c}_{k,\beta}^{u,v}, \mathbf{d}_{k,\beta}^{u,v}, \mathbf{e}_{k,\beta}^{u,v}$ so that we obtain an isotopy of isotopies module T^{E_1} .

Therefore we pull back again by the inverse of h to obtain required isotopy of isotopies. The proof of Theorem 14.1 is complete. \square

Now the main result of this section is the following.

THEOREM 14.2

The cyclic, unital filtered A_∞ -structure in Theorem 12.1 is independent of the choices up to pseudoisotopy.

Proof

The proof is similar to that in [11, Section 7.2.13].

We take tame almost complex structures J_0 and J_1 and $E_0 < \dots < E_i \dots$. We use J_0 and E_i to define cyclic filtered A_∞ -structures $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(0,i)}\})$ modulo T^{E_i} and pseudoisotopies modulo T^{E_i} $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,(0,i)}\}, \{\mathbf{c}_{k,\beta}^{t,(0,i)}\})$ between $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(0,i)}\})$ and $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(0,i+1)}\})$. We then extend them to cyclic filtered A_∞ -structures and pseudoisotopies. We next use J_1 and E_i to define $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(1,i)}\})$ and $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,(1,i)}\}, \{\mathbf{c}_{k,\beta}^{t,(0,i)}\})$ and extend them to cyclic filtered A_∞ -structures and pseudoisotopies.

To prove Theorem 14.2, it suffices to show that the extension of $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(0,i)}\})$ is pseudoisotopic to the extension of $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(1,i)}\})$.

Theorem 11.1 implies that there exists a pseudoisotopy modulo T^{E_i} , $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{s,i}\}, \{\mathbf{d}_{k,\beta}^{s,i}\})$ between $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(0,i)}\})$ and $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{(1,i)}\})$.

LEMMA 14.1

There exists a pseudoisotopy of pseudoisotopies modulo T^{E_i} , $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{t,s;i}\}, \{\mathbf{c}_{k,\beta}^{t,s;i}\}, \{\mathbf{d}_{k,\beta}^{t,s;i}\}, \{\mathbf{e}_{k,\beta}^{t,s;i}\})$ such that its restriction to $t = j$ ($j = 0, 1$) coincides with $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{s,i+j}\}, \{\mathbf{c}_{k,\beta}^{s,i+j}\})$ as pseudoisotopy modulo $T^{E_{i+j}}$. Moreover, its restriction to s_j ($j = 0, 1$) coincides with $(\Lambda(L), \langle \cdot, \cdot \rangle, \{\mathbf{m}_{k,\beta}^{s,i}\}, \{\mathbf{d}_{k,\beta}^{s,i}\})$ as pseudoisotopy modulo T^{E_i} .

Proof

We construct a two-parameter family of Kuranishi structures and multisections in a way similar to that used in Section 11. Then we use it to construct pseudoisotopy of pseudoisotopies in the same way as (11.5). \square

By Lemma 14.1 and Theorem 14.1, we can extend the pseudoisotopy modulo $T^{E_i}(\Lambda(L), \langle \cdot \rangle, \{\mathfrak{m}_{k,\beta}^{s,i}\}, \{\mathfrak{d}_{k,\beta}^{s,i}\})$ to a pseudoisotopy. The proof of Theorem 14.2 is complete. \square

The uniqueness part of Theorem 1.1 follows from Theorems 14.2, 8.1, and 8.2.

Acknowledgments. The author thanks Y.-G. Oh, H. Ohta, and K. Ono, with whom he is happy to share many of the ideas of this article.

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