A sufficient condition for well-posedness for systems with time-dependent coefficients

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Abstract We consider linear, smooth, hyperbolic systems with time-dependent coefficients and size N. We give a condition sufficient for the well-posedness of the Cauchy Problem in some Gevrey classes. We present some Levi conditions to improve the Gevrey index of well-posedness for the scalar equation of order N, using the transformation in [DAS] and following the technique introduced in [CT]. By using this result and adding some assumptions on the form of the first-order term, we can improve the well-posedness for systems. A similar condition has been studied in [DAT] for systems with size 3.

1. Introduction

In this article we study the well-posedness of the Cauchy Problem for first-order $(N \times N)$-systems whose coefficients depend only on the time variable,

$$
\begin{cases} L(t,\partial_t,\partial_x)U(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ U(0,x) = U_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1}
$$

where

$$
LU = \partial_t U - \sum_{j=1}^{n} A_j(t)\partial_{x_j} - B(t)U, \quad A_j(t) \in M_N(\mathbb{R}), B(t) \in M_N(\mathbb{C});
$$

by $M_N(\mathbb{R})$ (resp., $M_N(\mathbb{C})$) we denote the space of the $(N \times N)$-matrices with entries valued in $\mathbb{R}$ (resp., $\mathbb{C}$). We assume that

$$
A(t,\xi) := |\xi|^{-1} \sum_{j=1}^{n} A_j(t)\xi_j
$$

is (weakly) hyperbolic; that is, its eigenvalues are real (not necessarily distinct) for any $\xi \in \mathbb{R}^n \setminus \{0\}$. We remark that A is real valued, whereas B may be complex valued. In the following, we assume that $A \in \mathcal{C}^N$; that is, each entry of A_j belongs to $\mathcal{C}^N([0,T])$, for $j = 1, \dots, n$. We assume also that $B \in \mathcal{C}^{N-1}$.

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NOTATION

Let $f(t)$ and $g(t)$ be defined in $[0,T]$; we write $f \lesssim g$, meaning that there exists a positive constant C such that

$$f(t) \le Cg(t) \quad \text{for } t \in [0,T].$$

Analogously, if $f(t,\xi)$ and $g(t,\xi)$ are symbols defined in $[0,T] \times \mathbb{R}^n$, we write $f \lesssim g$, meaning that there exists a positive constant C such that

$$f(t,\xi) \le Cg(t,\xi) \quad \text{for } (t,\xi) \in [0,T] \times \mathbb{R}^n.$$

In both cases, we write $f \approx g$, meaning that $f \lesssim g$ and $g \lesssim f$.

NOTATION

We denote with γ^d the Gevrey class with index $d \in (1,\infty)$, that is, the class of functions in $\mathcal{C}^\infty(\mathbb{R}^n)$ such that for any compact $K \subset \mathbb{R}^n$, there exists C_K such that

$$|\partial_x^\alpha f(x)| \le C_K^{|\alpha|+1}(|\alpha|!)^d \quad \text{for any } x \in K \text{ and } \alpha \in \mathbb{N}^n.$$

NOTATION

We put $I := \{1,\dots,N\}$. We denote by I_M the identity matrix of size M. We denote by $\|A\| := \max_{i,j} |a_{ij}|$ the norm of a matrix $A = (a_{ij})_{i,j}$.

DEFINITION 1

The Cauchy problem (1.1) is said to be well posed in γ^d with $1 < d < \infty$ if, for any choice of the data $f \in \mathcal{C}([0,T],\gamma^d(\mathbb{R}^n,\mathbb{C}^N))$ and $U_0 \in \gamma^d(\mathbb{R}^n,\mathbb{C}^N)$, it admits a unique solution $U \in \mathcal{C}^1([0,T],\gamma^d(\mathbb{R}^n,\mathbb{C}^N))$.

In this article, we say that (1.1) is strongly well posed in γ^d to mean that it is well posed for any choice of the lower-order term $B(t) \in C^{N-1}([0,T])$.

We refer the interested reader to [CI], [CJS], [CO], [D], [DAK], and [Y] for questions related to the well-posedness of weakly hyperbolic equations and systems.

It is acknowledged (see [B2]) that the Cauchy problem (1.1) for a system is well posed in γ^d for

$$(1.2) \qquad 1 < d < d_B \equiv d_B(r) := 1 + \frac{1}{r-1},$$

where r is the maximum multiplicity of the eigenvalues of A. If the multiplicities of the eigenvalues are not constant, then this Gevrey index may be improved. In this note, we consider matrices $A(t,\xi)$ whose eigenvalues have variable multiplicity and such that $A(0,\xi)$ has a unique eigenvalue with multiplicity N; hence, Bronšteĭn's index is

$$d_B(N) = 1 + \frac{1}{N-1}.$$

By Bronšteĭn's Lemma [B1, Theorem 2] the eigenvalues of $A(t,\xi)$, counted with their multiplicities, namely,

$$\{\lambda_i(t,\xi) : i = 1,\dots,N\},$$

are Lipschitz-continuous functions. In this article, we assume that $\lambda_1, \dots, \lambda_N$ satisfy the following.

ASSUMPTION 1

For any $i, j = 1, \dots, N$, there exists $\kappa_{ij} \in [1, \infty]$, such that either

$$\begin{aligned} &|\lambda_i - \lambda_j| \approx t^{\kappa_{ij}} \quad \text{if } \kappa_{ij} \in [1, \infty), \text{ or} \\ &|\lambda_i - \lambda_j| \lesssim t^m \quad \text{for any } m \in \mathbb{N} \quad \text{if } \kappa_{ij} = \infty. \end{aligned} \tag{1.3}$$

We remark that $\kappa_{ij} = \kappa_{ji}$ and that $\kappa_{jj} = \infty$; that is, $\kappa = (\kappa_{ij})_{i,j}$ is a symmetric matrix with ∞ as diagonal entries.

REMARK 1.1

If $\kappa_{ij} < \kappa_{jk}$ for some i, j, k, then $\kappa_{ik} = \kappa_{ij}$; indeed, in such a case, it holds that

$$t^{\kappa_{ij}} \lesssim |\lambda_i - \lambda_j| - |\lambda_j - \lambda_k| \le |\lambda_i - \lambda_k| \le |\lambda_i - \lambda_j| + |\lambda_j - \lambda_k| \lesssim t^{\kappa_{ij}}.$$

We look for a sufficient condition for the well-posedness of the Cauchy problem (1.1) in γ^d for any $1 < d < d^*$, for some $d^* > d_B(N)$.

It is easy to check that, if the system has size $N = 2$ and

$$|\lambda_1 - \lambda_2| \approx t^\alpha, \quad 1 < \alpha < \infty,$$

then the Cauchy problem (1.1) is well posed in γ^d for any $1 < d < d^*$, where

$$d^* = 1 + \frac{\alpha + 1}{\alpha - 1} = 2 + \frac{2}{\alpha - 1}.$$

We remark that $d^* > 2 = d_B(2)$. On the other hand, if $|\lambda_1 - \lambda_2| \approx t$, then (1.1) is well posed in any γ^d, $d > 1$, and in $\mathcal{C}^\infty$.

For the sake of brevity, in this article we assume $N \ge 3$ (in particular, for systems with size 2 or 3, see also [DAT]). F. Colombini and G. Taglialatela [CT] stated a condition for the well-posedness of the equation of order N in Gevrey classes; in [CT] the functions λ_j represent the roots of the characteristic equation.

With the notation in Assumption 1, they assumed the existence of $\kappa_1, \dots, \kappa_{N-1} \ge 1$, such that

$$\begin{aligned} &\kappa_1 \le \kappa_2 \le \cdots \le \kappa_{N-1} < \infty \qquad \text{and} \\ &\kappa_{ij} = \kappa_i, \quad \text{for any } j = i+1, \dots, N, \end{aligned} \tag{1.4}$$

and they proved, in [CT, Theorem 1.2], that the Cauchy problem for the equation of order N is strongly well posed in γ^d for any $1 < d < d^*$ with

$$d^* = 1 + \frac{\kappa_h + 1}{(N - h - 1)\kappa_h + s_h - 1}, \tag{1.5}$$

where

$$s_p = \kappa_1 + \cdots + \kappa_p \qquad \text{and} \qquad h := \min\{p : s_p + p \ge N\}.$$

Condition (1.4) on the structure of $\kappa = (\kappa_{ij})_{i,j}$ is very restrictive; in fact,

$$\kappa = \begin{pmatrix} \infty & \kappa_1 & \kappa_1 & \dots & \kappa_1 \\ \kappa_1 & \infty & \kappa_2 & \dots & \kappa_2 \\ \kappa_1 & \kappa_2 & \infty & \dots & \kappa_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_1 & \kappa_2 & \kappa_3 & \dots & \infty \end{pmatrix}.$$

In order to get a similar result of well-posedness without condition (1.4) on the matrix κ, we have to construct a suitable sequence $\kappa_1, \dots, \kappa_{N-1}$, which depends on the structure of the matrix $\kappa = (\kappa_{ij})$.

DEFINITION 2

Let $k \in [1, \infty]$. Then there exist a unique integer $m \le N$ and a unique partition $P_k(I) = \{I_1, \dots, I_m\}$ of the set $I = \{1, \dots, N\}$, namely,

$$I = \bigcup_{p=1}^{m} I_p \quad \text{with } I_p \neq \emptyset \text{ and } I_p \cap I_q = \emptyset \text{ for } p \neq q,$$

such that $\kappa_{ij} > k$ if and only if $i, j \in I_p$ for some p. We call such a partition the *k-partition* of the set I. We define

$$\alpha = \min\{\kappa_{ij} : i, j = 1, \dots, N\},$$

and we call the *minimum partition* of the set I the partition $P_{\min}(I) = P_\alpha(I)$.

REMARK 1.2

Let $k \in [1, \infty]$; then the *k-partition* of I is the trivial partition $P_k(I) = \{I\}$ if and only if $k < \alpha$. On the other hand, for $k = \infty$, the *∞-partition* of I is the trivial partition $P_\infty(I) = \{\{1\}, \{2\}, \dots, \{N\}\}$.

We remark that if $k_{ij} = \alpha$ for any $i \neq j$, then $P_{\min}(I) = P_\alpha(I) = \{\{1\}, \{2\}, \dots, \{N\}\}$, too.

REMARK 1.3

Let $P_{\min}(I) = \{I_1, \dots, I_m\}$ be the *minimum partition* of I. After a permutation on I, we can write

$$I_1 = \{1, \dots, \#(I_1)\}, \qquad I_2 = \{\#(I_1) + 1, \dots, \#(I_1) + \#(I_2)\}, \dots,$$
$$I_m = \{N - \#(I_m) + 1, \dots, N\}.$$

Therefore (see Remark 1.1) the matrix $\kappa = (\kappa_{ij})$ can be represented in the block form

$$\kappa = \begin{pmatrix} \kappa^{(1)} & C_\alpha & C_\alpha & \dots & C_\alpha \\ C_\alpha & \kappa^{(2)} & C_\alpha & \dots & C_\alpha \\ C_\alpha & C_\alpha & \kappa^{(3)} & \dots & C_\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_\alpha & C_\alpha & C_\alpha & \dots & \kappa^{(m)} \end{pmatrix}, \tag{1.6}$$

where C_α are blocks with suitable size and each entry equal to α, and $\kappa^{(p)} = (\kappa_{ij})_{i,j\in I_p}$ are square blocks with size $\#(I_p)$ and $\kappa_{ij}^{(p)} > \alpha$.

REMARK 1.4

Let $P_{\min}(I) = \{I_1,\dots,I_m\}$ be the *minimum partition* of I; if there exist two different subsets I_p, I_q with $\#(I_p), \#(I_q) \geq 2$, then condition (1.4) cannot be satisfied.

EXAMPLE 1.5

Let $N = 4$, and assume that

$$\kappa_{12} = \beta_1, \qquad \kappa_{34} = \beta_2, \qquad \kappa_{13} = \alpha$$

with $\alpha < \beta_1, \beta_2$; that is,

$$\kappa = \begin{pmatrix} \infty & \beta_1 & \alpha & \alpha \\ \dots & \infty & \alpha & \alpha \\ \dots & \dots & \infty & \beta_2 \\ \dots & \dots & \dots & \infty \end{pmatrix}.$$

It is clear that condition (1.4) is not satisfied. Nevertheless, in [CT, Theorem 1.3], it is proved that the Cauchy problem is strongly well posed in γ^d for any $1 < d < d^*$, where d^* is determined as in (1.5), provided that we put

$$\kappa_1 = \kappa_2 = \alpha, \qquad \kappa_3 = \max\{\beta_1, \beta_2\}.$$

Hence (1.5) gives

$$d^* = 1 + \frac{\alpha+1}{3\alpha-1}.$$

We are ready to state the following.

THEOREM 1

We assume that, for any $t > 0$, the matrix $A(t,\xi)$ has m distinct eigenvalues $\mu_1,\dots,\mu_m$, that each eigenvalue μ_p has constant multiplicity M_p, and that

$$|\mu_i - \mu_j| \approx t^\alpha, \quad i \neq j.$$

Let $M = \max_p M_p$. Then the Cauchy problem (1.1) is strongly well posed in γ^d for any $1 < d < d^*$ with

$$d^* := \begin{cases} 1 + \frac{\alpha+1}{(N-1)\alpha-1} & \text{if } (N-M)\alpha \geq M, \\ 1 + \frac{1}{M-1} \equiv d_B(M) & \text{if } (N-M)\alpha \leq M. \end{cases} \tag{1.7}$$

REMARK 1.6

More in general, if we assume that $A(t,\xi)$ verifies Assumption 1 and we put

$$M = \max_p \#(I_p),$$

where $P_{\min}(I) = P_\alpha(I) = \{I_1, \ldots, I_m\}$ is the *minimum partition* of I as in Definition 2, then the Cauchy problem (1.1) is strongly well posed in γ^d for any $1 < d < d^*$ with d^* as in (1.7).

However, in this case the Gevrey index d^* may be improved (see Theorem 2) by adding further assumptions on the blocks $\kappa^{(p)}$ in (1.6). In fact, the proof is an immediate consequence of Theorem 2.

EXAMPLE 1.7
Let

$$A = A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where the matrix A_2 has a unique eigenvalue

$$\mu = \lambda_{(N-M)+1} = \cdots = \lambda_N$$

with multiplicity $M \geq N/2$, and assume that

$$|\mu - \lambda_j| \approx t^\alpha \quad \text{for any } j = 1, \ldots, N-M,$$
$$|\lambda_i - \lambda_j| \lesssim t^\alpha \quad \text{for any } i, j = 1, \ldots, N-M.$$

Then Remark 1.6 gives $d^* = d_B(M)$, provided that α is sufficiently small, namely,

$$\alpha \leq \frac{M}{N-M}.$$

We remark that $A_2(t,\xi)$ has a unique eigenvalue with constant multiplicity M.

EXAMPLE 1.8
We can apply Theorem 1 to Example 1.5. Indeed, we have

$$P_{\min}(I) = P_\alpha(I) = \{I_1, I_2\} \quad \text{with } I_1 = \{1,2\} \text{ and } I_2 = \{3,4\};$$

hence we get the expected Gevrey index:

$$d^* = 1 + \frac{\alpha+1}{3\alpha-1}.$$

We notice that $d^* = d_B(2) = 2$ as in Example 1.7 if and only if $\alpha = 1$ since $N = 4$ and $M = 2$.

In order to improve the Gevrey index d^* in Theorem 1, we need to refine the partition of I.

DEFINITION 3
Let $J \subset I$, and let

$$\alpha_J := \min\{\kappa_{ij} : i, j \in J\}.$$

We call the *minimum partition* of J the *α_J-partition* of J (see Definition 2): $P_{\alpha_J}(J) = \{J_1, \ldots, J_m\}$. Now we define by induction

$$\Sigma_h[J] := \max_p s_h[J_p, \#(J \setminus J_p)], \quad h = 1, \ldots, \#(J) - 1,$$

where

$$s_h[J_p,l] := \begin{cases} h\alpha_J & \text{if } h \le l, \\ l\alpha_J + \Sigma_{h-l}[J_p] & \text{if } h > l. \end{cases}$$

We put $\Sigma_h := \Sigma_h[I]$; it is clear that $\Sigma_1[J] = \alpha_J$.

EXAMPLE 1.9

We consider Example 1.5; with the notation introduced in Definition 3, we easily get

$$\Sigma_1 = \alpha = \kappa_1, \qquad \Sigma_2 = 2\alpha = \kappa_1 + \kappa_2,$$

since $\#(I \setminus I_1) = \#(I \setminus I_2) = 2$, whereas

$$\Sigma_3 = \max\big\{s_3[I_1,2], s_3[I_2,2]\big\} = 2\alpha + \max\big\{\Sigma_1[I_1], \Sigma_1[I_2]\big\} = 2\alpha + \max\{\beta_1,\beta_2\}.$$

LEMMA 1.10

Let $J \subset I$. We assume that for some permutation on the set I, we have $J = \{1,\dots,M\}$ with $M = \#(J)$, and J satisfies condition (1.4); that is,

$$\kappa_1 \le \cdots \le \kappa_{M-1} < \infty \qquad \textit{and} \qquad \kappa_{ij} = \kappa_i, \quad \textit{for any } j = i+1,\dots,M.$$

Then

$$\Sigma_h[J] = s_h \equiv \kappa_1 + \cdots + \kappa_h, \quad h = 1,\dots,M-1.$$

Proof

We can prove the statement by induction on M. It is trivially true for $M = 2$; we assume that the thesis holds for $M-1$, and we prove it for M. Let $m \ge 2$ be such that

$$\kappa_1 = \cdots = \kappa_{m-1} < \kappa_m;$$

that is, $s_p = p\alpha_J$ for any $p \le m-1$ and $\kappa_m > \alpha_J$. Then J is partitioned in

$$J = \{1\} \cup \cdots \cup \{m-1\} \cup J_m \quad \text{with} \quad J_m = \{m,\dots,M\},$$

with the notation introduced in Definition 3. Hence it holds that

$$\Sigma_h = \max\big\{h\alpha_J, \sigma_h[J_m,m-1]\big\} = \begin{cases} s_h & \text{if } h \le m-1, \\ s_{m-1} + \Sigma_{h-(m-1)}[J_m] & \text{if } m \le h \le M-1. \end{cases}$$

Since $\#(J_m) \le M-1$, we can apply the hypothesis of induction and

$$\Sigma_{h-(m-1)}[J_m] = \kappa_m + \cdots + \kappa_h.$$

This concludes the proof. □

We proved that Definition 3 is consistent with the one given in (1.4). Moreover, it is easy to check that each $J \subset I$ with $\#(J) \le 3$ satisfies (1.4).

REMARK 1.11

With the notation in Definition 2, let $P_{\min}(I) = \{I_1, \dots, I_m\}$, and let $M = \max \#(I_p)$; then

$$\Sigma_h = h\alpha \quad \text{for any } h \le N - M.$$

Indeed, it holds that

$$s_h[I_p, \#(I \setminus I_p)] = h\alpha, \quad p = 1, \dots, m,$$

since $N - \#(I_p) \ge N - M$.

DEFINITION 4

We define $\kappa_1 := \Sigma_1 = \alpha$, and we put, for any $h \ge 2$,

$$\kappa_h := \begin{cases} \Sigma_h - \Sigma_{h-1} & \text{if } \Sigma_h < \infty, \\ \infty & \text{if } \Sigma_h = \infty. \end{cases}$$

Thanks to Lemma 1.10, Definition 4 is consistent with the one given in (1.4).

EXAMPLE 1.12

Let $N = 5$; with the notation in Definition 2, we assume that

$$P_{\min}(I) = P_\alpha(I) = \{I_1, I_2\} \qquad \text{with } I_1 = \{1, 2\} \text{ and } I_2 = \{3, 4, 5\}.$$

The set I_2 satisfies (1.4) since $\#(I_2) \le 3$; hence we can assume with no restriction that

$$\kappa = \begin{pmatrix} \infty & \beta_1 & \alpha & \alpha & \alpha \\ \beta_1 & \infty & \alpha & \alpha & \alpha \\ \alpha & \alpha & \infty & \beta_2 & \beta_2 \\ \alpha & \alpha & \beta_2 & \infty & \gamma \\ \alpha & \alpha & \beta_2 & \gamma & \infty \end{pmatrix}, \quad \alpha < \beta_1, \beta_2, \ \beta_2 \le \gamma.$$

With the notation in Definitions 3 and 4, we get

$$\Sigma_1 = \alpha, \qquad \Sigma_2 = 2\alpha, \qquad \Sigma_3 = \max\{3\alpha, 2\alpha + \beta_2\} = 2\alpha + \beta_2,$$

$$\Sigma_4 = \max\{3\alpha + \beta_1, 2\alpha + \beta_2 + \gamma\} = 2\alpha + \beta_2 + \max\bigl\{\gamma, \beta_1 - (\beta_2 - \alpha)\bigr\};$$

hence

$$\kappa_1 = \kappa_2 = \alpha, \qquad \kappa_3 = \beta_2, \qquad \kappa_4 = \max\bigl\{\gamma, \beta_1 - (\beta_2 - \alpha)\bigr\}.$$

In particular, if $\beta_1 > \beta_2 + \gamma - \alpha$, then κ_4 can be different from any of κ_{ij}. We remark that in such a case, it holds that $\gamma < \kappa_4 < \beta_1$.

Now we are ready to state our main result.

THEOREM 2

If Assumption 1 is satisfied, then the Cauchy problem (1.1) is strongly well posed *in* γ^d *for any* $1 < d < d^*$ *with*

$$(1.8) \qquad d^* := \begin{cases} 1 + \frac{\kappa_h + 1}{(N-h-1)\kappa_h + \Sigma_h - 1} & \textit{if } \kappa_h < \infty, \\ 1 + \frac{1}{N-h} = d_B\big(N - (h-1)\big) & \textit{if } \kappa_h = \infty, \end{cases}$$

where

$$(1.9) \qquad h := \min\{p = 1, \dots, N-1 : \Sigma_p + p \geq N\}.$$

We remark that $\Sigma_p \geq p$; hence $h \leq (N+1)/2$ in Theorem 2. Moreover, the Gevrey index d^* is greater than or equal to $d_B(N)$, and the equality holds if and only if $\alpha = \infty$.

REMARK 1.13

If $\alpha \geq N-1$, then $h = 1$; hence

$$d^* = 1 + \frac{\alpha + 1}{(N-1)\alpha - 1}.$$

On the other hand, if $\alpha < N-1$, that is, $h \geq 2$, then from (1.9) it follows that

$$\Sigma_{h-1} + (h-1) < N \leq \Sigma_h + h;$$

hence

$$d_B\big(N - (h-1)\big) \leq d^* \leq d_B(N-h).$$

This shows that d^* is *increasing with respect to* h.

By using Theorem 2, we can prove Theorem 1.

Proof of Theorem 1

We assume first that $(N-M)\alpha \geq M$; that is,

$$(N-M)(\alpha+1) \geq N.$$

Then in Theorem 2 we get $h \leq N-M$ since, thanks to Remark 1.11,

$$\Sigma_p + p = p(\alpha+1) \quad \text{for any } p \leq N-M.$$

Hence $\kappa_h = \alpha$ and

$$d^* = 1 + \frac{\alpha+1}{(N-h-1)\alpha + h\alpha - 1} = 1 + \frac{\alpha+1}{(N-1)\alpha - 1}.$$

On the other hand, if $(N-M)\alpha < M$, then $h \geq N-M+1$. Hence, thanks to Remark 1.13, Theorem 2 gives

$$d^* \geq d_B\big(N - (N-M+1-1)\big) = d_B(M).$$

This concludes the proof. □

EXAMPLE 1.14

Let $N = 3M$, and assume that $A(t,\xi)$ has three eigenvalues μ_1, μ_2, μ_3, each one with multiplicity M for $t>0$, such that

$$|\mu_1-\mu_2|, |\mu_1-\mu_3| \approx t^{\alpha}, \qquad |\mu_2-\mu_3| \approx t^{\beta}, \qquad \beta \geq \alpha \geq 1.$$

Then we can apply Theorem 2; the matrix κ can be written as

$$\kappa = \begin{pmatrix} C_\infty & C_\alpha & C_\alpha \\ C_\alpha & C_\infty & C_\beta \\ C_\alpha & C_\beta & C_\infty \end{pmatrix},$$

where C_α (resp., C_β, C_∞) is an $(M\times M)$-block with each entry equal to α (resp., β, ∞). We get

$$\Sigma_h = \begin{cases} h\alpha & \text{if } h\leq M, \\ M\alpha + (h-M)\beta & \text{if } M+1 \leq h \leq 2M, \\ \infty & \text{if } 2M+1 \leq h. \end{cases}$$

Hence

$$d^* = \begin{cases} 1+\frac{\alpha+1}{(3M-1)\alpha-1} & \text{if } \alpha \geq 2, \\ 1+\frac{\beta+1}{(2M-1)\beta+(M-1)} & \text{if } \alpha = 1. \end{cases}$$

Under additional assumptions on the form of $A(t,\xi)$ and on κ, we can improve Theorem 2.

ASSUMPTION 2

We consider the Cauchy problem (1.1) and Assumption 1 to be true. Moreover, we assume that there exists $0<\gamma\leq\alpha$ such that

$$\|\widetilde{A}(t,\xi)\| \lesssim t^{\gamma}, \quad \text{where } \widetilde{A} = A - \Big(\frac{\operatorname{tr} A}{N}\Big)\mathrm{I}_N. \tag{1.10}$$

REMARK 1.15

Condition (1.10) is equivalent to

$$\|A(t,\xi)-\lambda_i(t,\xi)\mathrm{I}_N\| \lesssim t^{\gamma} \quad \text{for some } i.$$

Indeed,

$$\|A(t,\xi)-\lambda_i(t,\xi)\mathrm{I}_N\| \leq \|\widetilde{A}(t,\xi)\| + \Big|\frac{\operatorname{tr} A}{N}-\lambda_i\Big| \lesssim \|\widetilde{A}(t,\xi)\| + t^{\alpha}$$

since $\gamma\leq\alpha$; analogously, we can prove the inverse inequality.

REMARK 1.16

Let $A(t,\xi)$ be a triangular matrix; that is, let $a_{ij}=0$ for $j<i$. Then, since $a_{ii}=\lambda_i$, condition (1.10) is equivalent to

$$|a_{ij}| \lesssim t^{\gamma} \quad \text{for } j>i.$$

Thanks to Assumption 2 we can state the following.

THEOREM 3

We consider the Cauchy problem (1.1) and Assumptions 1 and 2 to be true. We assume that (1.4) is satisfied. Let

$$h = \min\{p = 1, \dots, N-1 : \Sigma_p + p \geq N + (N-2)\gamma\}.$$

Then the Cauchy Problem is well posed in γ^d *for any* $1 < d < d^*$, *where*

$$d^* = \begin{cases} 1 + \frac{\kappa_h + 1}{(N-h-1)\kappa_h + \Sigma_h - (N-2)\gamma - 1} & \text{if } \kappa_h < \infty, \\ 1 + \frac{1}{N-h} = d_B(N-(h-1)) & \text{if } \kappa_h = \infty. \end{cases}$$

The proof of Theorem 3 follows as a corollary of Theorem 6 stated in Section 4.

Under additional assumptions on the Jordan canonical form of $A(t,\xi)$, we can improve Theorem 1.

ASSUMPTION 3

We assume that there exists a matrix $C \in \mathcal{C}^N$, homogeneous of degree zero in ξ and with $|\det C(t,\xi)| \geq c > 0$, such that

$$C(t,\xi)A(t,\xi)C^{-1}(t,\xi) = J_A(t,\xi)$$

is a Jordan matrix, that is, a block diagonal matrix whose blocks are Jordan blocks. The matrix J_A is the *Jordan canonical form* of A.

If Assumption 3 is satisfied, then J_A is smooth; that is, each nondiagonal term is constantly equal to 1 or zero. In particular, if $\lambda_i(t,\xi) \neq \lambda_j(t,\xi)$ for any $t > 0$, $\xi \neq 0$, and $i \neq j$, then A is uniformly diagonalizable. Moreover, if Assumption 3 is satisfied, by the subsitution $W = CV$ the Cauchy problem (2.1) is equivalent to

$$\begin{cases} W' = i|\xi|J_A(t,\xi)W + B_1(t,\xi)W, \\ W(0,\xi) = C^{-1}(0,\xi)V_0(\xi), \end{cases} \tag{1.11}$$

where

$$B_1 = \big(C'(t,\xi) + C(t,\xi)B(t)\big)C^{-1}(t,\xi) \in \mathcal{C}^{N-1}.$$

Hence the strong well-posedness for $\partial_t - i|\xi|A$ is equivalent to the strong well-posedness for $\partial_t - i|\xi|J_A$. We are ready to state the following result, which improves Theorem 1.

THEOREM 4

We assume that for any $t > 0$, *the matrix* $A(t,\xi)$ *has* m *distinct eigenvalues* $\mu_1, \dots, \mu_m$, *each one with constant multiplicity* M_p, *and that*

$$|\mu_i - \mu_j| \approx t^\alpha, \quad i \neq j,$$

for some $\alpha \geq 1$. *Let* $M = \max_p M_p$, *with* $M \geq 2$, *be the maximum multiplicity of the eigenvalues of* $A(t,\xi)$ *for* $t > 0$. *If Assumption 3 is satisfied, then the Cauchy* *problem (1.1) is strongly well posed in* γ^d *for any* $1 < d < d^*$ *with*

$$d^* := \begin{cases} 1 + \frac{\alpha+1}{(N-m-1)\alpha - 1} & \text{if } (N-M-m)\alpha \geq M, \\ 1 + \frac{1}{M-1} \equiv d_B(M) & \text{if } (N-M-m)\alpha \leq M, \end{cases} \tag{1.12}$$

provided that $M + m \leq N - 1$.

On the other hand, if $M + m \geq N - 1$*, then the Cauchy problem (1.1) is strongly well posed in* γ^d *for any* $1 < d < d^*$ *with*

$$d^* := \begin{cases} 1 + \frac{\alpha+1}{M\alpha - 1} & \text{if } \alpha \geq M, \\ 1 + \frac{1}{M-1} \equiv d_B(M) & \text{if } \alpha \leq M. \end{cases} \tag{1.13}$$

The proof of Theorem 4 follows from Theorem 6, stated in Section 4.

2. Proof of Theorem 2

Let U be a solution of the system (1.1), and let $V(t,\xi) := \widehat{U}(t,\xi)$ (resp., $V_0(\xi) = \widehat{U_0}(\xi)$) be the Fourier transform with respect to the x-variable of U (resp., U_0); using the Duhamel principle, we can assume $f \equiv 0$. Then V satisfies the system

$$\begin{cases} V' = iA(t,\xi)|\xi|V + B(t)V, \\ V(0,\xi) = V_0(\xi). \end{cases} \tag{2.1}$$

First, we assume that $A(t,\xi)$ is a Sylvester matrix; that is,

$$A(t,\xi) = \begin{pmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & 0 & \dots \\ & & \ddots & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ a_N & a_{N-1} & \dots & \dots & a_1 \end{pmatrix}, \quad a_j = (-1)^{j-1}\sigma_j, \tag{2.2}$$

where by σ_j we denote the elementary symmetric function

$$\begin{gathered} \sigma_j \equiv \sigma_j[I] = \sum_{I^{[j]}} \prod_{m=1}^{j} \lambda_{p(m)}, \\ I^{[j]} = \{p(1), \dots, p(j) \in I : p(1) < \dots < p(j)\}. \end{gathered} \tag{2.3}$$

Let

$$\omega[I] := -(a_N, \dots, a_1) = \big((-1)^N \sigma_N, (-1)^{N-1}\sigma_{N-1}, \dots, -\sigma_1\big).$$

We remark that $-\omega[I]$ is the Nth row of A.

DEFINITION 5

For any $K \subset I$ and $j \leq \#(K)$, we define the symmetric function

$$\sigma_j[K] := \sum_{K^{[j]}} \prod_{m=1}^{j} \lambda_{p(m)}, \quad K^{[j]} = \{p(1), \dots, p(j) \in K : p(1) < \dots < p(j)\}.$$

We put $\sigma_0[K]=1$, and $\sigma_j[K]=0$ for any $j>\#(K)$.

In order to establish our energy estimates by using the symmetric functions in Definition 5, we prepare some tools. It is clear that

$$\sigma_j[K]=\begin{cases}\lambda_i\sigma_{j-1}[K\setminus\{i\}] & \text{for } j=\#(K),\\ \sigma_j[K\setminus\{i\}]+\lambda_i\sigma_{j-1}[K\setminus\{i\}] & \text{for any } 1\le j<\#(K),\end{cases} \tag{2.4}$$

for any choice of $i\in K$. Moreover, $\partial_t\sigma_0[K]=0$, whereas

$$\partial_t\sigma_j[K]=\sum_{i\in K}(\partial_t\lambda_i)\sigma_{j-1}[K\setminus\{i\}] \quad \text{for any } 1\le j\le\#(K). \tag{2.5}$$

Now, for any $K\subsetneq I$, we put $k=\#(K)$ and we define the row vector

$$\omega[K]:=\big((-1)^k\sigma_k[K],(-1)^{k-1}\sigma_{k-1}[K],\dots,-\sigma_1[K],1,0,\dots,0\big);$$

that is,

$$\omega[K]=\begin{cases}(1,0,\dots) & \text{if } K=\emptyset,\\ (-\lambda_i,1,0,\dots) & \text{if } K=\{i\},\\ (\lambda_i\lambda_j,-(\lambda_i+\lambda_j),1,0,\dots) & \text{if } K=\{i,j\},\\ \quad\vdots & \quad\vdots\end{cases} \tag{2.6}$$

We denote also $\omega[K]=\omega_{i_1\cdots i_k}$, where $K=\{i_1,\dots,i_k\}$ and $\omega[\emptyset]=\omega$.

We can use (2.4) to prove that $\omega[I\setminus\{i\}]$ is the left eigenvector of A related to λ_i, that is, that

$$\omega[I\setminus\{i\}]A=\lambda_i\omega[I\setminus\{i\}] \quad \text{for any } i\in I.$$

On the other hand, using again (2.4), we can prove that for any $K\subsetneq I$, we have

$$\omega[K\setminus\{i\}]A=\omega[K]+\lambda_i\omega[K\setminus\{i\}] \quad \text{for any } i\in K. \tag{2.7}$$

It is clear that $\partial_t\omega=0$, whereas we can use (2.5) to prove that if $K\ne\emptyset$, then

$$\partial_t\omega[K]=\big((-1)^k\partial_t\sigma_k[K],\dots,-\partial_t\sigma_1[K],0,\dots,0\big)=-\sum_{i\in K}(\partial_t\lambda_i)\omega[K\setminus\{i\}]. \tag{2.8}$$

DEFINITION 6

For any vector $V\in\mathbb{C}^N$, we define

$$[V]_l^2:=\sum_{\#(K)=l}|\omega[K]V|^2, \quad l=0,\dots,N-1.$$

REMARK 2.1

Now let $K\subsetneq I$ with $\#(K)\le N-2$. It is clear that

$$\omega[K\cup\{i\}]-\omega[K\cup\{j\}]=(\lambda_j-\lambda_i)\omega[K] \quad \text{for any } i,j\notin K;$$

hence for any vector $V\in\mathbb{C}^n$, we get

$$|\omega[K]V|\lesssim\frac{|\omega[K\cup\{i\}]V|+|\omega[K\cup\{j\}]V|}{t^{\kappa_{ij}}},$$

provided that $\kappa_{ij} < \infty$.

We are ready to state the following.

LEMMA 2.2
Let $K \subsetneq I$ with $k = \#(K)$; we put $J = I \setminus K$. Then for any vector $V \in \mathbb{C}^n$, we have

$$|\omega[K]V| \lesssim \frac{[V]_l}{t^{\Sigma_{l-k}[J]}}, \quad l = k, \dots, N-1. \tag{2.9}$$

To prove Lemma 2.2, we need the following.

LEMMA 2.3
Let $J \subset I$, and let $J' = J \setminus \{j\}$ for some $j \in J$. Then we have

$$\Sigma_{h+1}[J] \geq \alpha_J + \Sigma_h[J'], \quad h = 0, \dots, \#(J) - 2.$$

Proof
We prove the statement by induction on $M = \#(J)$. It is trivially true for $M = 2$ since $\Sigma_1[J] = \alpha_J$. We assume that the thesis is satisfied for some $M - 1$, and we prove it for M.

With the notation introduced in Definition 3, let

$$P_{\min}(J) = P_{\alpha_J}(J) = \{J_1, \dots, J_m\}.$$

Now $j \in J_p$ for some p, say, $p = 1$. We recall that

$$\Sigma_{h+1}[J] = \max_p \sigma_{h+1}[J_p, \#(J \setminus J_p)];$$

we have

$$\begin{aligned} \sigma_{h+1}[J_p, \#(J \setminus J_p)] &= \sigma_{h+1}[J_p, M - \#(J_p)] \\ &= \alpha_J + \sigma_h[J_p, (M-1) - \#(J_p)] \\ &= \alpha_J + \sigma_h[J_p, \#(J' \setminus J_p)] \end{aligned}$$

for any $p \geq 2$, whereas if we put $l = M - \#(J_1)$, then

$$\sigma_{h+1}[J_1, l] = \begin{cases} (h+1)\alpha_J & \text{if } (h+1) \leq l, \\ l\alpha_J + \Sigma_{h+1-l}[J_1] & \text{if } (h+1) \geq l+1. \end{cases}$$

Now, being $\#(J_1) \leq M - 1$, we can apply the hypothesis of induction; hence

$$\Sigma_{h+1-l}[J_1] \geq \alpha_J + \Sigma_{h-l}[J_1'],$$

where we put $J_1' = J_1 \setminus \{j\}$. We remark that

$$\sigma_{h+1}[J_1, l] \geq \alpha_J + \begin{cases} h\alpha_J & \text{if } h \leq l, \\ l\alpha_J + \Sigma_{h-l}[J_1'] & \text{if } h \geq l+1; \end{cases}$$

that is,

$$\sigma_{h+1}[J_1, l] \geq \alpha_J + \sigma_h[J_1', \#(J' \setminus J_1')].$$

This concludes the proof. □

Proof of Lemma 2.2
We proceed by induction on $n = l - \#(K)$. If $n = 0$, then (2.9) is trivial. We assume that (2.9) is satisfied for some n, and we prove it for $n + 1$.

We put $J = I \setminus K$. Let α_J and

$$P_{\min}(J) = P_{\alpha_J}(J) = \{J_1, \ldots, J_m\}$$

be as in Definition 3. Let $i_1 \in J_1$ and $i_2 \in J_2$; then

$$|\omega[K]V| \lesssim \frac{|\omega[K \cup \{i_1\}]V| + |\omega[K \cup \{i_2\}]V|}{t^{\alpha_J}}. \tag{2.10}$$

Now we can apply the hypothesis of induction to the term $|\omega[K \cup \{i_p\}]V|$ since

$$l - \#(K \cup \{i_p\}) = n.$$

We get

$$|\omega[K \cup \{i_p\}]V| \lesssim \frac{[V]_l}{t^{\Sigma_{l-(k+1)}[J']}},$$

where $J' = J \setminus \{i_p\}$. Thanks to Lemma 2.3, we have

$$\alpha_J + \Sigma_{l-(k+1)}[J'] \leq \Sigma_{l-k}[J].$$

This concludes the proof. □

We show how Lemma 2.2 works with the following.

EXAMPLE 2.4
We consider Example 1.12:

$$P_{\min}(I) = \{I_1, I_2\}, \quad I_1 = \{1, 2\}, I_2 = \{3, 4, 5\},$$

with

$$\kappa = \begin{pmatrix} \infty & \beta_1 & \alpha & \alpha & \alpha \\ \beta_1 & \infty & \alpha & \alpha & \alpha \\ \alpha & \alpha & \infty & \beta_2 & \beta_2 \\ \alpha & \alpha & \beta_2 & \infty & \gamma \\ \alpha & \alpha & \beta_2 & \gamma & \infty \end{pmatrix},$$

and we directly prove the estimate

$$|\omega V| \lesssim [V]_3 \cdot t^{-\Sigma_3}.$$

In fact, we have

$$|\omega[\emptyset]V| \lesssim \frac{|\omega_1 V| + |\omega_3 V|}{t^{\alpha}} \lesssim \frac{|\omega_{12} V| + |\omega_{13} V|}{t^{2\alpha}} + \frac{|\omega_{31} V| + |\omega_{34} V|}{t^{2\alpha}}$$

$$\begin{aligned}
&= \frac{|\omega_{12}V| + 2|\omega_{13}V| + |\omega_{34}V|}{t^{2\alpha}} \\
&\lesssim \frac{|\omega_{123}V| + |\omega_{124}V|}{t^{2\alpha+\beta_2}} + \frac{2|\omega_{123}V| + 2|\omega_{134}V|}{t^{3\alpha}} + \frac{|\omega_{134}V| + |\omega_{345}V|}{t^{3\alpha}} \\
&\lesssim [V]_3 t^{-\max\{2\alpha+\beta_2, 3\alpha\}}.
\end{aligned}$$

LEMMA 2.5

If $V(t,\xi)$ satisfies (2.1), then we have the following estimates:

$$\begin{aligned}
&\partial_t [V]_0^2 \lesssim (|\xi|[V]_1 + |BV|)[V]_0, \\
&\partial_t [V]_l^2 \lesssim ([V]_{l-1} + |\xi|[V]_{l+1} + |BV|)[V]_l, \quad l = 1, \dots, N-2, \\
&\partial_t [V]_{N-1}^2 \lesssim ([V]_{N-2} + |BV|)[V]_{N-1}.
\end{aligned} \tag{2.11}$$

Proof

We fix l and $K \subset I$ to be such that $\#(K) = l$, as in Definition 6. We have to estimate

$$\begin{aligned}
\partial_t |\omega[K]V|^2 &= 2\,\mathrm{Re}\big(\partial_t(\omega[K]V), \omega[K]V\big) \\
&= 2\,\mathrm{Re}\big(i|\xi|\omega[K]AV + \omega[K]BV + (\partial_t\omega[K])V, \omega[K]V\big).
\end{aligned} \tag{2.12}$$

Let $j \in I \setminus K$; thanks to (2.7), we get

$$\mathrm{Re}(i|\xi|\omega[K]AV, \omega[K]V) = \mathrm{Re}(i|\xi|\lambda_j\omega[K]V, \omega[K]V) + \mathrm{Re}(i|\xi|\omega[K\cup\{j\}]V, \omega[K]V);$$

since λ_j is real valued, the first term vanishes, whereas

$$|\xi||\omega[K\cup\{j\}]V| \cdot |\omega[K]V| \lesssim |\xi|[V]_{l+1}[V]_l.$$

For the second term of (2.12), we simply estimate $|\omega[K]BV| \lesssim |BV|$, whereas for the third one, thanks to (2.8), we get

$$|(\partial_t\omega[K])V| \le \sum_{i\in K} |\partial_t\lambda_i||\omega[K\setminus\{i\}]V| \lesssim [V]_{l-1}$$

since λ_i is Lipschitz continuous. □

DEFINITION 7

Let $K^1 = \emptyset$. We define by induction a sequence of sets $K^k \subsetneq I$ with $\#(K^k) = k-1$, such that

$$K^1 \subset K^2 \subset \cdots \subset K^N.$$

It is clear that there exists a (unique) permutation π over I such that

$$K^k = \big\{\pi(1), \pi(2), \dots, \pi(k-1)\big\}.$$

We define the vectors

$$w_k := \omega[K^k] \equiv \omega_{\pi(1),\pi(2),\dots,\pi(k-1)}, \quad k = 1, \dots, N.$$

REMARK 2.6

With the notation in Definition 7, $(w_1,\dots,w_N)$ is a base of the vector space $\mathbb{C}^n$ over $\mathbb{C}$.

Moreover, if we denote by $(e_1,\dots,e_N)$ the canonical base, that is,

$$e_1=(1,0,\dots),\quad e_2=(0,1,0,\dots),\quad \dots,$$

then it can be proved that

$$w_i=\sum_{j=1}^{i}(-1)^{i-j}\sigma_{i-j}[K^i]e_j,\quad i=1,\dots,N, \tag{2.13}$$

$$e_j=\sum_{k=1}^{j}\widetilde{\sigma}_{j-k}[K^{k+1}]w_k,\quad j=1,\dots,N, \tag{2.14}$$

where for any $K\subset I$ we define the symmetric functions

$$\widetilde{\sigma}_j[K]:=\sum_{\widetilde{K^{[j]}}}\prod_{m=1}^{j}\lambda_{p(m)},\quad \widetilde{K^{[j]}}=\{p(1),\dots,p(j)\in K: p(1)\le\dots\le p(j)\}.$$

DEFINITION 8

Let $|\xi|\ge 1$. We set the energy

$$E_1(t,\xi):=|V|^2.$$

We remark that for any $t\ge 0$ and $\xi\in\mathbb{R}^n$, the energy $E_1(t,\xi)$ represents a norm for the vector $V(t,\xi)\in\mathbb{C}^N$. Since

$$\partial_t E_1(t,\xi)\lesssim|\xi||V|^2=|\xi|E_1(t,\xi),$$

by applying Grönwall's lemma for $t\le t_1(\xi)$, where we put

$$t_1(\xi):=|\xi|^{-1+(1/d^*)}, \tag{2.15}$$

we get

$$E_1(t,\xi)\lesssim\exp\Big(\int_0^{t_1(\xi)}C_1|\xi|\,ds\Big)E_1(0,\xi)\lesssim\exp(C_1|\xi|^{1/d^*})E_1(0,\xi). \tag{2.16}$$

DEFINITION 9

For any $t\in[t_1(\xi),T]$, we set the energy

$$E_2(t,\xi):=\sum_{l=0}^{N-1}\Big(\frac{t_1(\xi)}{t}\Big)^{2(N-(l+1))}[V]_l^2.$$

For any $t\in[t_1(\xi),T]$ and $\xi\in\mathbb{R}^n$, the energy $E_2(t,\xi)$ represents a norm for $V(t,\xi)\in\mathbb{C}^N$; moreover, thanks to Remark 2.6,

$$\begin{cases} E_2(t,\xi)\lesssim E_1(t,\xi),\\ E_1(t,\xi)\lesssim(t_1(\xi))^{-2(N-1)}E_2(t,\xi)\\ \qquad\qquad =|\xi|^C E_2(t,\xi),\quad C=2(N-1)\big(1-\frac{1}{d^*}\big).\end{cases} \tag{2.17}$$

We get

$$\partial_t E_2(t,\xi) = -\frac{2(N-(l+1))}{t} E_2(t,\xi) + \sum_{l=0}^{N-1} \Big(\frac{t_1(\xi)}{t}\Big)^{2(N-(l+1))} \partial_t [V]_l^2, \tag{2.18}$$

and we claim that the second term in (2.18) is estimated by

$$|\xi|^{1/d^*} \frac{1}{t} E_2(t,\xi).$$

This is sufficient to conclude the proof since, by applying Grönwall's lemma,

$$\begin{aligned} E_2(t,\xi) &\lesssim \exp\Big(\int_{t_1}^T C_1 |\xi|^{1/d^*} \frac{1}{s}\, ds\Big) E_2\big(t_1(\xi),\xi\big) \\ &= \exp\Big(C_1 |\xi|^{1/d^*} \log \frac{t}{t_1}\Big) E_2\big(t_1(\xi),\xi\big) \\ &\lesssim \exp\bigg(C_1 |\xi|^{1/d^*} \Big(\log T + \Big(1 - \frac{1}{d^*}\Big) \log|\xi|\Big)\bigg) E_2\big(t_1(\xi),\xi\big), \\ &\qquad t \in [t_1(\xi), T]. \end{aligned} \tag{2.19}$$

Therefore we should prove the following.

LEMMA 2.7

For any $t \in [t_1(\xi), T]$, we have

$$\Big(\frac{t_1(\xi)}{t}\Big)^{2(N-(l+1))} \partial_t [V]_l^2 \lesssim |\xi|^{1/d^*} \frac{1}{t} E_2(t,\xi), \quad l = 0, \ldots, N-1.$$

Before proving Lemma 2.7, we need the following.

DEFINITION 10

Let

$$\omega := \max\{\kappa_{ij} : i \neq j\} \in [\alpha, \infty];$$

we put

$$d_{\max} := \begin{cases} \infty & \text{if } \omega = 1, \\ 2 + \frac{2}{\omega - 1} & \text{if } 1 < \omega < \infty, \\ 2 & \text{if } \omega = \infty. \end{cases} \tag{2.20}$$

REMARK 2.8

The Gevrey index given by Theorem 2 satisfies $d^* \leq d_{\max}$.

In particular, if $N \geq 4$, then Theorem 2 gives $d^* \leq 2 \leq d_{\max}$, whereas if $N = 3$, then $\kappa_2 = \omega$ and Theorem 2 (see also [DAT]) gives

$$d^* = \begin{cases} 1 + \frac{\alpha+1}{2\alpha-1} & \text{if } \alpha \geq 2, \\ 2 & \text{if } \alpha = 1 \text{ and } \omega = \infty, \\ 2 + \frac{1}{\omega} & \text{if } \alpha = 1 \text{ and } \omega < \infty. \end{cases}$$

Proof of Lemma 2.7
Thanks to (2.11), it is sufficient to prove the following:

$$\left(\frac{t_1(\xi)}{t}\right)^{N-(l+1)}[V]_{l-1} \lesssim |\xi|^{1/d^*}\frac{1}{t}\sqrt{E_2}, \quad l=1,\dots,N-1, \tag{2.21}$$

$$\left(\frac{t_1(\xi)}{t}\right)^{N-(l+1)}|\xi|[V]_{l+1} \lesssim |\xi|^{1/d^*}\frac{1}{t}\sqrt{E_2}, \quad l=0,\dots,N-2, \tag{2.22}$$

$$|BV| \lesssim |\xi|^{1/d^*}\frac{1}{t}\sqrt{E_2}. \tag{2.23}$$

In order to prove (2.21), we set

$$t_2(\xi) := |\xi|^{-1/2\omega}$$

for $\omega < \infty$, and $t_2(\xi) = T$ for $\omega = \infty$. First, we assume that $t_1(\xi) \le t_2(\xi)$. Then, for any $t \in [t_1, t_2]$,

$$\frac{t^2}{t_1(\xi)} \le \frac{t_2^2}{t_1} = |\xi|^{-1/\omega}|\xi|^{(d^*-1)/d^*} \le |\xi|^{1/d^*}$$

since

$$-\frac{1}{\omega} \le -1 + \frac{2}{d^*}, \quad \text{where } d^* \le d_{\max} = \frac{2\omega}{\omega-1}.$$

Hence we get (2.21) in $[t_1, t_2]$; indeed,

$$\left(\frac{t_1(\xi)}{t}\right)^{N-(l+1)}[V]_{l-1} = \frac{t^2}{t_1(\xi)} \cdot \frac{1}{t}\left(\frac{t_1(\xi)}{t}\right)^{N-l}[V]_{l-1} \le |\xi|^{1/d^*}\frac{1}{t}\sqrt{E_2}.$$

On the other hand, for any $t \in [t_2, T]$, by applying Lemma 2.2 we can estimate

$$[V]_{l-1} \lesssim \frac{1}{t^\omega}[V]_l \le \frac{1}{t_2^{\omega-1}}\frac{1}{t}[V]_l = |\xi|^{(\omega-1)/2\omega}\frac{1}{t}[V]_l \le |\xi|^{1/d^*}\frac{1}{t}[V]_l. \tag{2.24}$$

Hence

$$\left(\frac{t_1(\xi)}{t}\right)^{N-(l+1)}[V]_{l-1} \lesssim \left(\frac{t_1(\xi)}{t}\right)^{N-(l+1)}|\xi|^{1/d^*}\frac{1}{t}[V]_l \le |\xi|^{1/d^*}\frac{1}{t}\sqrt{E_2}. \tag{2.25}$$

On the other hand, if $t_2(\xi) \le t_1(\xi)$, then for any $t \in [t_1, T]$ we can estimate

$$[V]_{l-1} \lesssim \frac{1}{t^\omega}[V]_l \le \frac{1}{t_1^{\omega-1}}\frac{1}{t}[V]_l \le \frac{1}{t_2^{\omega-1}}\frac{1}{t}[V]_l,$$

and the proof follows from (2.24) and (2.25).

In order to prove (2.22), it is sufficient to notice that $|\xi|t_1(\xi) = |\xi|^{1/d^*}$; hence

$$\left(\frac{t_1(\xi)}{t}\right)^{N-(l+1)}|\xi|[V]_{l+1} = t_1(\xi)|\xi|\frac{1}{t}\left(\frac{t_1(\xi)}{t}\right)^{N-(l+2)}[V]_{l+1} \lesssim |\xi|^{1/d^*}\frac{1}{t}\sqrt{E_2}.$$

We consider now (2.23). With the notation in Definition 7 we get

$$|BV| \lesssim |V| \lesssim \sum_{p=1}^{N}|\omega[K^p]V|. \tag{2.26}$$

It is immediate that

$$|\omega[K^N]V| \le [V]_{N-1} \le \sqrt{E_2}.$$

Let $p \le N-1$. Thanks to Lemma 2.2, if we put $J^p = I \setminus K^p$, then

$$|\omega[K^pV]| \lesssim \frac{[V]_l}{t^{\Sigma_{l-(p-1)}[J^p]}} \le \Big(\frac{t}{t_1(\xi)}\Big)^{N-(l+1)} \frac{1}{t^{\Sigma_{l-(p-1)}[J^p]-1}} \times \frac{1}{t}\sqrt{E_2(t,\xi)},$$
$$l = p-1, \dots, N-1.$$

From Lemma 2.3, since $\#(J^p) = N-(p-1)$, it follows that

$$\Sigma_{l-(p-1)}[J^p] \le \Sigma_l;$$

hence, for fixed l, we get

$$|\omega[K^p]V| \lesssim \Big(\frac{t}{t_1(\xi)}\Big)^{N-(l+1)} \frac{1}{t^{\Sigma_l -1}} \times \frac{1}{t}\sqrt{E_2(t,\xi)} \quad \text{for any } p \le l+1. \tag{2.27}$$

Moreover, (2.27) is trivially satisfied for $p > l+1$, too, since

$$|\omega[K^p]V| \le [V]_{p-1} \le \Big(\frac{t}{t_1(\xi)}\Big)^{N-p} \sqrt{E_2(t,\xi)} \lesssim \Big(\frac{t}{t_1(\xi)}\Big)^{N-(l+1)} \sqrt{E_2(t,\xi)}.$$

Let h be as in (1.9). We set

$$t_2(\xi) := \big(t_1(\xi)\big)^{1/(\kappa_h+1)} \equiv |\xi|^{-(d^*-1)/(\kappa_h+1)d^*} \tag{2.28}$$

for $\kappa_h < \infty$ and $t_2(\xi) = T$ for $\kappa_h = \infty$. For any $t \in [t_1, t_2]$, we take $l = h-1$ in (2.27); hence, thanks to (1.9), we obtain

$$|BV| \lesssim t_1^{-(N-h)} t_2^{N-h+1-\Sigma_{h-1}} \frac{1}{t}\sqrt{E_2}. \tag{2.29}$$

By using (2.28), since

$$\begin{aligned} -(N-h) + \frac{N-h+1-\Sigma_{h-1}}{\kappa_h+1} &= -\frac{(N-h)\kappa_h + (\Sigma_{h-1}-1)}{\kappa_h+1} \\ &= -\frac{1}{d^*-1}, \end{aligned}$$

we get

$$|V| \lesssim t_1^{-1/(d^*-1)} \frac{1}{t}\sqrt{E_2} = |\xi|^{1/d^*} \frac{1}{t}\sqrt{E_2}. \tag{2.30}$$

For any $t \in [t_2, T]$, we take $l = h$ in (2.27); hence, thanks to (1.9), we get

$$|BV| \lesssim t_1^{-(N-(h+1))} t_2^{N-(h+1)+1-\Sigma_h} \frac{1}{t}\sqrt{E_2}. \tag{2.31}$$

By using again (2.28), we find the same estimate in (2.30) since

$$-(N-h-1) + \frac{N-h-\Sigma_h}{\kappa_h+1} = -\frac{(N-h)\kappa_h + (\Sigma_{h-1}-1)}{\kappa_h+1} = -\frac{1}{d^*-1}.$$

This concludes the proof. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2

As in [DAS, Section 4], we transform the first-order system (1.1) into an Nth-order system whose principal part is a block Sylvester matrix. Using the Duhamel principle, we can assume $f \equiv 0$. Let

$$L(t,\tau,i\xi) \;= \tau - i|\xi|A(t,\xi) - B(t), \qquad \chi(t,\tau,i\xi) := \tau - i|\xi|A(t,\xi),$$
$$\Lambda(t,\tau,i\xi) := \big(\chi(t,\tau,i\xi)\big)^{\mathrm{adj}},$$

where $\chi(t,\partial_t,\partial_x)$ is the principal part of $L(t,\partial_t,\partial_x)$, and with the notation F^{adj} we denote the classical adjoint (or adjugate) matrix of F, that is, the transpose of the matrix of cofactors.

It is clear that the well-posedness for the systems

$$\begin{aligned}\mathcal{L}_1(t,\partial_t,\partial_x) &= \Lambda(t,\partial_t,\partial_x)L(t,\partial_t,\partial_x),\\ \mathcal{L}_2(t,\partial_t,\partial_x) &= L(t,\partial_t,\partial_x)\Lambda(t,\partial_t,\partial_x)\end{aligned} \tag{2.32}$$

implies the well-posedness for $L(t,\partial_t,\partial_x)$. The systems $\mathcal{L}_1(t,\partial_t,\partial_x)$ and $\mathcal{L}_2(t,\partial_t,\partial_x)$ are Nth-order systems with diagonal principal part $P(t,\partial_t,\partial_x)\mathrm{I}_N$, where $P(t,\tau,i\xi)$ is the characteristic polynomial of $\chi(t,\tau,i\xi) = \tau - i|\xi|A(t,\xi)$. Let

$$\mathcal{W} := \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ \vdots \\ W^{(N)} \end{pmatrix} \in \mathbb{C}^{N^2} \quad \text{with } W^{(j)} := \begin{pmatrix} (i|\xi|)^{N-1}V^{(j)} \\ (i|\xi|)^{N-2}\partial_t V^{(j)} \\ \vdots \\ \partial_t^{N-1}V^{(j)} \end{pmatrix};$$

then the Cauchy problem for $\mathcal{L}_1(t,\partial_t,i\xi)V(t,\xi) = 0$ (or $\mathcal{L}_2(t,\partial_t,i\xi)V(t,\xi) = 0$) is equivalent to the Cauchy problem for

$$\partial_t\mathcal{W} - i|\xi|\mathcal{A}(t,\xi)\mathcal{W} - \mathcal{B}(t,\xi)\mathcal{W} = 0, \tag{2.33}$$

where

$$\mathcal{A}(t,\xi) = \bigoplus_{i=1}^{N} A_{\mathrm{syl}}(t,\xi),$$

and by $A_{\mathrm{syl}}(t,\xi)$ we denote the Sylvester matrix with eigenvalues $\{\lambda_j(t,\xi)\}$, namely, (2.2), whereas $\mathcal{B}$ is an $((N^2)\times(N^2))$-matrix with the following block structure:

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_{[1,1]} & \mathcal{B}_{[1,2]} & \dots & \mathcal{B}_{[1,N]} \\ \mathcal{B}_{[2,1]} & \mathcal{B}_{[2,2]} & \dots & \mathcal{B}_{[2,N]} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{[N,1]} & \mathcal{B}_{[N,2]} & \dots & \mathcal{B}_{[N,N]} \end{pmatrix}; \tag{2.34}$$

each $(N\times N)$-block $\mathcal{B}_{[j,k]}$ has nonzero elements only on the last row and is bounded (for $|\xi|\ge 1$).

We remark that $W^{(j)}$ satisfies the $(N\times N)$-system

$$W_t^{(j)} - i|\xi|A_{\mathrm{syl}}(t,\xi)W^{(j)} - \mathcal{B}_{[j,j]}(t,\xi)W^{(j)} = \sum_{k\neq j}\mathcal{B}_{[j,k]}(t,\xi)W^{(k)}; \tag{2.35}$$

hence we may regard $\sum_{k\neq j}\mathcal{B}_{[j,k]}(t,\xi)W^{(k)}$ as a second member.

Let $|\xi|\ge 1$, and let $E_1[W^{(j)}](t,\xi)$ and $E_2[W^{(j)}](t,\xi)$ be the energies of the solution $W^{(j)}$ of (2.35), as in Definitions 8 and 9; we define the energies for the solution $\mathcal{W}$ of (2.33),

$$\mathcal{E}_1(t,\xi) := \sum_{j=1}^N E_1[W^{(j)}](t,\xi), \qquad \mathcal{E}_2(t,\xi) := \sum_{j=1}^N E_2[W^{(j)}](t,\xi).$$

For any $j = 1, \dots, N$, we have

$$E_1[W^{(j)}]'(t,\xi) \lesssim |\xi| E_1[W^{(j)}](t,\xi) + \Big[\sum_{k\neq j} \mathcal{B}_{[j,k]}(t,\xi) W^{(k)}\Big]^2,$$

and using

$$\Big[\sum_{k\neq j} \mathcal{B}_{[j,k]}(t,\xi) W^{(k)}\Big]^2 \lesssim \sum_{k\neq j} [W^{(k)}]^2 \lesssim \sum_{k\neq j} E_1[W^{(j)}],$$

we derive

$$\mathcal{E}_1'(t,\xi) \lesssim |\xi| \mathcal{E}_1(t,\xi).$$

Therefore, from (2.16) we obtain

$$\mathcal{E}_1(t,\xi) \lesssim \exp(C_1 |\xi|^{1/d^*}) \mathcal{E}_1(0,\xi), \quad t \in [0, t_1(\xi)],$$

and, analogously, from (2.19) we get

$$\mathcal{E}_2(t,\xi) \lesssim \exp\bigg(C_1 |\xi|^{1/d^*} \Big(\log T + \Big(1 - \frac{1}{d^*}\Big) \log|\xi|\Big)\bigg) \mathcal{E}_2\big(t_1(\xi),\xi\big), \quad t \in [t_1(\xi), T].$$

By using (2.17), we can prove that

$$\mathcal{E}_1(t,\xi) \lesssim |\xi|^C \exp\bigg(C_1 |\xi|^{1/d^*} \Big(\log T + \Big(1 - \frac{1}{d^*}\Big) \log|\xi|\Big)\bigg) \mathcal{E}_1(0,\xi), \quad t \in [0,T];$$

Therefore, for any $1 < d < d^*$,

$$\mathcal{E}_1(t,\xi) \lesssim \exp(C' |\xi|^{\frac{1}{d}}) \mathcal{E}_1(0,\xi), \quad t \in [0,T].$$

We conclude the proof by standard methods using a Paley-Wiener-Schwartz-type theorem (see [H]) for the characterization of functions in Gevrey classes via estimates of their Fourier-Laplace transforms. □

3. Levi conditions for the Nth order scalar equation

We consider the Cauchy problem

$$\begin{cases} L(t,\partial_t,\partial_x) u(t,x) = \sum_{j=0}^{N-1} M_j(t,\partial_t,\partial_x) u(t,x), \\ \partial_t^i u(0,x) = u_i(x), \quad i = 0, \dots, N-1, \end{cases} \tag{3.1}$$

where

$$L(t,\partial_t,\partial_x) = \partial_t^N + \sum_{0\leq k\leq N-1} a_k(t,\partial_x) \partial_t^k,$$

with $a_k(t,\xi)$ homogeneous of degree $N-k$ in ξ, is an Nth-order homogeneous operator in normal form and

$$M_j(t,\partial_t,\partial_x) = \sum_{0\leq l\leq j} b_{j,l}(t,\partial_x) \partial_t^l,$$

with $b_{j,l}(t,\xi)$ homogeneous of degree $j-l$ in ξ, is a lower-order term.

We assume that the roots λ_j of the characteristic equation $L(t,\lambda_j(t,\xi),\xi/|\xi|)=0$ verify Assumption 1.

We define the following vector functions, homogeneous of degree zero in ξ:

$$b_j(t,\xi) := \sum_{l=0}^{j} b_{j,l}(t,\xi/|\xi|)e_{l+1} = \sum_{l=0}^{j} |\xi|^{-(j-l)} b_{j,l}(t,\xi)e_{l+1},$$

where (e_l) denotes the canonical basis of $\mathbb{C}^N$.

Let u be a solution of the scalar equation in (3.1), and let $v(t,\xi) := \widehat{u}(t,\xi)$ (resp., $v_i(\xi)=\widehat{u_i}(\xi)$) be the Fourier transform with respect to the x-variable of u (resp., u_i); then v satisfies the system

$$\begin{cases} L(t,\partial_t,i\xi)v(t,\xi) = \sum_{j=0}^{N-1} M_j(t,\partial_t,i\xi)v(t,\xi), \\ \partial_t^i v(0,\xi) = v_i(\xi), \quad i=0,\dots,N-1. \end{cases} \tag{3.2}$$

We put, for $|\xi|\geq 1$,

$$V := \begin{pmatrix} (i|\xi|)^{N-1}v \\ (i|\xi|)^{N-2}\partial_t v \\ \dots \\ \partial_t^{N-1}v \end{pmatrix};$$

then the scalar equation in (3.2) is equivalent to the first-order $(N\times N)$-system

$$\partial_t V - i|\xi|A_{\mathrm{syl}}(t,\xi)V - B(t,\xi)V = 0,$$

where $A_{\mathrm{syl}}(t,\xi)$ is the Sylvester matrix in (2.2) and B is an $(N\times N)$-matrix with nonzero elements only on the last row, which can be written in the following form:

$$(B)_{N\cdot} = \sum_{j=0}^{N-1} (i|\xi|)^{-(N-1-j)} b_j(t,\xi).$$

In order to refine the estimate of $|BV|$ in Lemma 2.5 by using some Levi conditions, we introduce the following.

DEFINITION 11

Let $b=\sum_{l=1}^{N} b_l e_l$ be a vector in $\mathbb{C}^N$. We define by induction

$$\Delta_0[b](\tau) = \sum_{l=1}^{N} b_l \tau^{l-1},$$

$$\Delta_1[b](\tau_0,\tau_1) = \frac{\Delta_0[b](\tau_0)-\Delta_0[b](\tau_1)}{\tau_0-\tau_1},$$

$$\Delta_2[b](\tau_0,\tau_1,\tau_2) = \frac{\Delta_1[b](\tau_0,\tau_1)-\Delta_0[b](\tau_0,\tau_2)}{\tau_1-\tau_2},$$

$$\dots = \dots$$

$$\Delta_k[b](\tau_0,\dots,\tau_{k-2},\tau_{k-1},\tau_k) = \frac{\Delta_{k-1}[b](\tau_0,\dots,\tau_{k-3},\tau_{k-2},\tau_{k-1}) - \Delta_{k-1}[b](\tau_0,\dots,\tau_{k-3},\tau_{k-2},\tau_k)}{\tau_{k-1}-\tau_k}.$$

We remark that $\Delta_k[b](\tau_0,\dots,\tau_{k-2},\tau_{k-1},\tau_k)$ is bounded for any $k \geq 0$ and that

$$\Delta_0[b](\tau) = b\cdot V(\tau), \quad \text{where } V(\tau) = (1,\tau,\tau^2,\dots,\tau^{N-1}).$$

LEMMA 3.1

Let $b \in \mathbb{C}^N$ be as in Definition 11; we have

$$b \equiv \sum_{l=1}^N b_l e_l = \sum_{k=1}^N \Big(\sum_{l=k}^N b_l \widetilde{\sigma}_{l-k}[K^{k+1}]\Big) w_k = \sum_{k=1}^N \Delta_{k-1}[b](\lambda_{\pi(1)},\dots,\lambda_{\pi(k)}) w_k,$$

where we use the notation introduced in Definition 7.

Proof

See [CT, Proposition 3.2]. □

REMARK 3.2

We remark that

$$(b_k = \dots = b_N = 0) \Longrightarrow (\Delta_{k-1}[b] = \dots = \Delta_{N-1}[b] = 0).$$

Thanks to Lemma 3.1, we can write the vector $b_j(t,\xi) \in \mathbb{C}^N$ in the form

$$b_j(t,\xi) = \sum_{k=1}^N \Delta_{k-1}[b_j(t,\xi)](\lambda_{\pi(1)},\dots,\lambda_{\pi(k)}) w_k;$$

hence it follows that

$$\begin{aligned} |BV| &\leq \sum_{j=0}^{N-1} |\xi|^{-(N-1-j)} |b_j V| \\ &\leq \sum_{j=0}^{N-1} |\xi|^{-(N-1-j)} \sum_{k=1}^{j+1} |\Delta_{k-1}[b_j](\lambda_{\pi(1)},\dots,\lambda_{\pi(k)})||w_k V|, \end{aligned} \tag{3.3}$$

where π is the permutation introduced in Definition 7. We introduce the following.

ASSUMPTION 4

Fix a permutation π in Definition 7, and let $\gamma_{j,k} \in [0,\infty)$ be such that

$$|\Delta_{k-1}[b_j](\lambda_{\pi(1)},\dots,\lambda_{\pi(k)})| \lesssim t^{\gamma_{j,k}}, \quad j = 0,\dots,N-1,\ k = 1,\dots,j+1. \tag{3.4}$$

LEMMA 3.3

If we assume that

$$|\Delta_0[b_j](\lambda_{\pi(l)})| \lesssim t^{\gamma_j} \quad \textit{for any } l = 1,\dots,j+1, \tag{3.5}$$

then condition (3.4) is satisfied by

$$\gamma_{j,k} = [\gamma_j - \overline{\Sigma}_{k-1}]^+, \quad 1 \le k \le j+1, \tag{3.6}$$

where $[a]^+ := \max\{a, 0\}$ *is the positive part of* a*, and*

$$\begin{aligned}\overline{\Sigma}_{k-1} &:= \sum_{m=1}^{k-1} \max_{m+1 \le l \le k} \kappa_{\pi(m)\pi(l)} \\ &= \kappa_{\pi(k-1),\pi(k)} + \max\{\kappa_{\pi(k-2),\pi(k-1)}, \kappa_{\pi(k-2),\pi(k)}\} + \cdots + \max_{2 \le l \le k} \kappa_{\pi(1)\pi(l)}.\end{aligned}$$

Proof

For the sake of brevity, let π be the identical permutation on I.

If $\gamma_j \le \overline{\Sigma}_{k-1}$, then $\gamma_{j,k} = 0$ and (3.6) follows from the boundedness of $\Delta_{k-1}[b_j]$. If $\gamma_j > \overline{\Sigma}_{k-1}$, thanks to Definition 11, it is clear that

$$\begin{aligned}|\Delta_{k-1}[b_j](\lambda_1, \ldots, \lambda_k)| &\le \frac{|\Delta_{k-2}[b_j](\lambda_1, \ldots, \lambda_{k-1})| + |\Delta_{k-2}[b_j](\lambda_1, \ldots, \lambda_{k-2}, \lambda_k)|}{|\lambda_{k-1} - \lambda_k|} \\ &\lesssim \frac{|\Delta_{k-2}[b_j](\lambda_1, \ldots, \lambda_{k-1})| + |\Delta_{k-2}[b_j](\lambda_1, \ldots, \lambda_{k-2}, \lambda_k)|}{t^{\kappa_{k-1,k}}};\end{aligned}$$

on the other hand,

$$\begin{aligned}&|\Delta_{k-2}[b_j](\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1})| \\ &\quad \le \frac{|\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-2})| + |\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-3}, \lambda_{k-1})|}{|\lambda_{k-2} - \lambda_{k-1}|} \\ &\quad \lesssim \frac{|\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-2})| + |\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-3}, \lambda_{k-1})|}{t^{\kappa_{k-2,k-1}}}, \\ &|\Delta_{k-2}[b_j](\lambda_1, \ldots, \lambda_{k-2}, \lambda_k)| \\ &\quad \le \frac{|\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-2})| + |\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-3}, \lambda_k)|}{|\lambda_{k-2} - \lambda_k|} \\ &\quad \lesssim \frac{|\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-2})| + |\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-3}, \lambda_k)|}{t^{\kappa_{k-2,k}}}.\end{aligned}$$

Hence we may estimate $|\Delta_{k-1}[b_j](\lambda_1, \ldots, \lambda_k)|$ with

$$\begin{aligned}&\big(2|\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-2})| + |\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-3}, \lambda_{k-1})| \\ &\quad + |\Delta_{k-3}[b_j](\lambda_1, \ldots, \lambda_{k-3}, \lambda_k)|\big)/t^{\kappa_{k-1,k} + \max\{\kappa_{k-2,k-1}, \kappa_{k-2,k}\}}.\end{aligned}$$

By applying induction arguments, thanks to (3.5) we can prove

$$|\Delta_{k-1}[b_j](\lambda_1, \ldots, \lambda_k)| \lesssim \cdots \lesssim \frac{\sum_{l=1}^{k} |\Delta_0[b_j](\lambda_{\pi(l)})|}{t^{\overline{\Sigma}_{k-1}}} \lesssim \frac{t^{\gamma_j}}{t^{\overline{\Sigma}_{k-1}}};$$

that is, we have proved (3.6). □

REMARK 3.4

If condition (1.4) is satisfied, then

$$\overline{\Sigma}_{k-1} = \Sigma_{k-1} = \kappa_1 + \cdots + \kappa_{k-1}$$

by taking the identical permutation in Definition 7; that is, $K^k = \{1, \dots, k-1\}$.

REMARK 3.5
With the notation in Definition 2, let $P_{\min}(I) = P_\alpha(I) = \{I_1, \cdots, I_m\}$. Then one can take a permutation π in Definition 7 such that

$$\overline{\Sigma}_{k-1} = (k-1)\alpha = \Sigma_{k-1} \quad \text{for } k \le m.$$

Indeed, it is sufficient to take $\pi(p) \in I_p$ for $p \le m$.

DEFINITION 12
Let Assumptions 1 and 4 be satisfied. For any $1 \le k \le j+1 \le N$, we put

$$J^k := I \setminus K^k;$$

we remark that $\#(J^k) = N - (k-1)$. For any $p \ge k$, we denote

$$\Sigma_p^{j,k} := \Sigma_{p-(k-1)}[J^k] - \gamma_{j,k}, \tag{3.7}$$

$$\kappa_p^{j,k} := \begin{cases} \Sigma_1[J^k] - \gamma_{j,k} & \text{if } p = k, \\ \Sigma_p^{j,k} - \Sigma_{p-1}^{j,k} & \text{if } p > k \text{ and } \Sigma_p^{j,k} < \infty, \\ \infty & \text{otherwise.} \end{cases} \tag{3.8}$$

Now, let

$$h^{j,k} = \min\{p = k, \dots, N-1 : \Sigma_p^{j,k} + p \ge N\} \tag{3.9}$$

if the minimum exists, and let $h^{j,k} = N$ otherwise. We define

$$\Gamma^j = \max_{1 \le k \le j+1} \Gamma^{j,k}, \tag{3.10}$$

where

$$\Gamma^{j,k} := \begin{cases} \frac{(N-h)\kappa_h^{j,k} + \Sigma_h^{j,k}}{\kappa_h^{j,k}+1} & \text{if } \kappa_h^{j,k} < \infty, \\ N - (h-1) & \text{if } \kappa_h^{j,k} = \infty. \end{cases} \tag{3.11}$$

For the sake of brevity, we omitted the apexes in $h^{j,k}$ in (3.11).

REMARK 3.6
We remark that for any j, k, it holds that

$$N - h^{j,k} \le \Gamma^{j,k} \le N - (h^{j,k} - 1);$$

that is, $\Gamma^{j,k}$ is *decreasing* with respect to $h^{j,k}$. It is clear that if we put

$$h^j := \min_k h^{j,k},$$

then

$$\max_{1 \le k \le j+1} \Gamma^{j,k} = \max_{\mathcal{H}_j} \Gamma^{j,k}, \quad \mathcal{H}_j := \{k : h^{j,k} = h^j\};$$

it follows that

$$N - h^j \le \Gamma^j \le N - (h^j - 1).$$

Moreover, once we have fixed h^j, then for any $k \in \mathcal{H}_j$ we can write $\Gamma^{j,k}$ as

$$\Gamma^{j,k} = N - (h^j - 1) - \frac{N - (h^j - 1) - \Sigma^{j,k}_{h^j-1}}{\kappa^{j,k}_{h^j} + 1};$$

hence it is *increasing* with respect to $\Sigma^{j,k}_{h^j-1}$ and $\kappa^{j,k}_{h^j}$.

REMARK 3.7

Let (1.4) and (3.5) be satisfied; thanks to Remark 3.4, with the notation in Definition 12, we get

$$K^k = \{1, \dots, k-1\}, \qquad J^k = I \setminus K^k = \{k, \dots, N\}.$$

Hence it holds that

$$\begin{aligned}\Sigma^{j,k}_p &= \Sigma_{p-(k-1)}[J^k] - \left[\gamma_j - \overline{\Sigma}_{k-1}[K^k]\right]^+ \\ &\le \kappa_k + \dots + \kappa_p - \gamma_j + \Sigma_{k-1} = \Sigma_p - \gamma_j = \Sigma^{j,1}_p.\end{aligned}$$

It follows that $h^{j,k} \ge h^{j,1}$; that is, $h^j = h^{j,1}$. Moreover, $\kappa^{j,k}_p \le \kappa^{j,1}_p$. Therefore, from Remark 3.6, it follows that $\Gamma^j = \Gamma^{j,1}$.

We are ready to state the following.

THEOREM 5

Let Assumptions 1 and 4 be satisfied. Then the Cauchy problem (3.1) is well posed in γ^d for any $1 < d < \min\{d^*, d_{\max}\}$, where $d_{\max}$ is defined in (2.20), and

$$d^* = \min\{d_j : j = 1, \dots, N-1\}, \tag{3.12}$$

$$d_j := \begin{cases} \infty & \text{if } \Gamma^j \le N - j, \\ 1 + \frac{N-j}{\Gamma^j - (N-j)} & \text{otherwise.} \end{cases} \tag{3.13}$$

REMARK 3.8

We remark that $d_j = \infty$, that is $\Gamma^j \le N - j$, if and only if either $h^j \ge j + 1$, or $h^j = j$ and $\Sigma^{j,k}_j + j = N$ for any $k \in \mathcal{H}^j$, that is

$$\Sigma^{j,k}_j + j \le N, \qquad \text{for any } k.$$

REMARK 3.9

We notice that d_j can be written as

$$d_j = 1 + \frac{1}{\frac{\Gamma_j}{N-j} - 1} = d_B\left(\frac{\Gamma_j}{N-j}\right).$$

From Remark 3.6, being $\Gamma^j \le N$, it follows that

$$d_j \ge d_B\left(\frac{N}{N-j}\right),$$

namely, $d_{N-1} \ge d_B(N)$, $d_{N-2} \ge d_B(N/2), \dots$.

In particular, if $h^{N-1} \le N/2-1$, then $d^* = d_{N-1}$, since

$$d_{N-1} \le 1 + \frac{1}{N/2-1} \le d_j \qquad \text{for any } j \le N-2.$$

Proof of Theorem 5
The proof is based on the same energies $E_1(t,\xi)$ and $E_2(t,\xi)$ introduced in Definitions 8 and 9, but we have to replace (2.26) with (3.3) in the proof of Lemma 2.7. Therefore, in order to derive (2.23), we have to control the terms

$$|\xi|^{-(N-1-j)} |\Delta_{k-1}[b_j(t,\xi)]\big(\lambda_{\pi(1)}(t,\xi),\dots,\lambda_{\pi(k)}(t,\xi)\big)||w_k V|,$$
$$j = 0,\dots,N-1,\ 1 \le k \le j+1,$$

in $t \in [t_1(\xi),T]$, where $t_1(\xi) = |\xi|^{-1+\frac{1}{d^*}}$ as in (2.15).

For the sake of brevity, let π be the identical permutation in Definition 7. We fix j,k. Thanks to (3.4) and to Lemma 2.2, we get

$$\text{(3.14)} \quad |\Delta_{k-1}[b_j](\lambda_1,\dots,\lambda_k)||w_k V| \lesssim t^{\gamma_{jk}} |\omega[K^k]V| \lesssim \frac{[V]_l}{t^{\Sigma_{l-(k-1)}[J^k]}} t^{\gamma_{j,k}}$$
$$\lesssim \Big(\frac{t}{t_1(\xi)}\Big)^{N-(l+1)} \frac{1}{t^{\Sigma_l^{j,k}-1}} \times \frac{1}{t}\sqrt{E_2(t,\xi)}$$

for any $l \ge k-1$, where $\Sigma_l^{j,k}$ is defined in (3.7) for $p = l$.

Let $h = h^{j,k}$ be as in (3.9); we assume $\Gamma^{j,k} > N-j$, with the other case being trivial, and we set $t_2(\xi)$ as in (2.28); that is,

$$\text{(3.15)} \qquad t_2(\xi) := \big(t_1(\xi)\big)^{1/(\kappa_h^{j,k}+1)} \equiv |\xi|^{-(d^*-1)/(\kappa_h^{j,k}+1)d^*}$$

for $\kappa_h^{j,k} < \infty$ and $t_2(\xi) = T$ for $\kappa_h^{j,k} = \infty$, where $\kappa_h^{j,k}$ are defined in (3.8) for $p = h$.

For any $t \in [t_1,t_2]$, we take $l = h-1$ in (3.14) (we remark that $h^{j,k}-1 \ge k-1$ in (3.9)); hence, thanks to (3.9), analogously to (2.29), we get

$$|\Delta_{k-1}[b_j](\lambda_1,\dots,\lambda_k)||w_k| \lesssim t_1^{-(N-h)} t_2^{N-h-\Sigma_{h-1}^{j,k}+1} \times \frac{1}{t}\sqrt{E_2}.$$

We notice that

$$(N-h) - \frac{N-h-\Sigma_{h-1}^{j,k}+1}{\kappa_h+1} = \frac{(N-h)\kappa_h + \Sigma_{h-1}^{j,k}-1}{\kappa_h+1} = \Gamma^{j,k}-1;$$

hence, using (3.15), we get

$$\text{(3.16)} \qquad |\xi|^{-(N-1-j)} |\Delta_{k-1}[b_j](\lambda_1,\dots,\lambda_k)||w_k| \lesssim |\xi|^{-(N-1-j)} t_1^{1-\Gamma^{j,k}} \frac{1}{t}\sqrt{E_2}.$$

For any $t \in [t_2,T]$, we take $l = h$ in (3.14); hence, thanks to (3.9) and analogously to (2.31), we get

$$|\Delta_{k-1}[b_j](\lambda_1,\dots,\lambda_k)||w_k| \lesssim t_1^{-(N-(h+1))} t_2^{N-(h+1)+1-\Sigma_h^{j,k}} \frac{1}{t}\sqrt{E_2}.$$

By using (2.28) again we find the same estimate in (3.16) since

$$-(N-h-1) + \frac{N-h-\Sigma_h^{j,k}}{\kappa_h+1} = -\frac{(N-h)\kappa_h + \Sigma_{h-1}^{j,k}-1}{\kappa_h+1} = 1-\Gamma^{j,k}.$$

Now, from

$$d^* \le d_j = 1 + \frac{N-j}{\Gamma^j - (N-j)} \le 1 + \frac{N-j}{\Gamma^{j,k} - (N-j)} =: d_{j,k},$$

it follows that

$$\begin{aligned} |\xi|^{-(N-j-1)} t_1^{1-\Gamma^{j,k}} &= |\xi|^{-(N-j-1)+(\Gamma^{j,k}-1)(d^*-1)/d^*} \\ &\le |\xi|^{-(N-j-1)+(\Gamma^{j,k}-1)(N-j)/\Gamma^{j,k}} \\ &= |\xi|^{1/d_{j,k}} \le |\xi|^{1/d_j} \le |\xi|^{1/d^*}. \end{aligned}$$

This concludes the proof. □

4. Proof of Theorems 3 and 4

We come back to the the Cauchy problem (2.33), and we study more in detail the $((N^2)\times(N^2))$-matrix $\mathcal{B}$ in (2.34). In order to describe explicitly the last row of each $(N\times N)$-block $\mathcal{B}_{[j,k]}$ of $\mathcal{B}$ (the other rows are zero), we study more in detail the systems $\mathcal{L}_1$ and $\mathcal{L}_2$.

DEFINITION 13

We recall that

$$\Lambda(t,\tau,i\xi) = \chi^{\mathrm{adj}}(t,\tau,i\xi), \qquad \chi = \tau \mathrm{I}_N - i|\xi| A(t,\xi),$$

and we define

$$\begin{aligned} \Lambda^{\{j\}}(t,\tau,i\xi) &:= \frac{1}{j!}\partial_\tau^j \Lambda(t,\tau,i\xi), \quad j=1,\dots,N, \\ \Lambda'(t,\tau,i\xi) &:= \partial_t \Lambda(t,\tau,i\xi). \end{aligned}$$

We remark that $\Lambda^{\{N\}} \equiv 0$ and that $\Lambda^{\{N-1\}} \equiv \mathrm{I}_N$.

Now, since

$$L(t,\partial_t,i\xi) = \partial_t - i|\xi|A(t,\xi) - B(t),$$

with the notation in (2.32) and in Definition 13, we get

$$\mathcal{L}_1(t,\partial_t,i\xi) = \Lambda(t,\partial_t,i\xi)L(t,\partial_t,i\xi) = \mathrm{I}_N P(t,\partial_t,i\xi) - \sum_{j=0}^{N-1} M_j(t,\partial_t,i\xi),$$

where

$$\begin{aligned} M_j(t,\tau,i\xi) = i|\xi|\Lambda^{\{N-j\}}(t,\tau,i\xi)A^{(N-j)}(t,\xi) \\ + \Lambda^{\{N-1-j\}}(t,\tau,i\xi)B^{(N-1-j)}(t). \end{aligned}$$

Now, each of the N^2 entries of $M_j(t,\tau,i\xi)$, say, the (r,s)th, can be written in the form

$$\sum_{0\le l\le j} b_{j,l}[r,s](t,i\xi)\tau^l,$$

where $b_{j,l}[r,s](t,\xi)$ is homogeneous of degree $j-l$ in ξ. We put

$$\begin{aligned} b_j[r,s](t,\xi) &:= \sum_{l=0}^{j} b_{j,l}[r,s](t,\xi/|\xi|)e_{l+1} \\ &= \sum_{l=0}^{j} |\xi|^{-(j-l)} b_{j,l}[r,s](t,\xi)e_{l+1}, \end{aligned}$$

where (e_l) denotes the canonical basis of $\mathbb{C}^N$. Therefore the last row of the (r,s)th block of $\mathcal{B}$ is

$$(\mathcal{B}_{[r,s]})_{N\cdot} = \sum_{j=0}^{N-1} |\xi|^{-(N-1-j)} b_j[r,s](t,\xi),$$

and we have to estimate

$$|\xi|^{-(N-1-j)}|b_j[r,s](t,\xi)W^{(s)}|, \quad j=0,\dots,N-1,\ r,s=1,\dots,N,$$

as in (3.3). In order to apply Lemma 3.3 to the Cauchy problem (2.33) for $\mathcal{L}_1$, we look for indexes γ_j, not depending on r,s, such that (3.5) is satisfied for $b_j = b_j[r,s]$, for any $j=0,\dots,N-1$, and for any $r,s=1,\dots,N$. It is easy to check that

$$\Delta_0\big[b_j[r,s](t,\xi)\big](\lambda_{\pi(l)}) = \big(M_j(t,\lambda_{\pi(l)},\xi/|\xi|)\big)_{r,s} \tag{4.1}$$

for any $j=0,\dots,N-1$; hence we have to look for indexes γ_j such that

$$\|M_j(t,\lambda_{\pi(l)},\xi/|\xi|)\| \lesssim t^{\gamma_j}, \quad \text{for any } l=1,\dots,j+1;$$

with no assumption on the derivatives of $A(t,\xi)$ and $B(t)$, the estimate above is satisfied if

$$\begin{gathered} \|\Lambda^{\{N-j\}}(t,\lambda_{\pi(l)},\xi/|\xi|)\| + \|\Lambda^{\{N-1-j\}}(t,\lambda_{\pi(l)},\xi/|\xi|)\| \lesssim t^{\gamma_j} \\ \text{for any } l=1,\dots,j+1. \end{gathered} \tag{4.2}$$

Similarly, with the notation in (2.32) and in Definition 13, we get

$$\mathcal{L}_2(t,\partial_t,i\xi) = L(t,\partial_t,i\xi)\Lambda(t,\partial_t,i\xi) = \mathrm{I}_N P(t,\partial_t,i\xi) - M_{N-1}(t,\partial_t,i\xi)$$

with $M_{N-1} = -\Lambda' + B\Lambda$. In order to apply Lemma 3.3 to the Cauchy problem (2.33) for $\mathcal{L}_1$, we look for an index γ_{N-1}, not depending on r,s, such that (3.5) is satisfied for $b_{N-1} = b_{N-1}[r,s]$, for any $r,s=1,\dots,N$. Thanks to the equality (4.1) for $j=N-1$, and with no assumption on $B(t)$, we have to look for an index γ_{N-1} such that

$$\begin{gathered} \|\Lambda'(t,\lambda_{\pi(l)},\xi/|\xi|)\| + \|\Lambda(t,\lambda_{\pi(l)},\xi/|\xi|)\| \lesssim t^{\gamma_{N-1}} \\ \text{for any } l=1,\dots,j+1. \end{gathered} \tag{4.3}$$

We have proved the following corollary of Theorem 5.

THEOREM 6

Let Assumption 1 be satisfied, and assume that

$$(4.4)\qquad \begin{cases} \|\Lambda'(t,\lambda_l,\xi/|\xi|)\| + \|\Lambda^{\{1\}}(t,\lambda_l,\xi/|\xi|)\| + \|\Lambda(t,\lambda_l,\xi/|\xi|)\| \lesssim t^{\gamma_{N-1}}, \\ \|\Lambda^{\{N-j\}}(t,\lambda_l,\xi/|\xi|)\| + \|\Lambda^{\{N-1-j\}}(t,\lambda_l,\xi/|\xi|)\| \lesssim t^{\gamma_j}, \\ \quad j = 1,\dots,N-2, \end{cases}$$

for any $l = 1,\dots,N$*, with the notation in Definition 13. Moreover, as in (3.6), let*

$$\gamma_{j,k} = [\gamma_j - \overline{\Sigma}_{k-1}]^+.$$

Then the Cauchy problem (1.1) is strongly well posed in γ^d *for any* $1 < d < \min\{d^*, d_{\max}\}$*, with the notation in Theorem 5.*

By adding assumptions on the structure of A, we can obtain (4.4) and then apply Theorem 6.

LEMMA 4.1

If Assumptions 1 and 2 are satisfied, then we get (4.4) for

$$\gamma_j = (j-1)\gamma.$$

Proof

We notice that

$$\Lambda = \chi^{\mathrm{adj}} = \sigma^*_{N-1}\mathrm{I}_N - \sigma^*_{N-2}\chi + \cdots + (-1)^{N-2}\sigma^*_1\chi^{N-2} + (-1)^{N-1}\chi^{N-1},$$

where $\sigma^*_j(t,\tau,\xi)$ for $j = 1,\dots,N-1$ is the jth elementary symmetric function introduced in (2.3) associated to the eigenvalues of the matrix $\chi(t,\tau,i\xi)$, namely,

$$\sigma^*_j(t,\tau,\xi) = \sum_{I^{[j]}} \prod_{m=1}^{j} (\tau - \lambda_{p(m)}), \quad I^{[j]} = \{p(1),\dots,p(j) \in I : p(1) < \cdots < p(j)\}.$$

As in Definition 13, let

$$\sigma^{*\{p\}}_{N-1-r}(t,\tau,\xi) = \frac{1}{p!}\partial^p_\tau \sigma^*_{N-1-r}(t,\tau,\xi), \qquad (\chi^r)^{\{q\}} = \frac{1}{q!}\partial^q_\tau \chi^r;$$

it is clear that

$$\begin{aligned} \Lambda^{\{s\}} &= \frac{1}{s!}\partial^s_\tau \chi^{\mathrm{adj}} = \frac{1}{s!}\sum_{r=0}^{N-1}(-1)^r\Big(\sum_{q+p=s}\frac{s!}{p!q!}(\partial^p_\tau\sigma^*_{N-1-r})(\partial^q_\tau\chi^r)\Big) \\ &= \sum_{r=0}^{N-1}(-1)^r\Big(\sum_{q+p=s}\sigma^{*\{p\}}_{N-1-r}(\chi^r)^{\{q\}}\Big). \end{aligned}$$

We fix $l = 1,\dots,j+1$. For any $p \le N-1-r$, we have

$$|\sigma^{*\{p\}}_{N-1-r}(t,\lambda_l,\xi/|\xi|)| \lesssim (t^\alpha)^{N-1-r-p}.$$

To estimate the other terms we use (1.10); for any $q \le r$, we obtain

$$\big|(\chi^r)^{\{q\}}(t,\lambda_l,\xi/|\xi|)\big| \lesssim \|\chi(t,\lambda_l,\xi/|\xi|)\|^{r-q} \lesssim t^{(r-q)\gamma}$$

since

$$\Big|\frac{\operatorname{tr} A}{N} - \lambda_l\Big| \lesssim t^\alpha \lesssim t^\gamma.$$

Therefore we have proved (4.2) for $\gamma_j = ((N-1)-(N-j))\gamma = (j-1)\gamma$. Similarly, we prove (4.3) for $\gamma_{N-1} = (N-2)\gamma$. This concludes the proof. □

Thanks to Lemma 4.1, we can prove Theorem 3 as a consequence of Theorem 6.

Proof of Theorem 3
Thanks to Remark 3.7, we know that $\Gamma^j = \Gamma^{j,1}$. We claim that

$$\Gamma^{N-1} - 1 \ge \Gamma^j - (N-j) \tag{4.5}$$

for any j; hence $d_{N-1} \le d_j$. We prove (4.5) for $j = N-2$; that is, $\Gamma^{N-2} \le \Gamma^{N-1} + 1$, the other cases being analogous.

Let $h = h^{N-1}$ and $h' = h^{N-2}$. It is clear that either $h' = h$ or $h' = h-1$ since

$$0 \le \gamma_{N-1} - \gamma_{N-2} = \gamma \le \alpha.$$

If $h' = h$, then it trivially holds that

$$\Gamma^{N-2} \le N-(h-1) = (N-h)+1 \le \Gamma^{N-1}+1.$$

Let $h' = h-1$. From (1.4), it follows that $\kappa_h \ge \kappa_{h-1}$ and hence that

$$\kappa_h^{N-1,1} \ge \kappa_{h-1}^{N-2,1};$$

moreover, we have

$$\Sigma_h^{N-1,1} = \kappa_h + \Sigma_{h-1}^{N-2,1} - \gamma.$$

Therefore

$$\begin{aligned}\Gamma^{N-1} &= \frac{(N-h)\kappa_h + \Sigma_h - (N-2)\gamma}{\kappa_h+1}\\ &= \frac{(N-(h-1))\kappa_h + \Sigma_{h-1} - (N-3)\gamma - \gamma}{\kappa_h+1}\\ &\ge \Gamma^{N-2} - \frac{\gamma}{\kappa_h+1} \ge \Gamma^{N-2} - 1.\end{aligned}$$

In order to conclude the proof, we show that $d^* \le d_{\max}$, where d^* is as in Theorem 3. We distinguish two possibilities: if $h \le N-2$, then

$$d^* \le d_B\big(N-(N-2)\big) = 2 \le d_{\max},$$

whereas if $h = N-1$, then $\kappa_h = \omega$; therefore

$$d^* = 1 + \frac{\omega+1}{\Sigma_{N-1} - (N-2)\gamma - 1} \le 1 + \frac{\omega+1}{\omega-1} = d_{\max}$$

since

$$\Sigma_{N-1} = \kappa_1 + \cdots + \kappa_{N-2} + \omega \geq (N-2)\alpha + \omega \geq (N-2)\gamma + \omega.$$

This concludes the proof. □

In order to prove Theorem 4, we apply Theorem 6 to the Cauchy problem (1.11).

Proof of Theorem 4
Let J_A be the Jordan canonical form of A. Because $\mu_p - \mu_q \neq 0$ for any $t > 0$ and thanks to Assumption 3, we can write

$$J_A = \bigoplus_{q=1}^{m} J_q,$$

where J_q is the Jordan block matrix related to the eigenvalue μ_q, and it has size M_q. If we put

$$\nu_p(t,\tau,\xi) = \tau - \mu_p(t,\xi), \quad p = 1,\dots,m,$$

then

$$\Lambda(t,\tau,\xi/|\xi|) = (\tau \mathrm{I}_N - J_A)^{\mathrm{adj}} = \bigoplus_{q=1}^{m}\Big((\tau \mathrm{I}_{M_q} - J_q)^{\mathrm{adj}} \prod_{p\neq q} \nu_p^{M_p}(t,\tau,\xi)\Big).$$

Therefore, for any $l = 1,\dots,m$, since $\nu_l(t,\mu_l,\xi) \equiv 0$, we have

$$\Lambda(t,\mu_l,\xi/|\xi|) = 0_{M_1} \oplus \cdots \oplus 0_{M_{l-1}} \oplus (\mu_l \mathrm{I}_{M_l} - J_l)^{\mathrm{adj}} \prod_{p\neq l} (\mu_l - \mu_p)^{M_p}$$
$$\oplus\, 0_{M_{l+1}} \oplus \cdots \oplus 0_{M_m},$$

where we denote by 0_{M_p} a block with size M_p such that all entries are zero. Now, because $M = \max M_p$, it is easy to check that

$$\|\Lambda(t,\mu_l,\xi/|\xi|)\| \lesssim t^{(N-M)\alpha},$$
$$\|\Lambda'(t,\mu_l,\xi/|\xi|)\| \lesssim t^{(N-M-1)\alpha},$$
$$\|\Lambda^{\{N-j\}}(t,\mu_l,\xi/|\xi|)\| \lesssim t^{(j-M)\alpha} \quad \text{for any } j \geq M.$$

Thanks to Remark 3.5, we have

$$\overline{\Sigma}_{k-1} = \begin{cases} (k-1)\alpha & \text{if } k \leq m, \\ \infty & \text{otherwise;} \end{cases}$$

hence

$$\gamma_{j,k} = \begin{cases} [[j-M]^+ - (k-1)]^+\alpha & \text{if } k \leq m, \\ 0 & \text{if } k > m. \end{cases}$$

We can apply Theorem 6. Since

$$\Sigma_p = \begin{cases} p\alpha & \text{if } p \leq N-M, \\ \infty & \text{otherwise,} \end{cases}$$

we get $\Sigma_p^{j,1} = \infty$ for $p \geq N - M + 1$, that is,

$$\max_{k \leq p} \Sigma_p^{j,k} = \Sigma_p^{j,1} = \infty, \quad p \geq N - M + 1.$$

On the other hand, for any $k \leq p \leq N - M$,

$$\Sigma_p^{j,k} = \big(p - (k-1)\big)\alpha - \gamma_{j,k},$$

and analogously to Remark 3.7, it is easy to show that for any $k \leq p \leq N - M$, it holds that

$$\Sigma_p^{j,k} \leq \Sigma_p^{j,1} = (p - [j-M]^+)\alpha \quad \text{for } k \leq m,$$

whereas

$$\Sigma_p^{j,k} \leq \Sigma_p^{j,m+1} = (p-m)\alpha \quad \text{for } m+1 \leq k.$$

Thus we get

$$\max_{k \leq p} \Sigma_p^{j,k} = \begin{cases} (p - [j-M]^+)\alpha & \text{if } p \leq m, \\ cr(p - [j-M]^+)\alpha & \text{if } m \leq p \leq N-M \text{ and } j \leq M+m, \\ (p-m)\alpha & \text{if } m \leq p \leq N-M \text{ and } M+m \leq j, \\ \infty & \text{if } N-M+1 \leq p. \end{cases}$$

We distinguish three cases.

(1) Let $j \geq M + m$. Then $h^j = N - M + 1$ if

$$(N-M-m)\alpha + (N-M) < N, \quad \text{that is, if } (N-M-m)\alpha < M,$$

whereas $h^j \leq N - M$ otherwise. In the first case, it follows that

$$\Gamma^j = N - (N-M) = M,$$

whereas in the second one we get

$$\Gamma^j = \frac{(N-h)\alpha + (h-m)\alpha}{\alpha+1} = \frac{(N-m)\alpha}{\alpha+1}.$$

(2) Let $M \leq j \leq M + m$. Then $h^j = N - M + 1$ if

$$\big((N-M) - (j-M)\big)\alpha + (N-M) < N, \quad \text{that is, if } (N-j)\alpha < M,$$

whereas $h^j \leq N - M$ otherwise. In the first case, it follows $\Gamma^j = M$ again, whereas in the second one we get

$$\Gamma^j = \frac{(N-h)\alpha + (h-(j-M))\alpha}{\alpha+1} = \frac{(N-j+M)\alpha}{\alpha+1}.$$

(3) Let $j \leq M$. Then $\Gamma^j = M$ if $(N-M)\alpha < M$, whereas

$$\Gamma^j = \frac{N\alpha}{\alpha+1}$$

if $N\alpha \geq M(\alpha+1)$.

We assume first that $M + m \leq N - 1$. We distinguish three cases.

- If $(N-M)\alpha < M$, then $\Gamma^j = M$ for any j; hence

$$d^* = d_{N-1} = d_B(M).$$

- If there exists j^*, with $M \le j^* \le M+m-1$, such that

$$\big(N-(j^*+1)\big)\alpha < M \le (N-j^*)\alpha, \tag{4.6}$$

then $\Gamma^j = M$ for any $j \ge j^*+1$, whereas

$$\Gamma^j = \frac{(N-j+M)\alpha}{\alpha+1}, \quad M \le j \le j^*,$$

$$\Gamma^j = \frac{N\alpha}{\alpha+1}, \quad j \le M.$$

It follows that $d_{j^*} \le d_{j^*-1} \le \cdots$ since

$$\Gamma^{j^*} \ge \Gamma^{j^*-1} - 1 \ge \cdots.$$

On the other hand, from (4.6) it follows that

$$\Gamma^{j^*} < \frac{(M+1)\alpha+M}{\alpha+1} = M + \frac{\alpha}{\alpha+1} \le M+1 = \Gamma^{j^*+1}+1.$$

Therefore $d^* = d_{N-1} = d_B(M)$.

- If $(N-M-m)\alpha \ge M$, then

$$\Gamma^j = \frac{(N-m)\alpha}{\alpha+1}, \quad M+m \le j,$$

and

$$\Gamma^{N-1} = \cdots = \Gamma^{M+m} \ge \Gamma^{M+m-1} - 1 \ge \cdots.$$

Therefore

$$d^* = d_{N-1} = 1 + \frac{\alpha+1}{(N-m-1)\alpha-1}.$$

We have proved (1.12). Now we assume that $M+m \ge N$, and we prove (1.13).

Case (1) is verified for no j; hence we have to distinguish three cases.

- If $(N-M)\alpha < M$, then $\Gamma^j = M$ for any j; hence

$$d^* = d_{N-1} = d_B(M).$$

- If there exists $M \le j^* \le N-2$ such that

$$\big(N-(j^*+1)\big)\alpha < M \le (N-j^*)\alpha,$$

we obtain again

$$M+1 \ge \Gamma^{j^*} \ge \Gamma^{j^*-1} - 1 \ge \cdots.$$

Therefore $d^* = d_{N-1} = d_B(M)$.

- If $\alpha \ge M$, then

$$\Gamma^j = \frac{(N-j+M)\alpha}{\alpha+1}, \quad M \le j;$$

hence

$$\cdots \le \Gamma^{N-2} - 1 \le \Gamma^{N-1} = \frac{(M+1)\alpha}{\alpha+1}.$$

Therefore

$$d^* = d_{N-1} = 1 + \frac{\alpha+1}{M\alpha - 1}.$$

This concludes the proof. □

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