Another construction of a Cantor bouquet at a fixed indeterminate point

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Abstract In this article, we study the local dynamical structure of a rational mapping F of $\mathbf{P}^2$ at a fixed indeterminate point p. Using a sequence of blowups, we construct a family $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ of germs of holomorphic curve at the point p, where J is a subset of a Cantor set $\{1,2\}^{\mathbf{N}}$. This is a new construction for a Cantor bouquet.

1. Introduction

The dynamics of a rational mapping F on the 2-dimensional complex projective space $\mathbf{P}^2$ at an indeterminate point p have been studied by Y. Yamagishi [7], [8] and T. C. Dinh, R. Dujardin, and N. Sibony [2]. Roughly speaking, they showed that if F contracts some open neighborhood U_p of p in some direction, then there exists a family of uncountably many currents or stable manifolds of p which is called a *Cantor bouquet* of p. Their results show that a chaotic phenomenon occurs in a neighborhood of the indeterminate point at which the mapping is not continuous.

In this article, we try another approach to the construction of a Cantor bouquet. By using a sequence of blowups, we construct a family $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ of germs of holomorphic curve at the point p, where J is a subset of a Cantor set $\{1,2\}^{\mathbf{N}}$. We remark here that the family $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ contains not only stable manifolds of p but also center or unstable manifolds of p. Hence our $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ is a generalization of a Cantor bouquet.

This article is organized as follows. In Section 2, we state some preliminary facts and our main theorems. Section 3 is devoted to the construction of the family $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ of germs of holomorphic curve at the point p. In the final Section 4, as an application, we consider a specific rational mapping F and completely determine the number of germs of $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$. In particular, J is a proper subset of the Cantor set $\{1,2\}^{\mathbf{N}}$, and every $\tilde{W}_{\mathbf{j}}$ is an unstable manifold of p. This is a new dynamical structure at an indeterminate point p where F is not continuous.

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2. Preliminaries and main theorems

In this section, we fix the notation that is used throughout this article and state our main theorems. First, we fix once and for all a homogeneous coordinate system $[x:y:z]$ in $\mathbf{P}^2$; we often use the natural identification given by

$$\mathbf{C}^2 = \{[x:y:z] \in \mathbf{P}^2 \mid z \neq 0\} \qquad \text{and} \qquad (x,y) = [x:y:1].$$

Let $f_i(x,y,z)$ $(i=0,1,2)$ be homogeneous polynomials of degree d. Then, by setting

$$F([x:y:z]) = [f_0:f_1:f_2] \qquad \text{and} \qquad \hat{F}(x,y,z) = (f_0,f_1,f_2),$$

we have a rational mapping F on $\mathbf{P}^2$ and a polynomial mapping $\hat{F}$ on $\mathbf{C}^3$ with $\hat{\pi} \circ \hat{F} = F \circ \hat{\pi}$ on $\mathbf{C}^3$ outside some proper analytic sets, where $\hat{\pi} : \mathbf{C}^3 \setminus \{(0,0,0)\} \to \mathbf{P}^2$ is the canonical projection. A point $p \in \mathbf{P}^2$ is said to be an *indeterminate point* of F if $\hat{F}(\hat{p}) = (0,0,0)$ for some point $\hat{p} \in \hat{\pi}^{-1}(p)$. In this article, we assume that a rational mapping F has an indeterminate point $p = [0:0:1]$. In general, if p is an indeterminate point, then F is not continuous at p and $\bigcap_{N_p} \overline{F(N_p \setminus \{p\})}$ is not a singleton, where the intersection is taken over all open neighborhoods N_p of p. Moreover, p is said to be a *fixed indeterminate point* if $p \in \bigcap_{N_p} \overline{F(N_p \setminus \{p\})}$. We remark here that a fixed indeterminate point p is nonwandering; nevertheless, F is not continuous at p. Therefore, it is important to study the local dynamical structure at such a point.

Next, we introduce some notation and terminology from algebraic geometry. We refer the reader to [3, §2.4]. Consider the product space $\mathbf{C}^2 \times \mathbf{P}^1$, and define the subvariety $X \subset \mathbf{C}^2 \times \mathbf{P}^1$ as the following:

$$X := \{(x,y) \times [u:v] \in \mathbf{C}^2 \times \mathbf{P}^1 \mid xv - (y-\alpha)u = 0\}$$

for the point $(0,\alpha) \in \mathbf{C}^2$.

DEFINITION 2.1

The mapping $\pi : X \to \mathbf{C}^2$ defined by restricting the first projection $\mathbf{C}^2 \times \mathbf{P}^1 \to \mathbf{C}^2$ to X is called the blowup of $\mathbf{C}^2$ centered at $(0,\alpha)$.

It follows from the definition that $\pi^{-1}(0,\alpha) = \{(0,\alpha)\} \times \mathbf{P}^1$ and that

$$\pi : X \setminus \pi^{-1}(0,\alpha) \to \mathbf{C}^2 \setminus \{(0,\alpha)\} \quad \text{is biholomorphic.}$$

Put $E := \pi^{-1}(0,\alpha)$; E is called the *exceptional curve*. X has the local chart $\{(U^i,\varphi^i)\}_{i=1,2}$ defined by

$$U^1 := \{(x,y) \times [u:v] \in X \mid u \neq 0\} = \Big\{(x,y) \times [u:v] \in X \mid y = \alpha + x\frac{v}{u}\Big\},$$

$$U^2 := \{(x,y) \times [u:v] \in X \mid v \neq 0\} = \Big\{(x,y) \times [u:v] \in X \mid x = (y-\alpha)\frac{u}{v}\Big\},$$

$$\text{(C.1)} \qquad \begin{cases} \varphi^1 : U^1 \ni (x,y) \times [u:v] \mapsto (x, v/u) \in \mathbf{C}^2, \\ \varphi^2 : U^2 \ni (x,y) \times [u:v] \mapsto (u/v, y) \in \mathbf{C}^2. \end{cases}$$

Observe that the restriction of π to U^i can be written as

$$\text{(C.2)} \qquad \begin{cases} \pi|_{U^1} : U^1 \ni (x,\eta) \mapsto (x, x\eta + \alpha) \in \mathbf{C}^2, \\ \pi|_{U^2} : U^2 \ni (\xi, y) \mapsto (\xi(y-\alpha), y) \in \mathbf{C}^2 \end{cases}$$

by using local charts φ^i. The verification of the following proposition is straightforward; therefore, the proof is left to reader.

PROPOSITION 2.1

We have the following:

(1) $X \setminus U^1 = \{(\xi, y) \in U^2 \mid \xi = 0\}$.

(2) $E \cap U^1 = \{(x,\eta) \in U^1 \mid x = 0\}$ *and* $E \cap U^2 = \{(\xi, y) \in U^2 \mid y = \alpha\}$.

(3) $E \cap (U^2 \setminus U^1) = \{(\xi, y) = (0, \alpha) \in U^2\}$.

By pasting $\mathbf{C}^2 = \{[x : y : z] \in \mathbf{P}^2 \mid z \neq 0\}$ on the other charts of $\mathbf{P}^2$, one can obtain the blowup of $\mathbf{P}^2$ centered at $[0 : \alpha : 1]$. To simplify our notation, we denote this also by $\pi : X \to \mathbf{P}^2$.

Throughout this article, we concentrate our attention on the dynamics of F in the chart $\mathbf{C}^2 = \{[x : y : z] \in \mathbf{P}^2 \mid z \neq 0\}$. Observe that $p = (0,0)$ is our indeterminate point. We also denote the restriction of F to $\mathbf{C}^2 = \{[x : y : z] \in \mathbf{P}^2 \mid z \neq 0\}$ by F.

The investigation of the local dynamical structure at an indeterminate point originated with Y. Yamagishi [7], [8]; we introduce his idea of Cantor bouquet here. Let us define a rational mapping

$$\tilde{F} : X \to \mathbf{C}^2 \quad \text{by } \tilde{F} := F \circ \pi,$$

where π is the blowup centered at $p = (0,0)$. Yamagishi assumed that $\tilde{F}$ satisfies the following:

$$\text{(A.0)} \qquad \begin{cases} \text{(1) } \tilde{F} \textit{ is a holomorphic mapping on a neighborhood of } E; \\ \text{(2) } \tilde{F}^{-1}(p) \cap E \textit{ consists of two points } p_{j_1} \ (j_1 = 1,2); \textit{ and} \\ \text{(3) } \textit{there exists an open neighborhood } N_{j_1} \textit{ of } p_{j_1} \ (j_1 = 1,2) \\ \qquad \textit{such that } \tilde{F} \textit{ is biholomorphic on } N_{j_1}. \end{cases}$$

Notice that p is a fixed indeterminate point of F under condition (2) of (A.0). Moreover, he showed that if F contracts some open neighborhood U_p of p in some direction, then there exists a family $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ of uncountably many local stable manifolds of p (for details, see [7]). $\{W_{\mathbf{j}}\}_{\mathbf{j} \in \{1,2\}^{\mathbf{N}}}$ is called a *Cantor bouquet* of p.

Instead of a Cantor bouquet, in this article, we consider the following family of germs of holomorphic curve.

DEFINITION 2.2

W_λ is a *holomorphic curve which passes through* p if there exist a holomorphic function ϕ_λ on Δ_{ρ_λ} and a holomorphic mapping $\Phi_\lambda : \Delta_{\rho_\lambda} \to \mathbf{C}^2, t \mapsto (t, \phi_\lambda(t))$ such that $\Phi_\lambda(0) = p$ and $\Phi_\lambda(\Delta_{\rho_\lambda}) = W_\lambda$, where $\Delta_{\rho_\lambda} := \{t \in \mathbf{C} \mid |t| < \rho_\lambda\}$.

Two holomorphic curves W_λ and $W_{\lambda'}$ which pass through p are called equivalent if there is an open neighborhood U of p such that $W_\lambda \cap U = W_{\lambda'} \cap U$, and it is denoted by $W_\lambda \sim W_{\lambda'}$. This is an equivalence relation, and an equivalence class is called a *germ of a holomorphic curve* at the point p. Any holomorphic curve W_λ which passes through p belongs to some equivalence class; this class is called the *germ of the holomorphic curve* W_λ and is denoted by $\tilde{W}_\lambda$.

DEFINITION 2.3

A family $\{\tilde{W}_\lambda\}_{\lambda\in\Lambda}$ of germs of holomorphic curve at the point p is *invariant* if for every $\lambda \in \Lambda$ there exist a unique $\lambda' \in \Lambda$ and a positive constant $\rho_{\lambda'}$ with $0 < \rho_{\lambda'} \le \rho_\lambda$ such that $F \circ \Phi_\lambda(\Delta_{\rho_{\lambda'}})$ is a holomorphic curve that passes through p and $F \circ \Phi_\lambda(\Delta_{\rho_{\lambda'}}) \in \tilde{W}_{\lambda'}$.

REMARK 1

The mapping $F \circ \Phi_\lambda$ is well defined at $t = 0$, although $p = \Phi_\lambda(0)$ is an indeterminate point of F. Indeed, there exists a unique holomorphic mapping $g : \Delta_{\rho_{\lambda'}} \to \mathbf{C}^2$ such that $g(t) = F \circ \Phi_\lambda(t)$ for any $t \in \Delta_{\rho_{\lambda'}} \setminus \{0\}$ (for more detail, see [1]).

REMARK 2

The family $\{\tilde{W}_\lambda\}_{\lambda\in\Lambda}$ contains not only stable manifolds of p but also center or unstable manifolds of p. Hence, our $\{\tilde{W}_\lambda\}_{\lambda\in\Lambda}$ is a generalization of a Cantor bouquet in the sense of Yamagishi.

In this article, we assume that a rational mapping F satisfies the condition (A.0). By Proposition 2.1, if $p_{j_1} \in E \cap U^1$, then one can set $p_{j_1} := (0, \alpha_{j_1}) \in U^1$; and if $p_{j_1} \in E \cap (U^2 \setminus U^1)$, then $p_{j_1} = (0,0) \in U^2$. Hence, one can put $p_{j_1} = (0, \alpha_{j_1}) \in U^i$ in any case. Together with the identification $U^i \cong \mathbf{C}^2$ $(i = 1,2)$, for $p_{j_1} \in U^i$ we can define the subvariety

$$X_{j_1} := \{(x,y) \times [u:v] \in U^i \times \mathbf{P}^1 \mid xv - (y - \alpha_{j_1})u = 0\},$$

the blowup $\pi_{j_1} : X_{j_1} \to U^i$ centered at p_{j_1}, and the exceptional curve $E_{j_1} := \pi_{j_1}^{-1}(p_{j_1})$ analogous to the definitions for X, π, and E. Moreover, by pasting the chart U^i which contains p_{j_1} on the other charts of X, one can obtain the blowup $\pi_{j_1} : X_{j_1} \to X$. By repeating this process inductively, we can obtain the following theorem (see Figure 1).

THEOREM 2.2

Assume that a rational mapping F with the indeterminate point p satisfies the condition (A.0). Then, for every $n \in \mathbf{N}$, $j_n = 1,2$, the following claims hold.

(1.1) *Define the composition $F_{j_1} := \pi^{-1} \circ \tilde{F} : N_{j_1} \to X$. Then, the point p_{j_1} is an indeterminate point of F_{j_1}.*

(1.2) *Let us define the mapping*

$$\tilde{F}_{j_1} := F_{j_1} \circ \pi_{j_1} : \pi_{j_1}^{-1}(N_{j_1}) \to X,$$

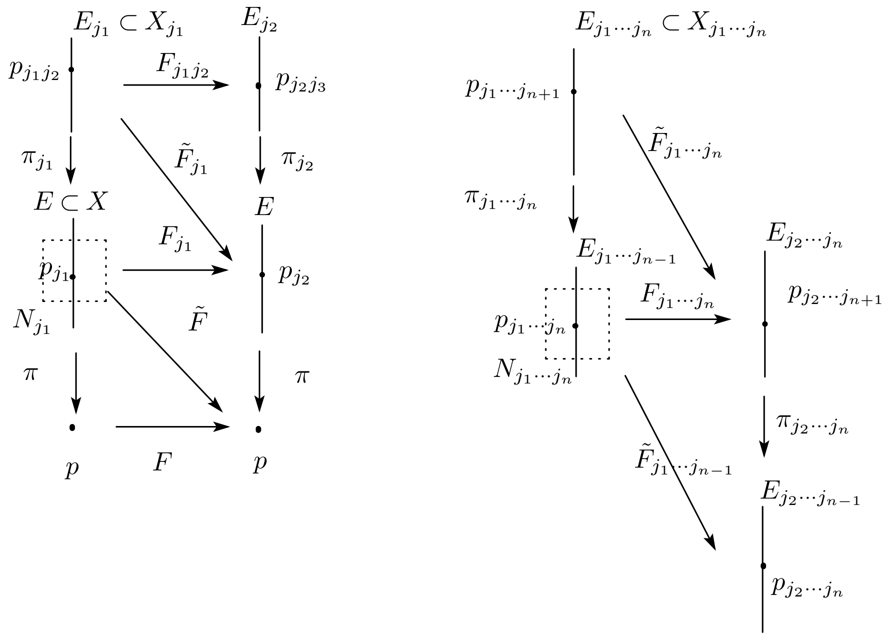

Figure 1

where $\pi_{j_1} : X_{j_1} \to X$ *is the blowup of* X *centered at* p_{j_1}. *Then, the exceptional curve* $E_{j_1} \subset \pi_{j_1}^{-1}(N_{j_1})$.

(1.3) *It is seen that* $\tilde{F}_{j_1}|_{E_{j_1}} : E_{j_1} \to E$ *is bijective. Hence, we can put* $p_{j_1 j_2} := \tilde{F}_{j_1}^{-1}(p_{j_2}) \in E_{j_1}$.

(1.4) *There exists an open neighborhood* $N_{j_1 j_2}$ *of* $p_{j_1 j_2}$ *such that* $\tilde{F}_{j_1}|_{N_{j_1 j_2}}$ *is biholomorphic.*

(2.1) *Define the composition* $F_{j_1 j_2} := \pi_{j_2}^{-1} \circ \tilde{F}_{j_1} : N_{j_1 j_2} \to X_{j_2}$. *Then, the point* $p_{j_1 j_2}$ *is an indeterminate point of* $F_{j_1 j_2}$.

(2.2) *Let us define the mapping*

$$\tilde{F}_{j_1 j_2} := F_{j_1 j_2} \circ \pi_{j_1 j_2} : \pi_{j_1 j_2}^{-1}(N_{j_1 j_2}) \to X_{j_2},$$

where $\pi_{j_1 j_2} : X_{j_1 j_2} \to X_{j_1}$ *is the blowup of* X_{j_1} *centered at* $p_{j_1 j_2}$. *Then, the exceptional curve* $E_{j_1 j_2} \subset \pi_{j_1 j_2}^{-1}(N_{j_1 j_2})$.

(2.3) *It is seen that* $\tilde{F}_{j_1 j_2}|_{E_{j_1 j_2}} : E_{j_1 j_2} \to E_{j_2}$ *is bijective. Hence, we can put* $p_{j_1 j_2 j_3} := \tilde{F}_{j_1 j_2}^{-1}(p_{j_2 j_3}) \in E_{j_1 j_2}$.

(2.4) *There exists an open neighborhood* $N_{j_1 j_2 j_3}$ *of* $p_{j_1 j_2 j_3}$ *such that* $\tilde{F}_{j_1 j_2}|_{N_{j_1 j_2 j_3}}$ *is biholomorphic.*

We can repeat this process inductively and define the following:

$$\text{the point } p_{j_1 \cdots j_n} := \tilde{F}_{j_1 \cdots j_{n-1}}^{-1}(p_{j_2 \cdots j_n}) \in E_{j_1 \cdots j_{n-1}},$$

$$F_{j_1 \cdots j_n} := \pi_{j_2 \cdots j_n}^{-1} \circ \tilde{F}_{j_1 \cdots j_{n-1}} : N_{j_1 \cdots j_n} \to X_{j_2 \cdots j_n},$$

$$\text{the blowup } \pi_{j_1 \cdots j_n} : X_{j_1 \cdots j_n} \to X_{j_1 \cdots j_{n-1}} \text{ centered at } p_{j_1 \cdots j_n},$$

and the mapping $\tilde{F}_{j_1\cdots j_n} := F_{j_1\cdots j_n} \circ \pi_{j_1\cdots j_n} : \pi^{-1}_{j_1\cdots j_n}(N_{j_1\cdots j_n}) \to X_{j_2\cdots j_n}$,

where $E_{j_1\cdots j_{n-1}}$ *is the exceptional curve of* $X_{j_1\cdots j_{n-1}}$ *and* $N_{j_1\cdots j_n}$ *is an open neighborhood of* $p_{j_1\cdots j_n}$ *such that* $\tilde{F}_{j_1\cdots j_{n-1}}|_{N_{j_1\cdots j_n}}$ *is biholomorphic. Then, the following claims hold.*

(1) *The point* $p_{j_1\cdots j_n}$ *is an indeterminate point of* $F_{j_1\cdots j_n}$.

(2) *The exceptional curve* $E_{j_1\cdots j_n} \subset \pi^{-1}_{j_1\cdots j_n}(N_{j_1\cdots j_n})$.

(3) *It is seen that* $\tilde{F}_{j_1\cdots j_n}|_{E_{j_1\cdots j_n}} : E_{j_1\cdots j_n} \to E_{j_2\cdots j_n}$ *is bijective. Hence, we can define the point*

$$p_{j_1\cdots j_{n+1}} := \tilde{F}^{-1}_{j_1\cdots j_n}(p_{j_2\cdots j_{n+1}}) \in E_{j_1\cdots j_n}.$$

(4) *There exists some open neighborhood* $N_{j_1\cdots j_{n+1}}$ *of* $p_{j_1\cdots j_{n+1}}$ *such that* $\tilde{F}_{j_1\cdots j_n}|_{N_{j_1\cdots j_{n+1}}}$ *is biholomorphic.*

We denote the local charts for the $X_{j_1\cdots j_n}$ by $U^i_{j_1\cdots j_n}$ $(i=1,2)$ defined similarly to U^i $(i=1,2)$ for X.

To state our Theorem 2.3, we need the following conditions:

$$\text{(A.1)}\qquad \begin{array}{c} p_{j_1} \in U^1 \cap E \qquad \text{and} \qquad p_{j_1\cdots j_{n+1}} \in U^1_{j_1\cdots j_n} \cap E_{j_1\cdots j_n} \\ \text{for any } n \in \mathbf{N}, j_n = 1,2. \end{array}$$

By using this chart, we can define $p_{j_1\cdots j_n} = (0, \alpha_{j_1\cdots j_n}) \in U^1_{j_1\cdots j_{n-1}}$. For every $n \in \mathbf{N}$ and $j_n = 1,2$, let us define the space of symbol sequences

$$\{1,2\}^{\mathbf{N}} := \big\{\mathbf{j} = (j_1, j_2, \ldots) \mid j_n = 1 \text{ or } 2\big\}.$$

For every $\mathbf{j} \in \{1,2\}^{\mathbf{N}}$, define formal power series

$$y = \phi_{\mathbf{j}}(x) := \alpha_{j_1}x + \alpha_{j_1 j_2}x^2 + \cdots$$

and

$$J := \big\{\mathbf{j} \in \{1,2\}^{\mathbf{N}} \mid \rho_{\mathbf{j}} > 0\big\},$$

where $\rho_{\mathbf{j}}$ is the radius of the domain of definition of $\phi_{\mathbf{j}}$. For all $\mathbf{j} \in J$, put a holomorphic mapping

$$\Phi_{\mathbf{j}} : \Delta_{\rho_{\mathbf{j}}} \to \mathbf{C}^2 \quad \text{by } t \mapsto \big(t, \phi_{\mathbf{j}}(t)\big) \qquad \text{and} \qquad W_{\mathbf{j}} := \Phi_{\mathbf{j}}(\Delta_{\rho_{\mathbf{j}}}).$$

Let us define $\sigma : \{1,2\}^{\mathbf{N}} \to \{1,2\}^{\mathbf{N}}$ to be the left shift mapping $\sigma(j_1, j_2, \ldots) = (j_2, j_3, \ldots)$. Then, we have the following theorem.

THEOREM 2.3

Assume that rational mapping F *with indeterminate point* p *satisfies conditions (A.0) and (A.1). Then, the following hold.*

(1) *The family* $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ *of germs of holomorphic curve at the point* p *is invariant and the maximal of such a family. Here, to say* $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ *is maximal means that every family* $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ *of germs of holomorphic curve at the point* p

which is invariant satisfies the following:

$$\text{for any } \lambda \in \Lambda, \text{ there exists a unique sequence } \mathbf{j} \in J \text{ such that } \tilde{V}_\lambda = \tilde{W}_{\mathbf{j}}$$

as a germ.

(2) *There exists an injective mapping* $\Psi : \{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J} \ni \tilde{W}_{\mathbf{j}} \mapsto \mathbf{j} \in \{1,2\}^{\mathbf{N}}$ *such that* $\Psi \circ F = \sigma \circ \Psi$.

3. Proof of Theorems 2.2 and 2.3

Proof of Theorem 2.2

For polynomials $p(x,y), q(x,y)$, let us denote by

$$O\big(p(x,y), q(x,y)\big) := \sum_{\substack{i+j\geq 2\\ i,j\geq 0}} \beta_{ij} p(x,y)^i q(x,y)^j$$

some formal power series of $p(x,y)$ and $q(x,y)$, where $\beta_{ij} \in \mathbf{C}$. Without loss of generality, we may assume that $p_{j_1} \in U^1 \cap E$ and identify it with $p_{j_1} := (0, \alpha_{j_1})$ by using the chart of U^1. From (A.0), $\tilde{F}$ has the following Taylor expansion on some open neighborhood N_{j_1} of p_{j_1}:

$$\begin{aligned}\tilde{F}(x,\eta) = \big(&a_{10}x + a_{01}(\eta - \alpha_{j_1}) + O(x, \eta - \alpha_{j_1}),\\ &b_{10}x + b_{01}(\eta - \alpha_{j_1}) + O(x, \eta - \alpha_{j_1})\big).\end{aligned}$$

Set the right-hand side of the above to $(f(x,\eta), g(x,\eta))$. Since $\tilde{F}$ is biholomorphic at p_{j_1}, $J\tilde{F}(0,\alpha_{j_1}) = a_{10}b_{01} - a_{01}b_{10} \neq 0$, where $J\tilde{F}(0,\alpha_{j_1})$ is the Jacobian determinant of $\tilde{F} : X \to \mathbf{C}^2$ at $(0,\alpha_{j_1})$. For $(x,\eta) \in N_{j_1} \cap U^1$, F_{j_1} has the form

(i) $$F_{j_1}(x,\eta) := \pi^{-1} \circ \tilde{F}(x,\eta) = \big(f(x,\eta), g(x,\eta)\big) \times [f(x,\eta) : g(x,\eta)].$$

In order to prove the assertion (1.1), we assume, to the contrary, that $p_{j_1} = (0, \alpha_{j_1})$ is not an indeterminate point of F_{j_1}. Then, the convergent power series $f(x,\eta)$ and $g(x,\eta)$ have a common factor $h(x,\eta)$ such that

$$\big\{(x,\eta) \in N_{j_1} \mid h(x,\eta) = 0\big\} \subset \tilde{F}^{-1}(p).$$

This contradicts the fact that $\tilde{F}^{-1}(p) = \{p_1, p_2\}$.

It is easy to see (1.2) from the definition of the blowup.

By (C.2) and (i), $\tilde{F}_{j_1}$ has the following form on $\pi_{j_1}^{-1}(N_{j_1}) \cap U^1_{j_1}$:

$$\begin{aligned}\tilde{F}_{j_1}(x,\eta) &:= F_{j_1} \circ \pi_{j_1}(x,\eta) = F_{j_1}(x, x\eta + \alpha_{j_1})\\ &= \big(f(x, x\eta + \alpha_{j_1}), g(x, x\eta + \alpha_{j_1})\big)\\ &\quad \times [a_{10} + a_{01}\eta + \tilde{O}(x, x\eta) : b_{10} + b_{01}\eta + \tilde{O}(x, x\eta)],\end{aligned}$$

where $\tilde{O}(x, x\eta) = O(x, x\eta)/x$, and we note here that $\tilde{O}(x, x\eta)$ is a convergent power series. Then, it follows from

$$a_{10}b_{01} - a_{01}b_{10} \neq 0 \qquad \text{and} \qquad \tilde{F}_{j_1}(0,\eta) = (0,0) \times [a_{10} + a_{01}\eta : b_{10} + b_{01}\eta]$$

that $\tilde{F}_{j_1}$ is injective on $U^1_{j_1} \cap E_{j_1} = \{(x,\eta) \in U^1_{j_1} \mid x = 0\}$. Similarly, by (C.2) and (i), we see that $\tilde{F}_{j_1}$ has the following form on $\pi^{-1}_{j_1}(N_{j_1}) \cap U^2_{j_1}$:

$$\begin{aligned}\tilde{F}_{j_1}(\xi, y) &= F_{j_1}\big(\xi(y-\alpha_{j_1}), y\big)\\ &= \big(f(\xi(y-\alpha_{j_1}), y), g(\xi(y-\alpha_{j_1}), y)\big)\\ &\quad \times \big[a_{10}\xi + a_{01} + \tilde{O}\big(\xi(y-\alpha_{j_1}), y-\alpha_{j_1}\big) :\\ &\qquad b_{10}\xi + b_{01} + \tilde{O}\big(\xi(y-\alpha_{j_1}), y-\alpha_{j_1}\big)\big],\end{aligned}$$

where $\tilde{O}(\xi(y-\alpha_{j_1}), y-\alpha_{j_1}) = O(\xi(y-\alpha_{j_1}), y-\alpha_{j_1})/(y-\alpha_{j_1})$ and it is a convergent power series. It implies that

$$\tilde{F}_{j_1}(0, \alpha_{j_1}) = (0,0) \times [a_{01} : b_{01}].$$

On the other hand, by Proposition 2.1(3), one can see that $E_{j_1} \cap (U^2_{j_1} \setminus U^1_{j_1}) = (0, \alpha_{j_1})$ and $\tilde{F}_{j_1}|_{E_{j_1}} : E_{j_1} \to E$ is bijective; this implies (1.3).

Since F_{j_1} is biholomorphic on $N_{j_1} \setminus \{p_{j_1}\}$ and π_{j_1} is biholomorphic on $\pi^{-1}_{j_1}(N_{j_1}) \setminus E_{j_1}$, $\tilde{F}_{j_1}$ is biholomorphic on $\pi^{-1}_{j_1}(N_{j_1}) \setminus E_{j_1}$. Together with (1.3), $\tilde{F}_{j_1}$ is biholomorphic on $\pi^{-1}_{j_1}(N_{j_1})$, and this shows (1.4). By repeating this process inductively, the proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3

First, we want to show that $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j} \in J}$ is a maximal family. For this purpose, fix any family $\{\tilde{V}_\lambda\}_{\lambda\in\Lambda}$ of germs of holomorphic curve at the point p which is invariant. Take a representative element $V_\lambda \in \tilde{V}_\lambda$. By Definition 2.2, for every V_λ there exists a holomorphic mapping

$$\Phi_\lambda : \Delta_{\rho_\lambda} \ni t \mapsto \big(t, \phi_\lambda(t)\big) \in \mathbf{C}^2$$

such that $V_\lambda = \Phi_\lambda(\Delta_{\rho_\lambda})$. Denote the Taylor expansion of $\phi_\lambda(t)$ at $t = 0$ by $\phi_\lambda(t) := c_1 t + c_2 t^2 + \cdots$. For V_λ, the following lemma holds.

LEMMA 3.1

(1) *For every V_λ, there exist an open neighborhood N_λ of p and a point $p_{j_1} \in \{p_1, p_2\} = \tilde{F}^{-1}(p)$ such that $\{p_{j_1}\} = \overline{\pi^{-1}(V_\lambda \cap N_\lambda \setminus \{p\})} \cap E$, where the closure is taken with respect to the relative topology of $\pi^{-1}(N_\lambda)$. Put*

$$V_{\lambda j_1} := \overline{\pi^{-1}(V_\lambda \cap N_\lambda \setminus \{p\})}.$$

(2) *There exists a holomorphic function $\phi_{\lambda j_1}$ on Δ_{ρ_λ} satisfying the following condition. Define a holomorphic mapping $\Phi_{\lambda j_1} : \Delta_{\rho_\lambda} \to U^1$ by $t \mapsto (t, \phi_{\lambda j_1}(t))$. Then, $V_{\lambda j_1} \sim \Phi_{\lambda j_1}(\Delta_{\rho_\lambda})$ and $c_1 = \alpha_{j_1}$.*

Proof

It follows from the definition of the blowup π that for $x \in \Delta_{\rho_\lambda}$,

$$\begin{aligned}\pi^{-1}(V_\lambda \cap N_\lambda) \cap U^1 &= \big\{(x,\eta) \in U^1 \mid x\eta = c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots\big\}\\ &= \big\{(x,\eta) \in U^1 \mid x = 0\big\}\end{aligned}$$

$$\cup \{(x,\eta) \in U^1 \mid \eta = c_1 + c_2 x + \cdots + c_n x^{n-1} + \cdots\},$$

$$\pi^{-1}(V_\lambda \cap N_\lambda \setminus \{p\}) \cap U^1 = \{(x,\eta) \in U^1 \mid \eta = c_1 + c_2 x + \cdots\} \setminus \{(0,c_1)\}.$$

Hence, we obtain the result that for $x \in \Delta_{\rho_\lambda}$,

$$\overline{\pi^{-1}(V_\lambda \cap N_\lambda \setminus \{p\}) \cap U^1} = \{(x,\eta) \in U^1 \mid \eta = c_1 + c_2 x + \cdots + c_n x^{n-1} + \cdots\}.$$

Put $\phi_{\lambda j_1}(t) := c_1 + c_2 t + \cdots + c_n t^{n-1} + \cdots$. It is clear that the radius of the domain of definition of $\phi_{\lambda j_1}$ is ρ_λ, too. Set $\tilde{p} := (0, c_1)$.

To complete the proof of Lemma 3.1, we need to show that $\tilde{p} \in \{p_1, p_2\} = \tilde{F}^{-1}(p)$. Since $\tilde{V}_\lambda$ is a germ of a holomorphic curve at the point p which is in an invariant family of germs, there exists some sequence of points $p_n \in V_\lambda$ such that $p_n \neq p$, $p_n \to p$, and $F(p_n) \to p$ as $n \to \infty$. Put $\tilde{p}_n := \pi^{-1}(p_n)$. Then, $\tilde{p}_n \in \pi^{-1}(V_\lambda \cap N_\lambda \setminus \{p\})$ and $\tilde{p}_n \to \tilde{p} = (0, c_1) \in E$ as $n \to \infty$. From the continuity of $\tilde{F}$, it follows that $\tilde{F}(\tilde{p}) = p$. By (2) of (A.0), $\tilde{F}^{-1}(p) \cap E = \{p_1, p_2\}$ and $\tilde{p} \in \{p_1, p_2\}$. □

From Definition 2.3, for every $\lambda \in \Lambda$ there exist $\lambda' \in \Lambda$ and a positive constant $\rho_{\lambda'}$ with $0 < \rho_{\lambda'} \le \rho_\lambda$ such that $F \circ \Phi_\lambda(\Delta_{\rho_{\lambda'}}) \in \tilde{V}_{\lambda'}$. Take a representative element $V_{\lambda'} \in \tilde{V}_{\lambda'}$. By Lemma 3.1(1), for $V_{\lambda'}$ there exist $p_{i_1} \in \{p_1, p_2\}$ and $V_{\lambda' i_1}$ such that $p_{i_1} \in V_{\lambda' i_1} \cap E$. Then, we obtain the following Lemma 3.2 (see Figure 2).

LEMMA 3.2
We have $F_{j_1} \circ \Phi_{\lambda j_1}(0) = p_{i_1}$ *and* $F_{j_1} \circ \Phi_{\lambda j_1}(\Delta_{\rho_{\lambda'}}) \in \tilde{V}_{\lambda' i_1}$, *where* $\tilde{V}_{\lambda' i_1}$ *is the germ of the holomorphic curve* $V_{\lambda' i_1}$ *at* p_{i_1}.

Proof
Since F_{j_1} is biholomorphic on $N_{j_1} \setminus \{p_{j_1}\}$, there exists an open neighborhood N_{i_1} of p_{i_1} such that

$$F_{j_1}(V_{\lambda j_1} \setminus \{p_{j_1}\}) \cap N_{i_1} \subset \pi^{-1}(V_{\lambda'} \setminus \{p\}),$$

$$\overline{F_{j_1} \circ \Phi_{\lambda j_1}(\Delta_{\rho_{\lambda'}} \setminus \{0\})} \cap N_{i_1} \subset \overline{\pi^{-1}(V_{\lambda'} \setminus \{p\})} = V_{\lambda' i_1}.$$

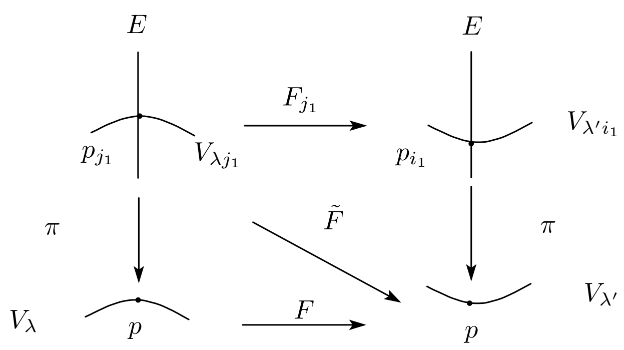

Figure 2

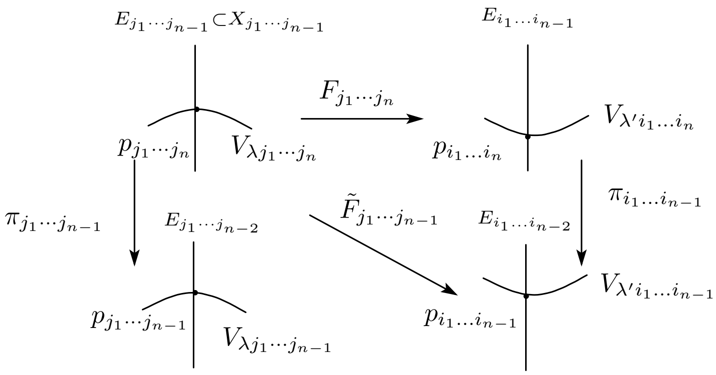

Figure 3

Moreover, by Remark 1, $F_{j_1} \circ \Phi_{\lambda j_1}$ is well defined on Δ_{ρ_λ} and

$$F_{j_1} \circ \Phi_{\lambda j_1}(\Delta_{\rho_{\lambda'}}) \cap N_{i_1} \subset V_{\lambda' i_1} \qquad \text{and} \qquad F_{j_1} \circ \Phi_{\lambda_{j_1}}(\Delta_{\rho_{\lambda'}}) \sim V_{\lambda' i_1}. \qquad \square$$

Inductively, for any $n \in \mathbf{N}$ and $j_n = 1, 2$, we can define curves

$$V_{\lambda j_1\cdots j_n} := \overline{(\pi \circ \cdots \circ \pi_{j_1\cdots j_{n-1}})^{-1}(V_\lambda \cap N_\lambda \setminus \{p\})} \quad \text{in } X_{j_1\cdots j_{n-1}}$$

and have the following Lemmas 3.3 and 3.4 (see Figure 3). Since the lemmas are proved by arguments similar to those for the proofs of Lemmas 3.1 and 3.2, we omit the proofs.

LEMMA 3.3

(1) *For every* $V_{\lambda j_1\cdots j_{n-1}}$*, there exist an open neighborhood* $N_{\lambda j_1\cdots j_{n-1}}$ *of* $p_{j_1\cdots j_{n-1}}$ *and a point* $p_{j_1\cdots j_n} \in \tilde{F}^{-1}_{j_1\cdots j_{n-1}}(p_{i_1\ldots i_{n-1}})$ *such that*

$$\{p_{j_1\cdots j_n}\} = \overline{\pi^{-1}_{j_1\cdots j_{n-1}}(V_{\lambda j_1\cdots j_{n-1}} \cap N_{\lambda j_1\cdots j_{n-1}} \setminus \{p_{j_1\cdots j_{n-1}}\})} \cap E_{j_1\cdots j_{n-1}}.$$

(2) *There exists a holomorphic function* $\phi_{\lambda j_1\cdots j_n}$ *on* Δ_{ρ_λ} *which has the following Taylor expansion at* $t = 0$*:*

$$\phi_{\lambda j_1\ldots j_n}(t) = c_n + c_{n+1}t + c_{n+2}t^2 + \cdots .$$

Define a holomorphic mapping $\Phi_{\lambda j_1\ldots j_n} : \Delta_{\rho_\lambda} \to U^1_{j_1\cdots j_{n-1}}$ *by* $t \mapsto (t, \phi_{\lambda j_1\cdots j_n}(t))$*. Then,* $V_{\lambda j_1\cdots j_n} \sim \Phi_{\lambda j_1\ldots j_n}(\Delta_{\rho_\lambda})$ *and* $c_n = \alpha_{j_1\ldots j_n}$*.*

From Definition 2.3, for every $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $F \circ \Phi_\lambda(\Delta_{\rho_{\lambda'}}) \in \tilde{V}_{\lambda'}$ as a germ. Take a representative element $V_{\lambda'} \in \tilde{V}_{\lambda'}$. By Lemma 3.3(1), there exist $p_{i_1\ldots i_n} \in V_{\lambda' i_1\ldots i_n} \cap E_{i_1\ldots i_{n-1}}$ and $V_{\lambda' i_1\ldots i_n}$. Then, we obtain the following.

LEMMA 3.4

We have

$$F_{j_1\cdots j_n} \circ \Phi_{\lambda j_1\cdots j_n}(0) = p_{i_1\ldots i_n}$$

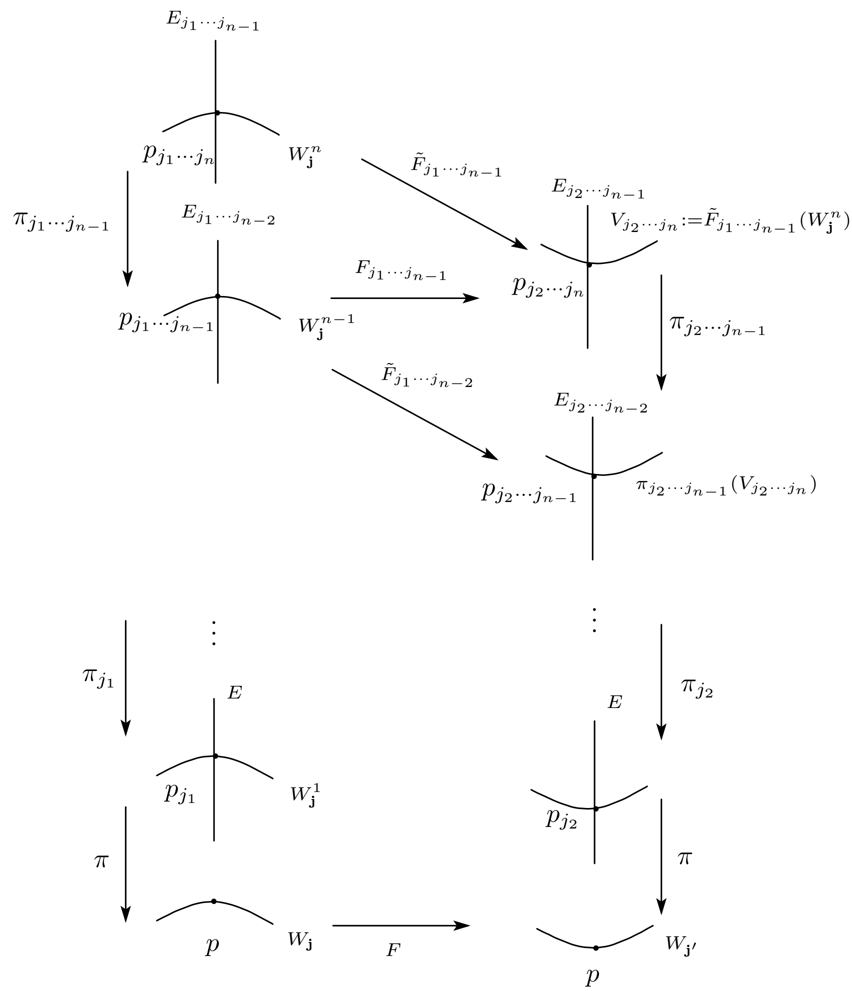

Figure 4

and

$$F_{j_1\cdots j_n} \circ \Phi_{\lambda j_1\cdots j_n}(\Delta_{\rho_{\lambda'}}) \in \tilde{V}_{\lambda' i_1\ldots i_n},$$

where $\tilde{V}_{\lambda' i_1\ldots i_n}$ *is the germ of the holomorphic curve* $V_{\lambda' i_1\ldots i_n}$.

From Lemma 3.3(2), $c_n = \alpha_{j_1\cdots j_n}$ for all n. Therefore, for every V_λ there exists a unique $\mathbf{j} \in J$ such that $V_\lambda \in \tilde{W}_{\mathbf{j}}$. Hence, we conclude that $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ is maximal.

To complete the proof of Theorem 2.3, we need to prove that $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$ is invariant. Select and fix a representative element $W_{\mathbf{j}} \in \tilde{W}_{\mathbf{j}}$. For the proof, it is enough to show that for every $\mathbf{j} \in J$, there exist $\mathbf{j}' \in J$ and a positive constant $\rho_{\mathbf{j}'}$ with $0 < \rho_{\mathbf{j}'} \le \rho_{\mathbf{j}}$ such that $F \circ \Phi_{\mathbf{j}}(\Delta_{\rho_{\mathbf{j}'}}) \in \tilde{W}_{\mathbf{j}'}$ (see Figure 4).

Inductively, let us set, for any $n \in \mathbf{N}$,

$$W^1_{\mathbf{j}} := \overline{\pi^{-1}(W_{\mathbf{j}}) \setminus E}, \qquad W^2_{\mathbf{j}} := \overline{(\pi \circ \pi_{j_1})^{-1}(W_{\mathbf{j}}) \setminus E_{j_1}}, \ldots$$
$$W^n_{\mathbf{j}} := \overline{(\pi \circ \pi_{j_1} \circ \cdots \circ \pi_{j_1 \cdots j_{n-1}})^{-1}(W_{\mathbf{j}}) \setminus E_{j_1 \cdots j_{n-1}}}.$$

By the same argument as in the proofs of Lemmas 3.1 and 3.2, one may obtain the following.

LEMMA 3.5

For every $n \geq 1$,

(1) $W^n_{\mathbf{j}} \cap E_{j_1 \cdots j_{n-1}} = \{p_{j_1 \cdots j_n}\}$;

(2) *Define a holomorphic function $\phi^n_{\mathbf{j}}$ on $\Delta_{\rho_{\mathbf{j}}}$ by $t \mapsto \alpha_{j_1 \cdots j_n} + \alpha_{j_1 \cdots j_{n+1}} t + \cdots$ and a holomorphic mapping*

$$\Phi^n_{\mathbf{j}} : \Delta_{\rho_{\mathbf{j}}} \to U^1_{j_1 \cdots j_{n-1}} \quad \text{by } t \mapsto \big(t, \phi^n_{\mathbf{j}}(t)\big).$$

Then $W^n_{\mathbf{j}} \sim \Phi^n_{\mathbf{j}}(\Delta_{\rho_{\mathbf{j}}})$.

By Theorem 2.2,

$$\tilde{F}_{j_1 \cdots j_{n-1}}(p_{j_1 \cdots j_n}) = p_{j_2 \cdots j_n} \in \tilde{F}_{j_1 \cdots j_{n-1}}(W^n_{\mathbf{j}}).$$

Using the finite symbol sequence $(j_2 \cdots j_n)$, we define the set

$$V_{j_2 \cdots j_n} := \tilde{F}_{j_1 \cdots j_{n-1}}(W^n_{\mathbf{j}}).$$

LEMMA 3.6

For all n, there exist a positive number $\rho_{j_2 \cdots j_n}$ and a holomorphic function $\psi_{j_2 \cdots j_n}$ on $\Delta_{\rho_{j_2 \cdots j_n}}$ satisfying the following condition. Define a holomorphic mapping $\Psi_{j_2 \cdots j_n} : \Delta_{\rho_{j_2 \cdots j_n}} \to U^1_{j_2 \cdots j_{n-1}}$ by $t \mapsto (t, \psi_{j_2 \cdots j_n}(t))$. Then,

$$V_{j_2 \cdots j_n} \sim \Psi_{j_2 \cdots j_n}(\Delta_{\rho_{j_2 \cdots j_n}}).$$

Proof

Here we use the same notation as in the proof of Theorem 2.2. Denote the Taylor expansion of $\tilde{F}_{j_1 \cdots j_{n-2}}$ at $p_{j_1 \cdots j_{n-1}} = (0, \alpha_{j_1 \cdots j_{n-1}})$ by

$$\tilde{F}_{j_1 \cdots j_{n-2}}(x, \eta) := \big(a_{10}x + a_{01}(\eta - \alpha_{j_1 \cdots j_{n-1}}) + \cdots, \\ \alpha_{j_2 \cdots j_{n-1}} + b_{10}x + b_{01}(\eta - \alpha_{j_1 \cdots j_{n-1}}) + \cdots\big).$$

Set the term of the above to $(\hat{f}(x,\eta), \hat{g}(x,\eta))$. Then,

$$F_{j_1 \cdots j_{n-1}} := \pi^{-1}_{j_2 \cdots j_{n-1}} \circ \tilde{F}_{j_1 \cdots j_{n-2}} = (\hat{f}, \hat{g}) \times [\hat{f} : \hat{g} - \alpha_{j_2 \cdots j_{n-1}}]$$

and

$$\begin{aligned} \tilde{F}_{j_1 \cdots j_{n-1}} &:= F_{j_1 \cdots j_{n-1}} \circ \pi_{j_1 \cdots j_{n-1}}(x, \eta) = F_{j_1 \cdots j_{n-1}}(x, x\eta + \alpha_{j_1 \cdots j_{n-1}}) \\ &= \big(\hat{f}(x, x\eta + \alpha_{j_1 \cdots j_{n-1}}), \hat{g}(x, x\eta + \alpha_{j_1 \cdots j_{n-1}})\big) \\ &\quad \times [a_{10} + a_{01}\eta + \cdots : b_{10} + b_{01}\eta + \cdots] \end{aligned}$$

for $(x,\eta) \in U^1_{j_1\cdots j_{n-1}}$. From the condition (A.1), $p_{j_1\cdots j_n} \in U^1_{j_1\cdots j_{n-1}}$ and $p_{j_2\cdots j_n} \in U^1_{j_2\cdots j_{n-1}}$. On the other hand,

$$\begin{aligned}\tilde{F}_{j_1\cdots j_{n-1}}(p_{j_1\cdots j_n}) &= \tilde{F}_{j_1\cdots j_{n-1}}(0,\alpha_{j_1\cdots j_n})\\ &= (0,0)\times[a_{10}+a_{01}\alpha_{j_1\cdots j_n} : b_{10}+b_{01}\alpha_{j_1\cdots j_n}]\\ &= p_{j_2\cdots j_n} \in U^1_{j_2\cdots j_{n-1}}.\end{aligned}$$

Hence, $a_{10}+a_{01}\alpha_{j_1\cdots j_n} \neq 0$.

By using the local chart, $\tilde{F}_{j_1\cdots j_{n-1}} : U^1_{j_1\cdots j_{n-1}} \to U^1_{j_2\cdots j_{n-1}}$ is written as

$$\tilde{F}_{j_1\cdots j_{n-1}}(x,\eta) = \Big(a_{10}x + a_{01}x\eta + O(x,x\eta), \frac{b_{10}+b_{01}\eta+\cdots}{a_{10}+a_{01}\eta+\cdots}\Big).$$

Set the right-hand side of the above to $(f(x,\eta), g(x,\eta)) := (u,v)$. By differentiating the holomorphic function $u = f(x,\phi^n_{\mathbf{j}}(x))$, with respect to the variable x and using the fact that $\phi^n_{\mathbf{j}}(0) = \alpha_{j_1\cdots j_n}$,

$$\frac{du}{dx}(0) = \frac{\partial f}{\partial x}(0,\alpha_{j_1\cdots j_n}) + \frac{\partial f}{\partial y}(0,\alpha_{j_1\cdots j_n})\frac{d\phi^n_{\mathbf{j}}}{dx}(0) = a_{10}+a_{01}\alpha_{j_1\cdots j_n} \neq 0.$$

Hence, the inverse function $x = \tilde{f}(u)$ of $u = f(x,\phi^n_{\mathbf{j}}(x))$ exists in some neighborhood of $u=0$. Therefore, there exists some positive constant $\rho_{j_2\cdots j_n}$ and some neighborhood $N_{j_2\cdots j_n}$ of $p_{j_2\cdots j_n}$ such that

$$\begin{aligned}&V_{j_2\cdots j_n} \cap N_{j_2\cdots j_n}\\ &\quad = \big\{(u,v) \in U^1_{j_2\cdots j_{n-1}} \mid v = g\big(\tilde{f}(u), \phi^n_{\mathbf{j}}(\tilde{f}(u))\big), u \in \Delta_{\rho_{j_2\cdots j_n}}\big\}.\end{aligned}$$

By setting

$$\psi_{j_2\cdots j_n}(x) := g\big(\tilde{f}(x), \phi^n_{\mathbf{j}}(\tilde{f}(x))\big),$$

the proof is completed. □

Put the Taylor expansions

$$\psi_{j_2\cdots j_n}(x) := \alpha_{j_2\cdots j_n} + \beta_1 x + \beta_2 x^2 + \cdots$$

and

$$\psi^n_{j_2\cdots j_n}(x) := \alpha_{j_2}x + \alpha_{j_2j_3}x^2 + \cdots + \alpha_{j_2\cdots j_n}x^{n-1} + \beta_1 x^n + \beta_2 x^{n+1} + \cdots$$

and the formal power series

$$\psi_{\sigma(\mathbf{j})} := \alpha_{j_2}x + \alpha_{j_2j_3}x^2 + \cdots, \quad \text{where } \sigma(\mathbf{j}) := (j_2, j_3, \ldots)$$

and the set

$$V_n := \pi \circ \pi_{j_2} \circ \cdots \circ \pi_{j_2\cdots j_{n-1}}(V_{j_2\cdots j_n} \cap N_{j_2\cdots j_n}).$$

LEMMA 3.7

(1) *For any $n \geq 2$, $\psi^n_{j_2\cdots j_n}$ is holomorphic on $\Delta_{\rho_{j_2\cdots j_n}}$. Define a holomorphic mapping $\Psi^n_{j_2\cdots j_n} : \Delta_{\rho_{j_2\cdots j_n}} \to \mathbf{C}^2$ by $t \mapsto (t, \psi^n_{j_2\cdots j_n}(t))$. Then,*

$$V_n \sim \Psi^n_{j_2\cdots j_n}(\Delta_{\rho_{j_2\cdots j_n}}).$$

(2) *There exists an open neighborhood* N_n *of* p *such that*

$$\overline{F(W_{\mathbf{j}} \setminus \{p\})} \cap N_n \subset \overline{V_n \setminus \{p\}}.$$

(3) *For any* $n, m \geq 2$, $V_n \sim V_m$. *In particular, there exists a positive constant* $\tilde{\rho}_{\sigma(\mathbf{j})}$ *such that* $0 < \tilde{\rho}_{\sigma(\mathbf{j})} \leq \rho_{j_2 \cdots j_n}$ *for any* $n \geq 2$ *and*

$$\psi_{\sigma(\mathbf{j})}(x) = \psi^n_{j_2 \cdots j_n}(x) \quad \textit{for } |x| < \tilde{\rho}_{\sigma(\mathbf{j})}.$$

(4) *Define a holomorphic mapping*

$$\Psi_{\sigma(\mathbf{j})} : \Delta_{\tilde{\rho}_{\sigma(\mathbf{j})}} \to \mathbf{C}^2 \quad \textit{by } t \mapsto \big(t, \psi_{\sigma(\mathbf{j})}(t)\big)$$

and a set

$$W_{\sigma(\mathbf{j})} := \Psi_{\sigma(\mathbf{j})}(\Delta_{\tilde{\rho}_{\sigma(\mathbf{j})}}).$$

Then, there exists an open neighborhood $N_{\sigma(\mathbf{j})}$ *of* p *such that*

$$\overline{F(W_{\mathbf{j}} \setminus \{p\})} \cap N_{\sigma(\mathbf{j})} \subset \overline{W_{\sigma(\mathbf{j})} \setminus \{p\}} = W_{\sigma(\mathbf{j})} \qquad \textit{and} \qquad W_{\sigma(\mathbf{j})} \sim \Psi_{\sigma(\mathbf{j})}(\Delta_{\tilde{\rho}_{\sigma(\mathbf{j})}}).$$

Proof
For every $(x, \eta) \in V_{j_2 \cdots j_n}$, define

$$\pi_{j_2 \cdots j_{n-1}}(x, \eta) = (x, x\eta + \alpha_{j_2 \cdots j_{n-1}}).$$

Set the right-hand side of the above to (X, Y). Then,

$$\eta = \psi_{j_2 \cdots j_n}(x), \quad x = X, \eta = \frac{Y - \alpha_{j_2 \cdots j_{n-1}}}{X}.$$

By eliminating x and η, we have the following equation:

$$Y = \alpha_{j_2 \cdots j_{n-1}} + \alpha_{j_2 \cdots j_n} X + \beta_0 X^2 + \cdots .$$

Repeating this process inductively, one can obtain the claim (1).

Since the blowup $\pi_{j_1 \cdots j_{n-1}} : X_{j_1 \cdots j_{n-1}} \setminus E_{j_1 \cdots j_{n-1}} \to X_{j_1 \cdots j_{n-2}} \setminus \{p_{j_1 \cdots j_{n-1}}\}$ is biholomorphic, there exists some open neighborhood N_n of p such that

$$F(W_{\mathbf{j}} \setminus \{p\}) \cap N_n \subset \pi \circ \pi_{j_2} \cdots \circ \pi_{j_2 \cdots j_{n-1}}$$
$$\big(\tilde{F}_{j_1 \cdots j_{n-1}} \circ (\pi \circ \pi_{j_1} \circ \cdots \circ \pi_{j_1 \cdots j_{n-1}})^{-1}(W_{\mathbf{j}} \setminus \{p\}) \cap N_{j_2 \cdots j_n}\big) = V_n \setminus \{p\}.$$

Then, one can see that claim (2) holds.

Together with the identity theorem, (3) and (4) follow immediately from (2). □

Using Lemma 3.7(4), we can complete the proof of (2) of Theorem 2.3. □

4. Example

In this section, as an application, consider the following rational map of $\mathbf{C}^2$:

$$F(x, y) = \Big(ax, \frac{y(y - x)}{x^2}\Big) \quad \text{with } |a| > 4.$$

We retain the notation from Section 3. It is easy to see that F satisfies conditions (A.0) and (A.1). Hence, Theorems 2.2 and 2.3 hold for F, and we have the family $\{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j} \in J}$ of germs of holomorphic curve at p which is invariant. It is remarked here that if $|a| < 1$, then there exists a Cantor bouquet of p in the sense of Yamagishi and $J = \{1,2\}^{\mathbf{N}}$.

Here, to say that there is a local unstable manifold $W^u_{\mathrm{loc}}(p)$ of p means that there exists an open neighborhood N_p of p such that

$$W^u_{\mathrm{loc}}(p) = \big\{(x,y) \in N_p \mid \text{ for all } n \geq 0, F^{-n}(x,y) \in N_p \text{ and } F^{-n}(x,y) \to p \text{ as } n \to \infty\big\} \cup \{p\}.$$

Define the set

$$J_0 := \big\{\mathbf{j} \in \{1,2\}^{\mathbf{N}} \mid n_0 \in \mathbf{N} \text{ with } j_{n_0} = 2 \text{ are finitely many}\big\}.$$

For our mapping, the following theorem holds.

THEOREM 4.1

For all symbol sequences $\mathbf{j} = (j_1, j_2, \ldots) \in \{1,2\}^{\mathbf{N}}$, *one of the following claims holds.*

(1) *If* $\mathbf{j} \in J_0$, *then there exists an integer* n_0 *such that* $j_n = 1$ *for any* $n \geq n_0$ *and* $\tilde{W}_{\mathbf{j}} \neq \emptyset$. *Take a representative element* $W_{\mathbf{j}} \in \tilde{W}_{\mathbf{j}}$. *Then,* $W_{\mathbf{j}} \subset F^{-n_0}(W_{11\ldots}) = F^{-n_0}(\{y = 0\})$, *and each* $W_{\mathbf{j}}$ *is a local unstable manifold of* p.

(2) *If* $\mathbf{j} \notin J_0$, *then* $\tilde{W}_{\mathbf{j}} = \emptyset$.

Hence, $J = J_0$.

Proof of Theorem 4.1

The remainder of this article is devoted to the proof of Theorem 4.1. For any $\mathbf{j} \in J$, there exists a positive constant $\rho_{\sigma(\mathbf{j})}$ with $0 < \rho_{\sigma(\mathbf{j})} \leq \rho_{\mathbf{j}}$ such that $F \circ \Phi_{\mathbf{j}}(\Delta_{\rho_{\sigma(\mathbf{j})}}) \in \tilde{W}_{\sigma(\mathbf{j})}$. Set $F(x,y) := (X,Y)$, and recall the notation from Section 3:

$$W_{\mathbf{j}} = \big\{(x,y) \in \mathbf{C}^2 \mid y = \phi_{\mathbf{j}}(x) = \alpha_{j_1} x + \alpha_{j_1 j_2} x^2 + \cdots, x \in \Delta_{\rho_{\mathbf{j}}}\big\}$$

and

$$W_{\sigma(\mathbf{j})} = \big\{(x,y) \in \mathbf{C}^2 \mid y = \phi_{\sigma(\mathbf{j})}(x) = \alpha_{j_2} x + \alpha_{j_2 j_3} x^2 + \cdots, x \in \Delta_{\rho_{\sigma(\mathbf{j})}}\big\}.$$

Then, we have

$$X = ax, \qquad Y = \frac{y(y-x)}{x^2}, y = \alpha_{j_1} x + \alpha_{j_1 j_2} x^2 + \cdots,$$
$$Y = \alpha_{j_2} X + \alpha_{j_2 j_3} X^2 + \cdots.$$

By eliminating y, X and Y, we have the equation

$$x^2(\alpha_{j_2} a x + \alpha_{j_2 j_3} a^2 x^2 + \cdots)$$
$$= (\alpha_{j_1} x + \alpha_{j_1 j_2} x^2 + \cdots)\big\{(\alpha_{j_1} - 1)x + \alpha_{j_1 j_2} x^2 + \cdots\big\}.$$

Therefore,

$$\alpha_{j_2} a x^3 + \alpha_{j_2 j_3} a^2 x^4 + \cdots = (\alpha_{j_1} x + \alpha_{j_1 j_2} x^2 + \cdots)^2 - (\alpha_{j_1} x^2 + \alpha_{j_1 j_2} x^3 + \cdots),$$

$$\sum_{n=2} \alpha_{j_2 \cdots j_n} a^{n-1} x^{n+1} = \sum_{n=2} \sum_{\substack{k+l=n \\ k,l \geq 1}} \alpha_{j_1 \cdots j_k} \alpha_{j_1 \cdots j_l} x^n - \sum_{n=1} \alpha_{j_1 \cdots j_n} x^{n+1},$$

$$\sum_{n=1} \alpha_{j_2\cdots j_{n+1}} a^n x^{n+2}$$
$$= (\alpha_{j_1}^2 - \alpha_{j_1})x^2 + \sum_{n=1} \sum_{\substack{k+l=n+2 \\ k,l\geq 1}} \alpha_{j_1\cdots j_k}\alpha_{j_1\cdots j_l} x^{n+2} - \sum_{n=1} \alpha_{j_1\cdots j_{n+1}} x^{n+2}.$$

Hence, we have the following:

$$\alpha_{j_1}^2 - \alpha_{j_1} = 0,$$
$$\alpha_{j_2\cdots j_{n+1}} a^n = \sum_{\substack{k+l=n+2 \\ k,l\geq 1}} \alpha_{j_1\cdots j_k}\alpha_{j_1\cdots j_l} - \alpha_{j_1\cdots j_{n+1}}$$
$$= \alpha_{j_1\cdots j_{n+1}}(2\alpha_{j_1} - 1) + \sum_{\substack{k+l=n+2 \\ k,l\geq 2}} \alpha_{j_1\cdots j_k}\alpha_{j_1\cdots j_l} \quad \text{for } n \geq 1.$$

As a result, we have the following recurrence system:

$$\alpha_1 = 0, \qquad \alpha_2 = 1,$$
$$\alpha_{j_1\cdots j_{n+1}} = \frac{1}{2\alpha_{j_1} - 1}\Big\{\alpha_{j_2\cdots j_{n+1}} a^n - \sum_{\substack{k+l=n+2 \\ k,l\geq 2}} \alpha_{j_1\cdots j_k}\alpha_{j_1\cdots j_l}\Big\} \quad \text{for } n \geq 1.$$

By a direct calculation, one can check that $\alpha_{11\ldots 1} = 0$,

$$W_{11\ldots} = \big\{(x,y) \in \mathbf{C}^2 \mid y = 0\big\},$$

and that $W_{11\ldots}$ is a local unstable manifold of p.

First, we show that the inverse image of $W_{11\ldots}$ with respect to F is the graph of some holomorphic function of x. To do this, we prove the following lemma. Denote the Taylor expansion of $\tilde{F}$ at $p_{j_1} = (0, \alpha_{j_1})$ by

$$\tilde{F}(x,y) = \big(a_{10}x + a_{01}(y - \alpha_{j_1}) + \cdots, b_{10}x + b_{01}(y - \alpha_{j_1}) + \cdots\big).$$

Set the right-hand side of the above to $(f(x,y), g(x,y))$.

LEMMA 4.2

For every $\tilde{W}_{\sigma(\mathbf{j})} \in \{\tilde{W}_{\mathbf{j}}\}_{\mathbf{j}\in J}$, take a representative element $W_{\sigma(\mathbf{j})} \in \tilde{W}_{\sigma(\mathbf{j})}$. If $b_{01} - \alpha_{j_2}a_{01} \neq 0$, then $F^{-1}(W_{\sigma(\mathbf{j})})$ is given by the graph of a holomorphic function of x.

Proof

From the definition,

$$\tilde{F}^{-1}(W_{\sigma(\mathbf{j})}) = \big\{(x,y) \in N_{j_1} \mid g(x,y) - \phi_{\sigma(\mathbf{j})}\big(f(x,y)\big) = 0\big\}.$$

Define $\Psi(x,y) := g(x,y) - \phi_{\sigma(\mathbf{j})}(f(x,y))$, and differentiate with respect to y. Then,

$$\frac{\partial \Psi}{\partial y}(x,y) = \frac{\partial g}{\partial y}(x,y) - \frac{d\phi_{\sigma(\mathbf{j})}}{dx}\big(f(x,y)\big)\frac{\partial f}{\partial y}(x,y),$$
$$\frac{\partial \Psi}{\partial y}(0, \alpha_{j_1}) = b_{01} - \alpha_{j_2}a_{01}.$$

It follows from the condition $b_{01} - \alpha_{j_2} a_{01} \neq 0$ that $\tilde{F}^{-1}(W_{\sigma(\mathbf{j})})$ is given by the graph of a holomorphic function of x on a neighborhood of $x = 0$. This result, together with the facts $\pi \circ \tilde{F}^{-1}(W_{\sigma(\mathbf{j})}) = F^{-1}(W_{\sigma(\mathbf{j})})$ and (C.2), proves the lemma. □

We remark here that $\tilde{F}$ has the following Taylor expansion:

$$\tilde{F}(x,y) = (ax, -y + y^2) \quad \text{on } N_1$$

and

$$\tilde{F}(x,y) = \big(ax, (y-1) + (y-1)^2\big) \quad \text{on } N_2,$$

where N_1 and N_2 are open neighborhoods of p_1 and p_2, respectively. Hence, $a_{01} = 0, b_{01} \neq 0$, and our F satisfies the condition of Lemma 4.2. By applying Lemma 4.2 for $W_{11\dots}$ repeatedly, we can prove the claim of Theorem 4.1(1).

To complete the proof of Theorem 4.1, we need the following lemma.

LEMMA 4.3

(1) *For every* $\mathbf{j} = (j_1, j_2, \dots) \in \{1,2\}^{\mathbf{N}}$, *there exist sequences* $\{M_n\}_{n \geq 2}$ *and* $\{M'_n\}_{n \geq 2}$ *of positive constants such that*

$$M_2 = 1, \qquad M'_2 = \frac{3}{2}, \qquad M_{n+1} = M_n M'_n, \quad 1 \leq M'_n \leq \frac{3}{2},$$
$$|\alpha_{j_1 \dots j_n}| \leq M_n |a|^{(n(n-1))/2} \quad \textit{for } n \geq 2.$$

(2) *For any finite sequence* $(j_1, \dots, j_{n_0}) \in \{1,2\}^{n_0}$ *with* $j_{n_0} = 2$, *there exist sequences* $\{m_n\}_{2 \leq n \leq n_0}$ *and* $\{m'_n\}_{2 \leq n \leq n_0}$ *of positive constants such that*

$$m_2 = 1, \qquad m'_2 = \frac{3}{4}, \qquad m_{n+1} = m_n m'_n, \quad \frac{1}{2} \leq m'_n \leq 1,$$
$$m_n |a|^{(n(n-1))/2} \leq |\alpha_{j_1 \dots j_n}| \quad \textit{for } 2 \leq n \leq n_0.$$

Proof

To prove this lemma, we proceed by induction on n. By a direct calculation, we have $\alpha_{11} = 0, \alpha_{12} = -a, \alpha_{21} = 0$, $\alpha_{22} = a$, and if $n = 2$, then (1) holds. Assume that claim (1) is proved for $n \geq 2$. We then have the following inequalities:

$$\begin{aligned} |\alpha_{j_1 \cdots j_{n+1}}| &\leq |\alpha_{j_2 \cdots j_{n+1}} a^n| + \sum_{\substack{k+l=n+2 \\ k,l \geq 2}} |\alpha_{j_1 \dots j_k} \alpha_{j_1 \cdots j_l}| \\ &\leq |a^n \alpha_{j_2 \cdots j_{n+1}}| + |\alpha_{j_1 j_2} \alpha_{j_1 \cdots j_n}| + |\alpha_{j_1 j_2 j_3} \alpha_{j_1 \cdots j_{n-1}}| + \cdots \\ &\quad + |\alpha_{j_1 \cdots j_n} \alpha_{j_1 j_2}| \\ &\leq |a|^n M_n |a|^{(n(n-1))/2} + M_2 M_n |a| |a|^{(n(n-1))/2} \\ &\quad + M_3 M_{n-1} |a|^3 |a|^{((n-1)(n-2))/2} + \cdots \\ &\quad + M_n M_2 |a|^{(n(n-1))/2} |a|. \end{aligned} \tag{ii}$$

Put $n_0 := [(n+2)/2]$. Since M_k is not decreasing,

$$\text{if } 2 \le k \le n_0, \text{ then } M_k \le M_{n_0} \text{ and if } n_0 \le k \le n, \text{ then } M_k \le M_n, \text{ and}$$

$$\text{(ii)} \le M_n\big\{|a|^{n+(n(n-1))/2} + M_{n_0}|a|^{1+(n(n-1))/2}$$

$$+ M_{n_0}|a|^{3+((n-1)(n-2))/2} + \cdots + M_{n_0}|a|^{1+(n(n-1))/2}\big\}. \tag{iii}$$

Set $\beta(k) := k(k-1)/2$, and for $2 \le k \le n_0$ and $n-k+2 \ge n_0$,

$$\gamma(k) := -\beta(k) - \beta(n-k+2) + \frac{n^2+n}{2} = (k-1)(n+1-k).$$

Then,

$$\text{(iii)} \le M_n\big\{|a|^{(n^2+n)/2} + M_{n_0}|a|^{\beta(2)+\beta(n)} + M_{n_0}|a|^{\beta(3)+\beta(n-1)} + \cdots$$

$$+ M_{n_0}|a|^{\beta(n)+\beta(2)}\big\}$$

$$= M_n|a|^{(n^2+n)/2}\Big\{1 + \frac{M_{n_0}}{|a|^{\gamma(2)}} + \frac{M_{n_0}}{|a|^{\gamma(3)}} + \cdots + \frac{M_{n_0}}{|a|^{\gamma(2)}}\Big\}. \tag{iv}$$

Moreover, $\gamma(k) - \gamma(2) = (n-k)(k-2) \ge 0$ for $n \ge k \ge 2$. It implies from $|a| \ge 4$ that

$$\frac{1}{|a|^{\gamma(2)}} \ge \frac{1}{|a|^{\gamma(k)}} \qquad \text{and}$$

$$\text{(iv)} \le M_n|a|^{(n^2+n)/2}\Big\{1 + \frac{M_{n_0}}{|a|^{\gamma(2)}}(n-1)\Big\}.$$

To complete the proof of (1), it is enough to show that for $n \ge 2$,

$$\frac{M_{n_0}}{|a|^{n-1}}(n-1) \le \frac{1}{2}.$$

Since

$$M_{n_0} \le M_{n_0-1}M'_{n_0-1} \le M_{n_0-1}\frac{3}{2} \le \cdots \le M_3\Big(\frac{3}{2}\Big)^{n_0-3} \le \Big(\frac{3}{2}\Big)^{(n-2)/2},$$

$$\frac{M_{n_0}}{|a|^{n-1}}(n-1) \le \frac{(3/2)^{(n-2)/2}(n-1)}{4^{n-1}} \le \frac{(5/4)^{n-2}(n-1)}{4^{n-1}} = \frac{1}{4}\Big(\frac{5}{16}\Big)^{n-2}(n-1).$$

Hence, we need to show, for $n \ge 2$,

$$\beta_n := \frac{1}{4}\Big(\frac{5}{16}\Big)^{n-2}(n-1) \le \frac{1}{2}.$$

By a direct calculation, one knows that $\beta_2 = 1/4$ and $\beta_3 = 5/32$. For $n \ge 3$,

$$\frac{\beta_{n+1}}{\beta_n} = \frac{5}{16}\Big(\frac{n}{n-1}\Big) = \frac{5}{16}\Big(1 + \frac{1}{n-1}\Big) \le \frac{5}{16} \times \frac{3}{2} = \frac{15}{32}.$$

Therefore, β_n is monotone decreasing. This completes the proof of (1).

Next, we prove (2). Put $m_2 := 1$. From the facts $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_{12} = -a$, $\alpha_{22} = a$, we obtain the claim for $n = 2$. For $n = 3$, it follows from a direct calculation that $\alpha_{112} = a^3, \alpha_{122} = -a^3 + a^2$, $\alpha_{212} = -a^3, \alpha_{222} = a^3 - a^2$, and

$$|\alpha_{122}| = |\alpha_{222}| = |a^3 - a^2| = |a|^3\Big(1 - \frac{1}{|a|}\Big) \ge \frac{3}{4}|a|^3.$$

Put $m'_2 = 3/4$; the claim follows. Assume that claim (2) is proved for $n \geq 3$. By using (1) of Lemma 4.3, we have the following inequalities:

$$\begin{aligned}
|\alpha_{j_1\cdots j_{n+1}}| &\geq |\alpha_{j_2\cdots j_{n+1}}a^n| - \sum_{\substack{k+l=n+2\\k,l\geq 2}} |\alpha_{j_1\cdots j_k}\alpha_{j_1\cdots j_l}| \\
&\geq m_n|a|^{(n(n-1))/2}|a|^n - |\alpha_{j_1j_2}\alpha_{j_1\cdots j_n}| - |\alpha_{j_1j_2j_3}\alpha_{j_1\cdots j_{n-1}}| - \cdots \\
&\quad - |\alpha_{j_1\cdots j_n}\alpha_{j_1j_2}| \\
&\geq m_n|a|^{(n(n-1))/2+n} - M_2M_n|a|^{1+(n(n-1))/2} \\
&\quad - M_3M_{n-1}|a|^{3+((n-1)(n-2))/2} - \cdots - M_nM_2|a|^{(n(n-1))/2+1} \\
\text{(v)}\qquad &= m_n|a|^{(n(n+1))/2}\Big\{1 - \frac{M_2M_n}{m_n|a|^{\gamma(2)}} - \frac{M_3M_{n-1}}{m_n|a|^{\gamma(3)}} - \cdots - \frac{M_nM_2}{m_n|a|^{\gamma(2)}}\Big\}.
\end{aligned}$$

From the facts

$$|a| \geq 4, \qquad M_n \leq \Big(\frac{3}{2}\Big)^{n-2}, \qquad m_n \geq \Big(\frac{1}{2}\Big)^{n-2} \qquad \text{and}$$

$$\gamma(k) \geq \gamma(2) \quad \text{for } n \geq k \geq 2,$$

it follows that

$$\begin{aligned}
\text{(v)} &\geq m_n|a|^{(n(n+1))/2}\Big\{1 - \frac{(3/2)^{n-2}}{(1/2)^{n-2}4^{n-1}} - \frac{(3/2)^{n-2}}{(1/2)^{n-2}4^{n-1}} - \cdots \\
&\quad - \frac{(3/2)^{n-2}}{(1/2)^{n-2}4^{n-1}}\Big\} \\
&= m_n|a|^{(n(n+1))/2}\Big\{1 - \frac{3^{n-2}}{4^{n-1}}(n-1)\Big\}.
\end{aligned}$$

Put $\delta_n := 3^{n-2}(n-1)/4^{n-1}$. To prove claim (2), we need to show $\delta_n \leq 1/2$. By a direct calculation, $\delta_3 = 3/8$ and $\delta_4 = 27/64$. For $n \geq 3$, we have

$$\frac{\delta_{n+1}}{\delta_n} = \frac{3}{4}\frac{n}{n-1} \leq \frac{3}{4}\Big(1 + \frac{1}{3}\Big) = 1.$$

Therefore, δ_n is monotone decreasing. This completes the proof of (2). □

By using Lemma 4.3, for every infinite sequence $\mathbf{j} = (j_1, j_2, \ldots) \in \{1,2\}^{\mathbf{N}}$ with infinitely many $j_{n_0} = 2$, we obtain the result that the radius of the domain of definition R of ϕ_j is equal to zero. Indeed, for $\alpha_{j_1\cdots j_{n_0}}$ with $j_{n_0} = 2$, from Lemma 4.3(2),

$$\Big(\frac{1}{2}\Big)^{n_0-2}|a|^{(n_0(n_0-1))/2} \leq |\alpha_{j_1\cdots j_{n_0}}|.$$

Hence,

$$\begin{aligned}
\frac{1}{R} &= \limsup_{k\to\infty, n\geq k} |\alpha_{j_1\ldots j_n}|^{1/n} \geq \lim_{k\to\infty, n_0\geq k}\Big\{\Big(\frac{1}{2}\Big)^{n_0-2}|a|^{(n_0(n_0-1))/2}\Big\}^{1/n_0} \\
&= \lim_{k\to\infty, n_0\geq k}\Big(\frac{1}{2}\Big)^{1-(2/n_0)}|a|^{(n_0-1)/2} = \infty.
\end{aligned}$$

The proof of Theorem 4.1 is now complete. □

REMARK 3

Suppose that $|a| < 1$ in our map F; then there exist a Cantor bouquet of p in the sense of Yamagishi and $J = \{1,2\}^{\mathbf{N}}$. In particular, $\phi_{\mathbf{j}}$ gives the Taylor expansion of the function which defines the stable manifold $W_{\mathbf{j}}$ of indeterminate point p. On the other hand, if $|a| > 4$, then the set J_0 is countable. Therefore, it may not be appropriate to call $\{W_{\mathbf{j}}\}_{\mathbf{j}\in J}$ a generalized Cantor bouquet. In [6], we construct another family, $\{W_{\mathbf{j}}\}_{\mathbf{j}\in\{1,2\}^{\mathbf{N}}}$, of curves which consist of a center manifold of an indeterminate point p. In the article, we show that $\phi_{\mathbf{j}}$ does not necessarily have a positive radius of the domain of definition, but $\phi_{\mathbf{j}}$ gives the asymptotic expansion of the function which defines $W_{\mathbf{j}}$.

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