

A Gauss-Bonnet-type formula on Riemann-Finsler surfaces with nonconstant indicatrix volume

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In memory of Professor Makoto Matsumoto

Abstract We prove a Gauss-Bonnet-type formula for Riemann-Finsler surfaces of non-constant indicatrix volume and with regular piecewise C^∞ -boundary. We give a Hadamard-type theorem for N -parallels of a Landsberg surface.

1. Introduction

A major topic in Riemannian geometry is the study of the relation between the curvature of the Riemannian metric and the topology of the manifold. This is mainly achieved through the well-known Gauss-Bonnet-Chern theorem. The theorem and its consequences are especially interesting in the case of Riemannian surfaces (see [SST] for a comprehensive exposition).

The Gauss-Bonnet theorem was extended for the first time by D. Bao and S. S. Chern to the case of boundaryless Finsler manifolds of Landsberg type and Finsler manifolds of constant volume (see for details [BC]). In the case of Landsberg surfaces the Gauss-Bonnet-Chern theorem is stated in a particular form that can be regarded as a direct generalization to the Finslerian case of the Riemannian classical result. In [SS] we have extended the Gauss-Bonnet-Chern theorem for boundaryless Landsberg surfaces to the case of Landsberg surfaces with smooth boundary.

The reason to restrict the considerations to Landsberg surfaces is that on these surfaces the Riemannian volume of the indicatrix is constant and therefore the Euler-Poincaré characteristic of the manifold can be related to the curvature in a similar way to the Riemannian case. However, the Landsberg structures include the Berwald ones (see [I], [BCS]), which, at least in the case of surfaces, are known to be locally Minkowski in the flat case, or Riemannian otherwise.

Recently, there are many suspicions about the existence of regular Landsberg structures that are not Berwald (see [Sz2], [Ma], [Sz3]), but the existence of such structures is still an open problem.

However, the Finsler structures more general than the Landsberg ones can have very interesting geometrical properties, and a Gauss-Bonnet-type formula might be a useful tool in the study of their geometry.

In [Sh1] are proved some Gauss-Bonnet-type formulas for $2n$ -dimensional Riemann-Finsler manifolds whose indicatrix volume is constant. The article also contains interesting information on different attempts to extend the Gauss-Bonnet theorem to the Finslerian setting (see also [BS] for several discussions on the constancy of the indicatrix volume).

On the other hand, M. Matsumoto studies a Gauss-Bonnet formula for bounded regions on a Finsler surface, but he uses a completely different approach than ours (see [M1]). Matsumoto's normals and curvatures have different geometrical meanings than the ones in this article. For his setting, S. S. Chern's transgression method used by us cannot be employed.

In this article we are concerned with the following question.

QUESTION

Does a Gauss-Bonnet-type formula hold in the case of Riemann-Finsler domains with regular piecewise C^∞ -boundary?

The lack of angles is a sort of peculiarity of traditional Finsler geometry. We show in the present study that the so-called *Landsberg angles* can be very useful in the study of the geometry near a "corner" of a regular piecewise C^∞ -curve.

The article is organized as follows. We recall some basic facts on the geometry of Riemann-Finsler manifolds in §2. We discuss here the Landsberg angles defined as the Riemannian length of the indicatrix curve arc defined by the tips of two unit vectors. In §3 we treat the normal lift of a curve to the indicatrix Σ which is different from the canonical lift of a curve usually used (see, e.g., [BCS, p. 112]). We are led in this way to the notion of N -parallels and N -parallel curvature of a curve γ on the surface M . The difference with the Finslerian geodesics is also discussed. An existence and unicity theorem for N -parallels is given in the appendix.

Theorem 4.2, proved in §4, gives a partial answer to the question above. We give here a topological lemma that allows us to relate the Euler characteristic of M with the Chern connection 1-form in the case when the indicatrix length is not constant, that is, in a more general case than the Landsberg structures.

The regular piecewise C^∞ -boundary case is discussed in §5, where we construct a variation curve near the given boundary. The Gauss lemma for Riemann-Finsler manifolds is the one that makes all the machinery work. Here is where we prove Theorem 5.1, which gives the final affirmative answer to the question above.

We finally show how the Gauss-Bonnet theorem controls the behavior of N -parallels by proving a Hadamard-type theorem in §6 for Landsberg surfaces.

2. The geometry of Riemann-Finsler surfaces

This chapter follows closely [BCS, Chapter 4].

A *Finsler norm*, or metric, on a real, smooth, 2-dimensional manifold M is a function $F : TM \rightarrow [0, \infty)$ that is positive and smooth away from the zero section, has the *homogeneity property* $F(x, \lambda v) = \lambda F(x, v)$ for all $\lambda > 0$ and all $v \in T_x M$, and has the *strong convexity property* such that the Hessian matrix

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}$$

is positive definite at every point of $\widetilde{TM} = TM \setminus \{0\}$.

This implies that the Finslerian unit sphere, or the *indicatrix*

$$(2.1) \quad \Sigma_x := \{x \in T_x M \mid F(x, v) = 1\} \subset T_x M$$

at $x \in M$, is a smooth, closed, strictly convex hypersurface in $T_x M$. In addition, if $F(x, -v) = F(x, v)$, then F is said to be reversible, or absolutely homogeneous (see [M2], [BCS], [Sh2] for the basics of Riemann-Finsler manifolds).

REMARK

We gave here the definition for the case of a surface M because in this article we deal only with surfaces, but the above definition can be easily extended to the arbitrary dimensional case.

A smooth 2-dimensional manifold endowed with a Finsler norm is called a *Finsler structure on the surface* M , or simply a *Finsler surface*.

In other words, a Finsler surface is a pair (M, F) , where $F : TM \rightarrow [0, \infty)$ is \mathbf{C}^∞ on $\widetilde{TM} := TM \setminus \{0\}$ and whose restriction to each tangent plane $T_x M$ is a Minkowski norm (see [SS] for a detailed discussion).

A Finsler structure (M, F) on a surface M is also equivalent to a smooth hypersurface (i.e., 3-dimensional submanifold) $\Sigma \subset TM$ for which the canonical projection $\pi : \Sigma \rightarrow M$ is a surjective submersion and having the property that for each $x \in M$, the π -fiber $\Sigma_x = \pi^{-1}(x)$ is a strictly convex smooth curve including the origin $O_x \in T_x M$.

Recall that in order to study the geometry of the surface (M, F) , one considers the pullback bundle $\pi^* TM$ with base manifold Σ and fibres $(T_x M)|_u$, where $u \in \Sigma$ such that $\pi(u) = x$ (see [BCS, Chapter 2]). In general, this is not a principal bundle.

Let us remark that if we denote the projection by $p : TM \rightarrow M$, then one can start with the pullback bundle $p^* TM$ constructed over the slit tangent bundle \widetilde{TM} . This is also a vector bundle whose fiber over a typical point $u = (x, y) \in \widetilde{TM}$ is a copy of $T_x M$, where $p(x, y) = x \in M$.

However, since the majority of our geometrical objects are sections of the pullback bundle $\pi^* TM$ with base manifold Σ , we prefer to use this one instead of $p^* TM$ over \widetilde{TM} .

We point out that we are in fact using the same theory as in [BCS], but we have switched the notation for $p : TM \rightarrow M$ to $\pi : \Sigma \rightarrow M$.

It is also known (see [BCS, p. 30]) that the vector bundle π^*TM has a distinguished global section $l := \frac{y^i}{F(y)} \frac{\partial}{\partial x^i}$.

Using this section, one can construct a positively oriented g -orthonormal frame $\{e_1, e_2\}$ for π^*TM , where $g = g_{ij}(x, y) dx^i \otimes dx^j$ is the induced Riemannian metric on the fibers of π^*TM . The frame $\{u; e_1, e_2\}$ for any $u \in \Sigma$ is a globally defined g -orthonormal frame field for π^*TM called the *Berwald frame*.

Locally, we have

$$e_1 := \frac{1}{\sqrt{g}} \left(\frac{\partial F}{\partial y^2} \frac{\partial}{\partial x^1} - \frac{\partial F}{\partial y^1} \frac{\partial}{\partial x^2} \right) = m^1 \frac{\partial}{\partial x^1} + m^2 \frac{\partial}{\partial x^2},$$

$$e_2 := \frac{y^1}{F} \frac{\partial}{\partial x^1} + \frac{y^2}{F} \frac{\partial}{\partial x^2} = l^1 \frac{\partial}{\partial x^1} + l^2 \frac{\partial}{\partial x^2},$$

where g is the determinant of the Hessian matrix g_{ij} .

The corresponding dual coframe is locally given by

$$\omega^1 = \frac{\sqrt{g}}{F} (y^2 dx^1 - y^1 dx^2) = m_1 dx^1 + m_2 dx^2,$$

$$\omega^2 = \frac{\partial F}{\partial y^1} dx^1 + \frac{\partial F}{\partial y^2} dx^2 = l_1 dx^1 + l_2 dx^2.$$

Next, one defines a moving coframing $(u; \omega^1, \omega^2, \omega^3)$ on π^*TM , orthonormal with respect to the Riemannian metric on Σ induced by the Finslerian metric F , where $u \in \Sigma$ and $\{\omega^1, \omega^2, \omega^3\} \in T^*\Sigma$. The moving equations on this frame lead to the so-called Chern connection. This is an almost metric compatible, torsion-free connection of the vector bundle (π^*TM, π, Σ) .

Indeed, by a theorem of Cartan it follows that the coframe $(\omega^1, \omega^2, \omega^3)$ must satisfy the structure equations

$$(2.2) \quad \begin{aligned} d\omega^1 &= -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3, \\ d\omega^2 &= -\omega^1 \wedge \omega^3, \\ d\omega^3 &= K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3. \end{aligned}$$

The functions I, J, K are smooth functions on Σ called the *invariants* of the Finsler structure (M, F) in the sense of Cartan's equivalence problem (see, e.g., [BCS], [Br1], [Br2]).

This implies that on the vector bundle π^*TM there exists a unique torsion-free and almost metric compatible connection $\nabla : C^\infty(T\Sigma) \otimes C^\infty(\pi^*TM) \rightarrow C^\infty(\pi^*TM)$, given by

$$(2.3) \quad \nabla_{\hat{X}} Z = \{ \hat{X}(z^i) + z^j \omega_j^i(\hat{X}) \} e_i,$$

where \hat{X} is a vector field on Σ , $Z = z^i e_i$ is a section of π^*TM , and $\{e_i\}$ is the g -orthonormal frame field on π^*TM .

The 1-forms ω_j^i define the *Chern connection* of the Finsler structure (M, F) , where

$$(2.4) \quad (\omega_j^i) = \begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix} = \begin{pmatrix} -I\omega^3 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix},$$

and $I := A_{111} = A(e_1, e_1, e_1)$ is the *Cartan scalar* for Finsler surfaces. Note that $I = 0$ is equivalent to the fact that the Finsler structure is Riemannian.

REMARKS

(1) We remark that the Chern connection gives a decomposition of the tangent bundle $T\Sigma$ by

$$T\Sigma = H\Sigma \oplus V\Sigma,$$

where the $H\Sigma$ is the horizontal distribution generated by e_1, e_2 and $V\Sigma$ is the vertical distribution generated by \hat{e}_3 , where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ is the dual frame of the coframe $\omega^1, \omega^2, \omega^3$.

(2) For comparison, recall the structure equations of a Riemannian surface. They are obtained from (2.2) by putting $I = J = 0$.

(3) The scalar K is called the *Gauss curvature* of the Finsler surface. In the case when F is Riemannian, K coincides with the usual Gauss curvature of a Riemannian surface.

Differentiating again (2.2), one obtains the *Bianchi identities*

$$(2.5) \quad J = I_2 = \frac{1}{F} \left(y^1 \frac{\delta I}{\delta x^1} + y^2 \frac{\delta I}{\delta x^2} \right),$$

$$K_3 + KI + J_2 = 0,$$

where $\left\{ \frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i} \right\}$ is the adapted basis of $T\Sigma$, given by

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

The functions N_i^j are called the coefficients of the *nonlinear connection* of (M, F) (see [BCS, p. 33] for details).

The linear indices in I_2, K_3, J_2 , and so on, indicate differential terms with respect to $\omega_1, \omega_2, \omega_3$. For example, $dK = K_1\omega^1 + K_2\omega^2 + K_3\omega^3$. The scalars K_1, K_2, K_3 are called the directional derivatives of K .

Nevertheless, note that the scalars $I = I(x, y), J = J(x, y), K = K(x, y)$, and their derivatives live on Σ , not on M as in the Riemannian case!

More generally, given any function $f : \Sigma \rightarrow \mathcal{R}$, one can write its differential in the form

$$df = f_1\omega_1 + f_2\omega_2 + f_3\omega_3.$$

Taking one more exterior differentiation of this formula, one obtains the following *Ricci identities*:

$$(2.6) \quad \begin{aligned} f_{21} - f_{12} &= -K f_3, \\ f_{32} - f_{23} &= -f_1, \\ f_{31} - f_{13} &= I f_1 + f_2 + J f_3. \end{aligned}$$

One defines *the curvature* of the Finsler structure (M, F) as usual by

$$(2.7) \quad \Omega_j^i = d\omega_j^i - \omega_j^k \wedge \omega_k^i,$$

where $i, j, k \in \{1, 2, 3\}$, and ω_j^i is the Chern connection matrix (2.4). It easily follows that the only essential entry in the matrix Ω_j^i is

$$(2.8) \quad \Omega_2^1 = d\omega_2^1 = d\omega^3 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3.$$

We remark that the fact that the curvature 2-form is closed is a peculiarity of Finslerian surfaces that is very useful in deriving the Gauss-Bonnet formula in the following sections.

Recall that a Finsler surface is called *Landsberg* if the invariant J vanishes. Bianchi identities imply that in this case $I_2 = 0$ and $K_3 = -KI$. A Finsler structure having $I_1 = 0$, $I_2 = 0$ is called a *Berwald* surface (see [BCS, Lemma 10.3.1, p. 267] for details).

It is known that Berwald surfaces are, in fact, Riemannian surfaces if $K \neq 0$ or locally Minkowski flats if $K = 0$ (see [Sz1] and [BCS, p. 278]).

We also remark that on a Landsberg surface, even though both K and g are quantities defined on the 3-dimensional manifold Σ , the product $K\sqrt{g}$ lives on M (see [BCS, p. 106]).

Recall that the restriction of a Finsler norm to a tangent plane $T_x M$ gives a Minkowski norm on $T_x M$. For an arbitrary fixed $x \in M$, this Minkowski norm induces a Riemannian metric \hat{g} on the punctured plane $\widetilde{T_x M}$ by

$$(2.9) \quad \hat{g} := g_{ij}(y) dy^i \otimes dy^j,$$

where $y = (y^i)$ are the global coordinates in $T_x M$.

Note that the Riemannian manifold $(\widetilde{T_x M}, \hat{g})$ is flat; that is, the Gaussian curvature of \hat{g} vanishes on $\widetilde{T_x M}$. This is a peculiarity of the two-dimensional case (see [BCS, p. 388]).

The outward-pointing normal to the indicatrix is

$$(2.10) \quad \hat{n}_{\text{out}} = \frac{y}{F(y)} = \frac{y^i}{F(y)} \cdot \frac{\partial}{\partial y^i}.$$

Indeed, let us consider $y^i = y^i(t)$ to be a unit speed parameterization of the indicatrix Σ . By derivation with respect to t of the formula $g_{ij}(y)y^i y^j = 1$, one obtains

$$g_{ij}(y)y^i \dot{y}^j = 0,$$

where the dot notation means derivative with respect to t .

In the following, let us consider the indicatrix Σ_x as a Riemannian submanifold of the punctured Riemannian manifold $(\widetilde{T_x M}, \hat{g})$ with the induced Riemannian metric h , and let $y(t) = (y^1(t), y^2(t))$ be a unit speed (with respect to h) parameterization of Σ_x .

Obviously, F is Euclidean if and only if the main scalar I restricted to Σ_x vanishes. In other words, $I|_{\Sigma_x}$ “measures” the deviation of F on $T_x M$ from an Euclidean inner product.

The volume form of the Riemannian metric \hat{g} on $T_x M$ is

$$(2.11) \quad dV = \sqrt{\hat{g}} dy^1 \wedge dy^2,$$

where $\sqrt{\hat{g}} = \sqrt{\det(g_{ij})}$, and the induced Riemannian volume form on the submanifold Σ_x is

$$(2.12) \quad ds = \frac{\sqrt{\hat{g}}}{F} (y^1 \dot{y}^2 - y^2 \dot{y}^1) dt.$$

Along Σ_x the 1-form ds coincides with

$$(2.13) \quad d\theta = \frac{\sqrt{\hat{g}}}{F^2} (y^1 dy^2 - y^2 dy^1).$$

The parameter θ is called the *Landsberg angle*.

REMARKS

(1) The formula $ds = \frac{\sqrt{\hat{g}}}{F} (y^1 \dot{y}^2 - y^2 \dot{y}^1) dt$ is valid as long as the underlying parameterization traces Σ out in a positive manner.

(2) The *Riemannian length* of the indicatrix is therefore defined by

$$(2.14) \quad L := \int_{\Sigma_x} ds,$$

and it is typically *not* equal to 2π as in the case of Riemannian surfaces. This fact was noted for the first time by M. Matsumoto [M2]. Since the indicatrix is a 1-dimensional submanifold, its Riemannian length and the Riemannian volume are, in fact, identical.

The Riemannian length of the indicatrix Σ_x is an integral where the integration domain also depends on F . One would like, however, to work with integrals over the *standard unit circle*

$$(2.15) \quad \mathbf{S}^1 = \{y \in \widetilde{\mathbf{T}_x M} : (y^1)^2 + (y^2)^2 = 1\},$$

even with the price of a more complicated integrand.

It follows immediately that the indicatrix length in a Minkowski plane can be computed by

$$(2.16) \quad L = \int_{\mathbf{S}^1} \frac{\sqrt{\hat{g}}}{F^2} (y^1 dy^2 - y^2 dy^1).$$

Indeed, the 1-form

$$(2.17) \quad d\theta = \frac{\sqrt{\hat{g}}}{F^2} (y^1 dy^2 - y^2 dy^1)$$

is a closed 1-form on $\widetilde{T_x M}$. By the use of Stokes's theorem, one can easily see that integrating this over two corresponding arcs (see (2.21)) of S and \mathbf{S}^1 , one obtains the same answer (see [BCS, pp. 101, 102]).

One defines in this way the *length function* of the indicatrix Σ_x by

$$(2.18) \quad L : M \rightarrow [0, \infty), \quad x \mapsto L(x) = \int_{\Sigma_x} \frac{1}{F} ds,$$

or, equivalently,

$$(2.19) \quad L(x) = \int_{\Sigma_x} d\theta.$$

Let us also remark that

$$(2.20) \quad d\theta = \omega_{2|\Sigma_x}^1,$$

that is, $d\theta$ is equal to the pure part dy of ω_2^1 ; therefore there is no harm if we write

$$(2.21) \quad L(x) = \int_{\Sigma_x} \omega_2^1.$$

We define the *Landsberg angle* $\angle_x(X, Y)$ of two *Finslerian unit vectors* $X, Y \in \mathbf{T}_x M$ (with the same origin, say, $y = 0$, or glided to have the same origin) and the tips on the indicatrix curve, as the oriented Riemannian angle of X and Y measured with the induced Riemannian metric \hat{g} .

In other words, for any two unit vectors X, Y as above, their Finslerian angle is given by

$$(2.22) \quad \begin{aligned} \angle_x(X, Y) &:= \int_{\widehat{XY}} \frac{\sqrt{\hat{g}}}{F^2} (y^1 dy^2 - y^2 dy^1) = \int_{\Sigma_x|(X, Y)} \sqrt{\hat{g}(\dot{y}, \dot{y})} dt \\ &= \int_{\Sigma_x|(X, Y)} d\theta, \end{aligned}$$

where \widehat{XY} and $\Sigma_x|(X, Y)$ are the arcs on the unit Euclidean circle and the indicatrix curve described by the directions of the vectors X and Y , respectively. Since the angle $\angle_x(X, Y)$ is described by the integral of the 1-form $d\theta$, it is customary to call it the Landsberg angle.

REMARK

In this point it is important to remark that there are big differences between the Euclidean angles used in plane geometry and the Landsberg angles defined above (see Figure 1). Imagine the indicatrix of a Finsler space to be a translated ellipse (this is actually the case of a Randers metric) and the Euclidean unit circle inside it. We represented the Euclidean circle in the interior of the Finslerian indicatrix (they might actually intersect) only to make this explanation easy to follow. We denote the intersection points of the indicatrix with the coordinate axes by A, B, A', B' , and we denote the corresponding arcs by L_1, L_2, L_3, L_4 , respectively. Moreover, we denote by E_1, E_2, E_3, E_4 the corresponding arcs on the Euclidean

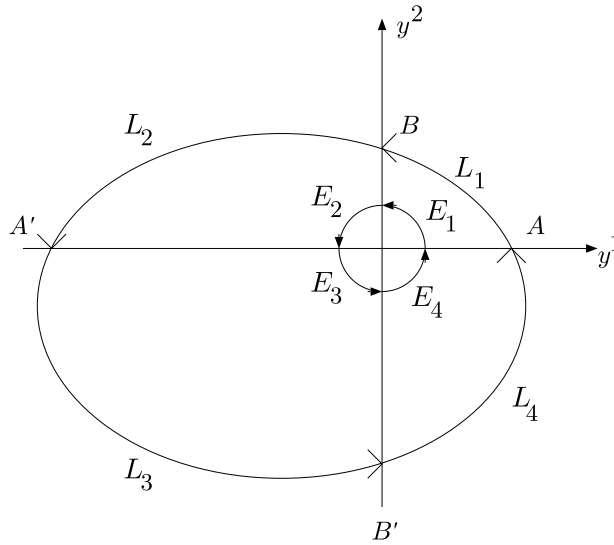


Figure 1. The Landsberg angle

unit circle. Obviously, the four Euclidean angles determined by the coordinate axes are all equal to $\pi/2$, and their sum equals 2π .

On the other hand, the Landsberg angles determined by the coordinate axes are described by the \hat{g} -Riemannian lengths of the indicatrix arcs L_1, L_2, L_3, L_4 , respectively. One can easily see that the usual properties of angles known to hold good in the Euclidean plane do not hold anymore. Indeed, note, for example, that the opposite angles are not equal anymore, $L_1 \neq L_3, L_2 \neq L_4$, nor does the sum of adjacent angles equal π . However, we do know that the sum of L_1, L_2, L_3, L_4 equals the total length of indicatrix L .

A special case would be the case of an absolutely homogeneous Finsler norm, that is, the case when the induced Minkowski norm satisfies the condition $F(-y) = F(y)$. In this case, the indicatrix, without being an ellipse, it is still a central symmetric curve, and therefore, the opposite angles are equal! In particular, $L_1 = L_3$ and $L_2 = L_4$.

3. The normal lift of a curve

Let us consider a smooth (or piecewise C^∞) curve $\gamma : [0, r] \rightarrow M$ with the tangent vector $\dot{\gamma}(t) = T(t)$, parameterized such that $F(\gamma(t), \dot{\gamma}(t)) = 1$, and let N be the *normal vector* along γ defined by

$$\begin{aligned}
 g_N(N, N) &= 1, \\
 g_N(T, N) &= 0, \\
 g_N(T, T) &= \sigma^2(t).
 \end{aligned}
 \tag{3.1}$$

We point out that here g_N means

$$(g_N)_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(\gamma(t), N(t)).$$

This kind of normal vector was introduced by Z. Shen ([Sh2, p. 27]) and used by us in the formulation of the Gauss-Bonnet theorem for Landsberg surfaces with smooth boundary (see [SS]).

The *normal lift* (shortly *N-lift*) $\hat{\gamma}^\perp(t)$ of $\gamma(t)$ to Σ is given by

$$(3.2) \quad \begin{aligned} \hat{\gamma}^\perp &: [0, r] \rightarrow \Sigma, \\ t &\mapsto \hat{\gamma}^\perp(t) = (\gamma(t), N(t)). \end{aligned}$$

The tangent vector \hat{T}^\perp to the normal lift $\hat{\gamma}^\perp$ is given by

$$(3.3) \quad \begin{aligned} \hat{T}^\perp(t) &= \hat{\gamma}^\perp(t) = \frac{d}{dt} \hat{\gamma}^\perp(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i} \Big|_{(x, N)} + \frac{d}{dt} N^i(t) \frac{\partial}{\partial y^i} \Big|_{(x, N)} \\ &= T^i(t) \frac{\delta}{\delta x^i} \Big|_{(x, N)} + (D_T^{(N)} N)^i \frac{\partial}{\partial y^i} \Big|_{(x, N)}. \end{aligned}$$

The local coefficients of the *covariant derivative with reference vector* N along γ are given by

$$(3.4) \quad D_T^{(N)} U = (D_T^{(N)} U)^i \cdot \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} = \left[\frac{dU^i}{dt} + T^j U^k \Gamma_{jk}^i(x, N) \right] \cdot \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}$$

for any $U = U^i(x) \frac{\partial}{\partial x^i}$ vector field along γ , where Γ_{jk}^i are the Chern connection coefficients; that is, $\omega_i^j = \Gamma_{ik}^j dx^k$.

Note that the term $D_T^{(N)} N$ in (3.3) means the covariant derivative of N along γ with reference vector N .

We recall here a useful lemma ([SS, Lemma 7.1]):

LEMMA 3.1

We have

$$(3.5) \quad \frac{d}{dt} g_N(V, W) = g_N(D_T^{(N)} V, W) + g_N(V, D_T^{(N)} W) + 2A(V, W, D_T^{(N)} N) \Big|_{(x, N)},$$

where A is the Cartan tensor (see [BCS, p. 30]).

Using this, we obtain

$$(3.6) \quad \begin{aligned} g_N(D_T^{(N)} N, N) &= 0, \\ g_N(D_T^{(N)} T, N) + g_N(T, D_T^{(N)} N) &= 0, \\ g_N(D_T^{(N)} T, T) &= \sigma(t) \frac{d\sigma}{dt} - A(T, T, D_T^{(N)} N). \end{aligned}$$

Similarly to the notion of Finslerian geodesics, we can define the notion of the N -parallel of a Finsler structure.

DEFINITION 3.1

A curve γ on the surface M , in Finslerian natural parameterization, is called an N -parallel of the Finslerian structure (M, F) if and only if we have

$$(3.7) \quad D_T^{(N)} N = 0.$$

It follows from (3.3) that the tangent vector to the normal lift of an N -parallel curve γ on M is given by $\hat{T}^\perp = T^i \frac{\delta}{\delta x^i}|_{(x, N)}$. In other words, we obtain the following characterization of an N -parallel of a Finsler surface.

PROPOSITION 3.2

A curve on M is an N -parallel curve if and only if its normal vector N is transported parallel along γ .

REMARKS

(1) If γ is an N -parallel, then we have

$$(3.8) \quad \begin{aligned} g_N(D_T^{(N)} T, N) &= 0, \\ g_N(D_T^{(N)} T, T) &= \sigma(t) \frac{d\sigma}{dt}. \end{aligned}$$

(2) The curve γ is an N -parallel if and only if $\nabla_{\hat{T}^\perp} l = D_T^{(N)} N = 0$. This implies that $g_N(\nabla_{\hat{T}^\perp} l, l) = 0$; that is, $\nabla_{\hat{T}^\perp} l$ is orthogonal to the indicatrix.

In case of an arbitrary curve γ on M , from

$$g_N(D_T^{(N)} N, N) = 0, \quad g_N(T, N) = 0$$

it follows that the vector $D_T^{(N)} N$ is proportional to T ; that is, there exists a non-vanishing function $k_T^{(N)}(t)$ such that

$$(3.9) \quad D_T^{(N)} N = -\frac{k_T^{(N)}(t)}{\sigma^2(t)} T, \quad \sigma(t) \neq 0.$$

The function $k_T^{(N)}(t)$ is called the N -parallel curvature of γ . The minus sign is put only in order to obtain the same formulas as in the classical theory of Riemannian manifolds.

In other words, we have

$$(3.10) \quad g_N(D_T^{(N)} N, T) = -g_N(D_T^{(N)} T, N) = -k_T^{(N)}(t).$$

Since $\{N, T\}$ is a basis, we also obtain

$$(3.11) \quad D_T^{(N)} T = k_T^{(N)}(t) N + B(t) T,$$

where we put

$$(3.12) \quad B(t) = \frac{1}{\sigma(t)} \frac{d\sigma(t)}{dt} - \frac{1}{\sigma^2(t)} A_{|(x, N)}(T, T, D_T^{(N)} N).$$

By making use of the cotangent map of $\hat{\gamma}^\perp$, we compute

$$\begin{aligned}
 \hat{\gamma}^{\perp*} \omega^1 \frac{\partial}{\partial t} &= \omega^1(\hat{T}^\perp)|_{(x,N)} = \frac{\sqrt{g}}{F}(N^2 T^1 - T^2 N^1) = \sigma(t), \\
 \hat{\gamma}^{\perp*} \omega^2 \frac{\partial}{\partial t} &= \omega^2(\hat{T}^\perp)|_{(x,N)} = g_N(N, T) = 0, \\
 \hat{\gamma}^{\perp*} \omega^3 \frac{\partial}{\partial t} &= \omega^3(\hat{T}^\perp)|_{(x,N)} = \frac{\sqrt{g}}{F}[N^2(D_T^{(N)}N)^1 - N^1(D_T^{(N)}N)^2] \\
 &= -\frac{k_T^{(N)}(t)}{\sigma(t)}
 \end{aligned}
 \tag{3.13}$$

(for details see also [SS]).

Therefore we obtain

$$\begin{aligned}
 \hat{\gamma}^{\perp*} \omega^1 &= \sigma(t) dt, \\
 \hat{\gamma}^{\perp*} \omega^2 &= 0, \\
 \hat{\gamma}^{\perp*} \omega^3 &= -\frac{k_T^{(N)}}{\sigma(t)} dt.
 \end{aligned}
 \tag{3.14}$$

If we denote by $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ the dual frame on Σ of the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$, then we obtain the fact that the tangent vector to the normal lift of $\hat{\gamma}^\perp$ is

$$\hat{T}^\perp = \sigma(t)\hat{e}_1 - \frac{k_T^{(N)}}{\sigma(t)}\hat{e}_3 \in \langle \hat{e}_1, \hat{e}_3 \rangle,
 \tag{3.15}$$

where $\langle \hat{e}_1, \hat{e}_3 \rangle$ is the 2-plane generated by \hat{e}_1, \hat{e}_3 .

Note that in the case when γ is an N -parallel, we have

$$D_T^{(N)}T = \frac{1}{\sigma(t)} \frac{d\sigma(t)}{dt} T,
 \tag{3.16}$$

and

$$\begin{aligned}
 \hat{\gamma}^{\perp*} \omega^1 &= \sigma(t) dt, \\
 \hat{\gamma}^{\perp*} \omega^2 &= 0, \\
 \hat{\gamma}^{\perp*} \omega^3 &= 0.
 \end{aligned}
 \tag{3.17}$$

Finally, we remark that the tangent vector to the normal lift of an N -parallel is

$$\hat{T}^\perp = \sigma(t)\hat{e}_1.
 \tag{3.18}$$

REMARK

Let us remark that the N -lift used in this section is different from the *canonical lift* (or *tangential lift*) of a curve. Indeed, for an arbitrary curve $\gamma : [0, r] \rightarrow M$

with the usual properties, the *canonical lift* of γ to Σ is given by

$$(3.19) \quad \begin{aligned} \hat{\gamma} &: [0, r] \rightarrow \Sigma, \\ t &\mapsto \hat{\gamma}(t) = (\gamma(t), \dot{\gamma}(t)). \end{aligned}$$

This is well defined because $(\gamma(t), \dot{\gamma}(t)) \in \Sigma$ because of the Finslerian natural parameterization of γ (see [BCS, p. 112]).

Let us consider now the normal vector \mathcal{V} along γ with respect to the tangent vector T defined by

$$g_T(T, \mathcal{V}) = 0.$$

Here, by g_T we mean

$$(g_T)_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(\gamma(t), T(t)).$$

We have the fundamental relations

$$\begin{aligned} g_T(T, T) &= 1, \\ g_T(T, \mathcal{V}) &= 0, \end{aligned}$$

and let us put

$$\mu^2(t) := g_T(\mathcal{V}, \mathcal{V}).$$

We also obtain

$$(3.20) \quad \begin{aligned} g_T(D_T^{(T)}T, T) &= 0, \\ g_T(D_T^{(T)}T, \mathcal{V}) + g_T(D_T^{(T)}\mathcal{V}, T) &= 0, \\ g_T(D_T^{(T)}\mathcal{V}, \mathcal{V}) &= \mu(t) \frac{d\mu}{dt} - A(\mathcal{V}, \mathcal{V}, D_T^{(T)}T). \end{aligned}$$

The local coefficients of the *covariant derivative with reference vector T* along γ are given by

$$(3.21) \quad D_T^{(T)}U = (D_T^{(T)}U)^i \cdot \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} = \left[\frac{dU^i}{dt} + T^j U^k \Gamma_{jk}^i(x, T) \right] \cdot \frac{\partial}{\partial x^i} \Big|_{\gamma(t)}$$

for any $U = U^i(x) \frac{\partial}{\partial x^i}$ vector field along γ , where Γ_{jk}^i are the Chern connection local coefficients.

One can see that the term $D_T^{(T)}T$ in (3.20) means the covariant derivative of T along γ with reference vector T .

From

$$g_T(D_T^{(T)}T, T) = 0, \quad g_T(T, \mathcal{V}) = 0$$

it follows that the vector $D_T^{(T)}T$ is proportional to \mathcal{V} ; that is, there exists a nonvanishing function $k_{\mathcal{V}}^{(T)}(t)$ such that

$$D_T^{(T)}T = k_{\mathcal{V}}^{(T)}(t)\mathcal{V}.$$

The function $k_{\mathcal{V}}^{(T)}(t)$ is called the *signed curvature of γ over T* .

On the other hand, note that from $g_T(D_T^{(T)}T, \mathcal{V}) + g_T(D_T^{(T)}\mathcal{V}, T) = 0$, we obtain $g_T(D_T^{(T)}\mathcal{V}, T) = -g_T(D_T^{(T)}T, \mathcal{V}) = -k_{\mathcal{V}}^{(T)}(t)$.

We also obtain

$$(3.22) \quad \begin{aligned} \hat{\gamma}^* \omega^1 &= 0, \\ \hat{\gamma}^* \omega^2 &= dt, \\ \hat{\gamma}^* \omega^3 &= -\frac{k_{\mathcal{V}}^{(T)}}{\mu} dt \quad (\mu \neq 0). \end{aligned}$$

In the case when γ is a Finslerian geodesic, we have by definition $D_T^{(T)}T = 0$, and therefore

$$\begin{aligned} g_T(T, D_T^{(T)}\mathcal{V}) &= 0, \\ g_T(\mathcal{V}, D_T^{(T)}\mathcal{V}) &= \mu(t) \frac{d\mu(t)}{dt}. \end{aligned}$$

It follows that

$$D_T^{(T)}\mathcal{V} = \frac{1}{\mu(t)} \frac{d\mu(t)}{dt} \mathcal{V}$$

and

$$(3.23) \quad \begin{aligned} \hat{\gamma}^* \omega^1 &= 0, \\ \hat{\gamma}^* \omega^2 &= dt, \\ \hat{\gamma}^* \omega^3 &= 0. \end{aligned}$$

The tangent vector to the tangential lift $\hat{\gamma}$ of a Finslerian geodesic γ is

$$\hat{T} = \hat{e}_2.$$

We end this section by pointing out that this theory reduces to the classical theory in the case of a *Riemannian surface*.

Let us assume that our Finslerian structure on M is actually a Riemannian one, and let us denote the Riemannian metric on the surface M by g . Then the normal along a curve γ on M , naturally parameterized, is defined by

$$(3.24) \quad \begin{aligned} g(T, T) &= 1, \\ g(T, N) &= 0, \\ g(N, N) &= 1. \end{aligned}$$

Therefore the two types of normals N and \mathcal{V} defined above coincide, and $\sigma = \mu = 1$.

The tangent lift of γ to Σ^g is

$$\begin{aligned} \hat{\gamma} &: [0, r] \rightarrow \Sigma^g, \\ t &\mapsto \hat{\gamma}(t) = (\gamma(t), T(t)), \end{aligned}$$

where Σ^g is the total space of the unit sphere bundle of the Riemannian structure (M, F) . Its tangent vector is

$$\begin{aligned}\hat{T}(t) &= \hat{\gamma}(t) = \frac{d}{dt}\hat{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i} \Big|_{(x,T)} + \frac{d}{dt}T^i(t) \frac{\partial}{\partial y^i} \Big|_{(x,T)} \\ &= T^i(t) \frac{\delta}{\delta x^i} \Big|_{(x,T)} + (D_T T)^i \frac{\partial}{\partial y^i} \Big|_{(x,T)},\end{aligned}$$

where $D_T T$ is the usual covariant derivative along γ with respect to the Levi-Civita connection of g .

By derivation, we obtain

$$\begin{aligned}g(D_T T, T) &= 0, \\ g(D_T T, N) + g(D_T N, T) &= 0, \\ g(D_T N, N) &= 0.\end{aligned}$$

From $g(D_T T, T) = 0$, $g(T, N) = 0$, it follows that

$$D_T T = k_N(t)N,$$

where the function $k_N(t)$ is the usual Riemannian signed curvature of γ .

On the other hand, from

$$\begin{aligned}g(T, D_T N) &= -g(D_T T, N) = -k_N(t), \\ g(D_T N, N) &= 0,\end{aligned}$$

we obtain

$$D_T N = -k_N(t)T.$$

Let us consider now the N -lift of γ to Σ^g defined as above. By computations similar to those in the Finslerian case, in the Riemannian case we obtain

$$k_T^{(N)} = k_N = k_N^{(T)};$$

that is, the N -parallel curvature and the signed curvature over T coincide with the usual Riemannian signed curvature.

Moreover, the curve γ is a Riemannian geodesic if and only if one of the following relations hold:

- (1) $k_N = 0$,
- (2) $D_T T = 0$,
- (3) $D_T N = 0$;

that is, on a Riemannian geodesic the vectors T and N are equally parallel transported along γ . In other words, on a Riemannian manifold, the Riemannian geodesics, and the N -parallel curves coincide.

4. The Gauss-Bonnet theorem for Finsler surfaces with smooth boundary

The proof of the Gauss-Bonnet theorem for Finsler manifolds without boundary was given by D. Bao and S. S. Chern in [BC] using the transgression method.

Using their method we have extended the result to Landsberg surfaces with smooth boundary (see [SS]).

In the present article, we give a Gauss-Bonnet-type formula for Riemann-Finsler surfaces where the indicatrix volume does not need to be constant anymore, using an idea of B. Lackey [L].

We start by discussing the case of a Riemann-Finsler surface with *smooth boundary*.

Let (M, F) be a compact Riemann-Finsler surface, and let $D \subset M$ be a domain with smooth boundary $\partial D = \gamma : [a, b] \mapsto M$, given by $x^i = x^i(t)$. We assume γ to be unit speed, that is, $F(\gamma(t), T(t)) = 1$, where $T(t) = \dot{\gamma}(t)$.

PROPOSITION 4.1

Let (M, F) be a compact oriented Finslerian surface, and let $D \subset M$ be a domain with boundary ∂D . Let $N : \partial D \rightarrow \Sigma$ be the inward-pointing Finslerian unit normal on ∂D .

Then, we have

$$(4.1) \quad \int_D \frac{1}{L(x)} [X^*(K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3) - d \log L(x) \wedge X^*(\omega^3)] \\ + \int_{N(\partial D)} \frac{1}{L(x)} \omega_1^2 = \mathcal{X}(D),$$

where $L(x)$ is the Riemannian length of the indicatrix Σ_x , X is a unit prolongation of N , K is the Gauss curvature, and $\mathcal{X}(D)$ is the Euler characteristic of D .

Proof

The proof follows [BCS, p. 106] or [SS]. Indeed, note first that we can extend the normal vector field N on γ to a vector field V on M with only finitely many zeros x_1, x_2, \dots, x_k in $D \setminus \partial D$. It is then known that the sum of indices of X is equal to the Euler characteristic $\mathcal{X}(D)$ (see, e.g., [Spiv, p. 561]).

By removing from D the interiors of the geodesic circles S_α^ε (centered at x_α of radius $\varepsilon > 0$), one obtains the manifold with boundary D_ε . Note that in this case, the boundary of D_ε consists of the boundaries of the geodesic circles S_α^ε and the boundary of D .

Since V has all zeros in $D \setminus \partial D$, it follows that V has no zeros on D_ε and therefore we can normalize it, obtaining in this way the application

$$(4.2) \quad X = \frac{V}{F(V)} : D_\varepsilon \rightarrow \Sigma, \quad x \mapsto \frac{V(x)}{F(V(x))}.$$

Using X , we can lift D_ε to Σ constructing in this way the 2-dimensional submanifold $X(D_\varepsilon)$ of Σ such that we can integrate formula (2.8) over this submanifold.

However, before doing this, we make the following remark.

From the degree theory (see, e.g., [Mil]) it results that

$$\lim_{\varepsilon \rightarrow 0} \int_{X(S_\alpha^\varepsilon)} \omega_1^2 = -i_\alpha(X) \int_{\Sigma_{x_\alpha}} \omega_1^2 = -i_\alpha(X)L(x_\alpha),$$

where $i_\alpha(X)$ is the index of X at x_α . Here the indicatrix Σ_{x_α} is traced in the counterclockwise orientation.

Since all the indicatrices are smooth closed convex curves enclosing the origin, it follows that $L(x) \neq 0$ for any $x \in M$. Using Lackey's idea (see [L]), we compute the index of X at an arbitrary fixed zero point x_α by

$$(4.3) \quad -i_\alpha(X) = \frac{1}{L(x_\alpha)} \lim_{\varepsilon \rightarrow 0} \int_{X(S_\alpha^\varepsilon)} \omega_1^2 = \lim_{\varepsilon \rightarrow 0} \int_{X(S_\alpha^\varepsilon)} \frac{1}{L(x_\alpha)} \omega_1^2,$$

where we have used the fact that when taking the limit of the integral ω_1^2 , the x -terms actually do not contribute anymore because the metric radius continuously shrinks.

By summing over the zeros of X and using Stokes's theorem, it follows that

$$(4.4) \quad \begin{aligned} -\mathcal{X}(D) &= -\sum_{\alpha=1}^k i_\alpha(X) = \sum_{\alpha=1}^k \lim_{\varepsilon \rightarrow 0} \int_{X(S_\alpha^\varepsilon)} \frac{1}{L(x_\alpha)} \omega_1^2 = \sum_{\alpha=1}^k \lim_{\varepsilon \rightarrow 0} \int_{X(S_\alpha^\varepsilon)} \Pi \\ &= \sum_{\alpha=1}^k \lim_{\varepsilon \rightarrow 0} \int_{X(S_\alpha^\varepsilon)} \Pi + \int_{N(\partial D)} \Pi - \int_{N(\partial D)} \Pi = \int_{\partial X(D_\varepsilon)} \Pi - \int_{N(\partial D)} \Pi \\ &= \int_D X^*(d\Pi) - \int_{N(\partial D)} \Pi, \end{aligned}$$

where we put

$$(4.5) \quad \Pi := \frac{1}{L(x)} \omega_1^2.$$

In this way we obtain the following.

TOPOLOGICAL LEMMA

Let (M, F) be a compact oriented Finslerian surface, and let $D \subset M$ be a domain with smooth boundary ∂D . Let $N : \partial D \rightarrow \Sigma$ be the inward-pointing Finslerian unit normal on ∂D .

Then, we have

$$(4.6) \quad -\int_D X^*(d\Pi) + \int_{N(\partial D)} \Pi = \mathcal{X}(D),$$

where the notation is the same as above.

This is the extension of the topological lemma in [L, page 331] to the case of Finsler surfaces with smooth boundary.

We need now to compute the first term of the sum in the left-hand side of (4.6).

In order to prove (4.1), a straightforward computation gives

$$\begin{aligned} d\Pi &= d\left[\frac{1}{L(x)}\omega_1^2\right] = \frac{1}{L(x)}[d\omega_1^2 - d\log L(x) \wedge \omega_1^2] \\ &= \frac{1}{L(x)}[-K\omega^1 \wedge \omega^2 + J\omega^1 \wedge \omega^3 - d\log L(x) \wedge \omega_1^2], \end{aligned}$$

and Proposition 4.1 is proved. □

Let us remark that in the first integral of the sum in the left-hand side of (4.1), we should have written $L \circ X(x)$ instead of $L(x)$. However, since $L(x)$ is the length of the indicatrix at $x \in M$, and X is a unit vector field on M , one can easily see that $L(x)$ and $L \circ X(x)$ give actually the same value. The same is true for the unit vector N , and we simplify the notation in this article by writing simply $L(x)$.

We can evaluate the second term in the left-hand sum of (4.1) as

$$\begin{aligned} (4.7) \quad \int_{N(\partial D)} \frac{1}{L(x)}\omega_1^2 &= \int_{\partial D} N^*\left(\frac{1}{L(x)}\omega_1^2\right) = \int_{\partial D} \frac{1}{L(x)}N^*(\omega_1^2) \\ &= \int_{\gamma} \frac{1}{L(x)} \frac{k_T^{(N)}(t)}{\sigma(t)} dt, \end{aligned}$$

where we have used (3.14).

From Proposition 4.1 and formula (4.7) we conclude the following.

THEOREM 4.2 (THE GAUSS-BONNET FORMULA FOR FINSLER SURFACES WITH SMOOTH BOUNDARY)

Let (M, F) be a compact oriented Finslerian surface, and let $D \subset M$ be a domain with boundary $\partial D = \gamma$. Let $N : \partial D \rightarrow \Sigma$ be the inward-pointing Finslerian unit normal on ∂D .

Then, we have

$$\begin{aligned} (4.8) \quad \int_D \frac{1}{L(x)}[X^*(K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3) - d\log L(x) \wedge X^*(\omega^3)] \\ + \int_{\gamma} \frac{1}{L(x)} \frac{k_T^{(N)}(t)}{\sigma(t)} dt = \mathcal{X}(D), \end{aligned}$$

where $L(x)$ is the Riemannian length of the indicatrix Σ_x , N is the inward-pointing normal to the boundary ∂D , X is a unit prolongation of N , K is the Gauss curvature, and $\mathcal{X}(D)$ is the Euler characteristic of D .

REMARKS

(1) If (M, F) is a Landsberg surface, then $J = 0$, $L(x) = L = \text{constant}$ and therefore (4.8) gives the Gauss-Bonnet formula for Landsberg surfaces (see [BC], [BCS] for the boundaryless case, and see [SS] for the smooth boundary case). In

other words, we have

$$(4.9) \quad \frac{1}{L} \int_D K \sqrt{g} dx^1 \wedge dx^2 + \frac{1}{L} \int_\gamma \frac{k_T^{(N)}(t)}{\sigma(t)} dt = \mathcal{X}(D)$$

with the same notation as above and where g is the determinant of the induced metric g_{ij} .

(2) If M is a compact orientable boundaryless Finsler manifold, then the Gauss-Bonnet formula reads

$$(4.10) \quad \int_D \frac{1}{L(x)} [X^*(K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3) - d \log L(x) \wedge X^*(\omega^3)] = \mathcal{X}(D).$$

One can see that this formula agrees with [Sh1].

(3) If (M, F) is Riemannian, then one obtains immediately the usual Gauss-Bonnet formula for Riemannian surfaces with smooth boundary (see, e.g., [Spiv, p. 558], [SST, p. 34], and many other places).

5. The Gauss-Bonnet theorem for Finsler surfaces with regular piecewise C^∞ -boundary

Let (M, F) be a compact Finsler surface, and let $D \in M$ be a domain with regular piecewise C^∞ -boundary $\partial D = \gamma : [a, b] \mapsto M$, given by $x^i = x^i(t)$. Let $a = t_0 < t_1 < \dots < t_k = b$ be a partition of $[a, b]$ such that γ is C^∞ on each closed subinterval $[t_{s-1}, t_s]$, $s \in \{1, 2, \dots, k\}$. We assume γ to be unit speed, that is, $F(\gamma(t), T(t)) = 1$, where $T(t) = \dot{\gamma}(t)$.

For the sake of simplicity, let us assume that our boundary curve γ has only one corner $x_0 = x(t_0)$ for some $t_0 \in [a, b]$. In the case of k corners, we sum the quantities to be obtained below.

As in the proof of Theorem 4.2, we take the N -lift of γ to Σ :

$$(5.1) \quad \hat{\gamma}^\perp : [a, b] \rightarrow \Sigma, \quad t \mapsto (x(t), N(t)),$$

where N is defined as above by $g_{N(t)}(T(t), N(t)) = 0$ for all $t \in [a, b] \setminus \{t_0\}$.

Note that in the case of one corner, the N -lift $\hat{\gamma}^\perp$ is not a closed curve anymore.

Indeed, let us denote by T^- and T^+ the tangent vectors to γ in x_0 ; that is,

$$(5.2) \quad T^- = \lim_{t \nearrow t_0} T(t), \quad T^+ = \lim_{t \searrow t_0} T(t),$$

and define the corresponding normals at x_0 by

$$(5.3) \quad g_{N^-}(N^-, T^-) = 0, \quad g_{N^+}(N^+, T^+) = 0,$$

respectively.

It follows that, at the point x_0 , the tangent vector $T(t)$ has a discontinuous jump from T^- to T^+ , and similarly, the normal vector $N(t)$ also has a discontinuous jump from N^- to N^+ .

When lifting the curve γ to Σ , we obtain a C^∞ -curve $\hat{\gamma}$ in Σ with the ends (x_0, N^-) and (x_0, N^+) . Note that $N^-, N^+ \in \widetilde{T_{x_0} M}$ and that $F(x_0, N^-) = F(x_0, N^+) = 1$ (see Figure 2).

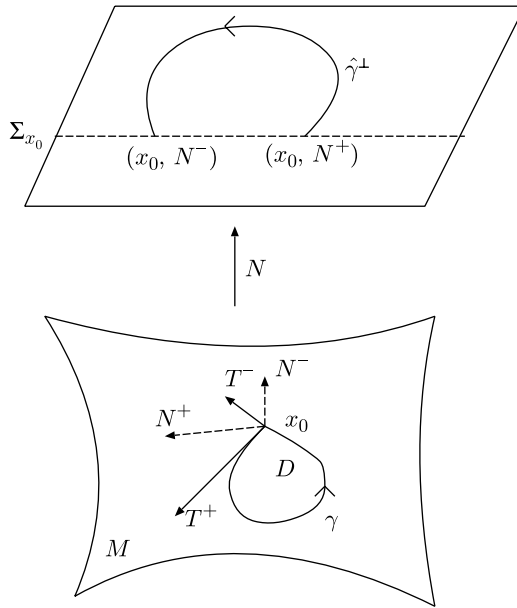


Figure 2. The normal lift of a regular C^∞ piecewise curve with a corner

Now, since N^- and N^+ are two vectors in $T_{x_0}M$ with the origin in x_0 and the tips on the indicatrix, their Landsberg angle is

$$\begin{aligned}
 \angle_{x_0}(N^-, N^+) &= \int_{N^-N^+} d\theta = \int_{N^-N^+} \frac{\sqrt{g}}{F^2} (y^1 dy^2 - y^2 dy^1) \\
 (5.4) \qquad \qquad \qquad &= \int_{\tau_1}^{\tau_2} \sqrt{\hat{g}(\dot{y}, \dot{y})} d\tau,
 \end{aligned}$$

where $y = y(\tau)$ is a unit speed parameterization of the indicatrix and $N^- = y(\tau_1)$, $N^+ = y(\tau_2)$. Here the Landsberg angle is always evaluated using the positive oriented indicatrix arc joining the points N^- , N^+ . Here the positive orientation on the indicatrix is given by ds .

We proceed further and extend the normal vector field N along γ to a smooth section of TM defined along the subset $\gamma \subset M$.

Intuitively, the most natural way of doing this is to consider the set of vectors in $T_{x_0}M$ with the origin in x_0 and the tips on the indicatrix segment $N_{\tau_2}^{\tau_1} := \{N(\tau) : \tau \in [\tau_1, \tau_2]\}$ and to join the points $u_0^- := (x_0, N_{\tau_2}^-)$ and $u_0^+ := (x_0, N_{\tau_2}^+)$ in Σ by the arc of indicatrix curve $(x_0, N_{\tau_2}^{\tau_1})$ (see Figure 3). Unfortunately, this method is not yet good enough because one does not obtain in this way a smooth section of TM along γ and therefore, the existence of the prolongation vector field X is not guaranteed anymore.

However, this idea works well if we consider a smooth variation of γ on M .

Indeed, let us consider a variation $\tilde{\gamma}_\varepsilon : [0, 1] \rightarrow M$ of γ depending on a small $\varepsilon > 0$ such that $\lim_{\varepsilon \rightarrow 0} \tilde{\gamma}_\varepsilon = \gamma$ as set of points.

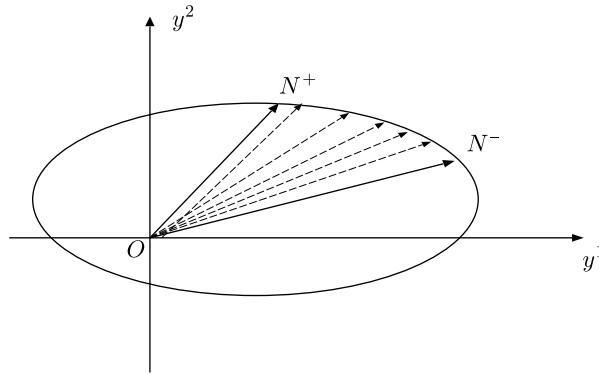


Figure 3. The Landsberg angle on a Finslerian indicatrix

We define

$$(5.5) \quad \tilde{\gamma}_1(\varepsilon, t) = \exp_{\gamma(t)}(\varepsilon N(t)),$$

where N is the normal vector field along γ . Since N has a discontinuous jump from N^- to N^+ at x_0 , the curve $\tilde{\gamma}_1$ also has a jump.

Indeed, let us remark that for a fixed, small enough $\varepsilon > 0$, we obtain a smooth curve $\tilde{\gamma}_1(\varepsilon, t)$ on M going around γ , while for a fixed t , we have a transversal Finslerian geodesic with initial conditions $(\gamma(t), N(t))$.

Note also that for a fixed t_1 the tangent vector $\tilde{T}_{t_1}(\varepsilon)$ of $\tilde{\gamma}_1$ at the point $\tilde{\gamma}_1(t_1, \varepsilon)$ is given by the parallel translation of the tangent vector $T(t_1)$ of γ at the point $\gamma(t_1)$ along the transversal geodesic $\exp_{\gamma(t_1)}(\varepsilon N(t_1))$, where $g_{N(t_1)}(N(t_1), T(t_1)) = 0$. Using now the properties of parallel displacement (see [BCS, pp. 140, 141]), it follows that at any small enough $\varepsilon > 0$ we have

$$g_{\tilde{N}_{t_1}(\varepsilon)}(\tilde{N}_{t_1}(\varepsilon), \tilde{T}_{t_1}(\varepsilon)) = 0,$$

where $\tilde{N}_{t_1}(\varepsilon)$ is the tangent vector of the transversal geodesic $\exp_{\gamma(t_1)}(\varepsilon N(t_1))$ at the point $\gamma_1(t_1, \varepsilon)$.

These remarks assure us that the variation curve $\tilde{\gamma}_1$ has its ends on the geodesics $\exp_{x_0}(\varepsilon N^-)$ and $\exp_{x_0}(\varepsilon N^+)$, for small enough $\varepsilon > 0$; that is, $\tilde{\gamma}_1$ is not a closed loop.

Next, we complete the curve $\tilde{\gamma}_1$ with an arc of curve $\tilde{\gamma}_2$ which connects smoothly the ends of $\tilde{\gamma}_1$ such that $\tilde{\gamma} = \tilde{\gamma}_1 \cup \tilde{\gamma}_2$ is a closed smooth variation of γ on M . The easiest way to do this is exponentiate the indicatrix arc between N^- and N^+ ; that is, we consider

$$(5.6) \quad \tilde{\gamma}_2(\varepsilon) = \exp_{x_0}(\varepsilon N_{\tau_2}^{r_1}).$$

One can now easily see that $\tilde{\gamma} = \tilde{\gamma}_1 \cup \tilde{\gamma}_2$ is a closed smooth variation near γ whose tangent vector \tilde{T} is given along $\tilde{\gamma}_1$ by the parallel displacement of T along the transversal geodesic $\sigma_t(\varepsilon) = \tilde{\gamma}_1(\varepsilon, t)$ and along $\tilde{\gamma}_2$ by $\exp_{x_0*} W$, where W is the tangent vector along the indicatrix curve. The Gauss lemma for Riemann-Finsler manifolds (see, e.g., [BCS, p. 140]) assures us that $\hat{g}_{x_0}(\varepsilon N, W) = 0$ and

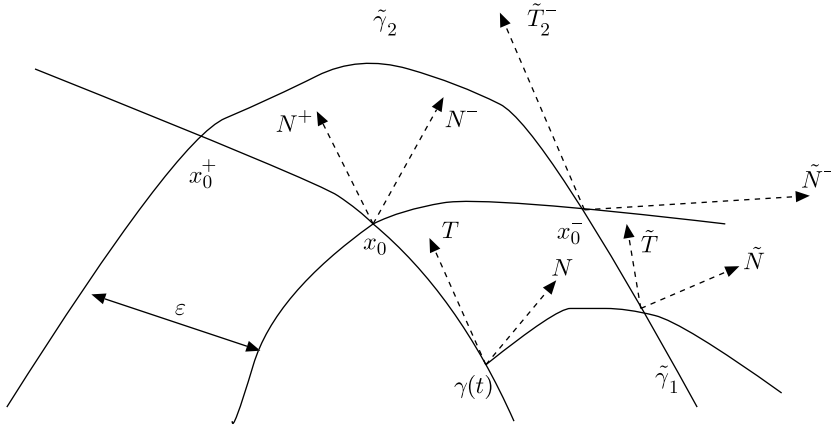


Figure 4. A magnified view of the landscape around the point x_0

$g_{\tilde{N}}(\tilde{N}, \tilde{T}_2) = 0$, where \tilde{N} and \tilde{T}_2 are the normal and tangent vectors, respectively, along $\tilde{\gamma}_2$.

From the discussion above, one can see now that the tangent vector of $\tilde{\gamma}_1$ at $x_0^- = \exp_{x_0}(\varepsilon N^-)$ is $g_{\tilde{N}^-}$ -orthogonal to $\tilde{N}^- := \frac{d}{d\varepsilon} \tilde{\gamma}_1(t, \varepsilon)$ and that the tangent vector of $\tilde{\gamma}_2$ at the same point x_0^- is also $g_{\tilde{N}^-}$ -orthogonal to \tilde{N}^- due to the Gauss lemma. Therefore the unitary left and right tangent vectors at x_0^- have the same direction, so they must coincide (see Figure 4).

Therefore we can conclude that the curve $\tilde{\gamma}$ is smooth at x_0^- when we take the limit $\varepsilon \rightarrow 0$. The same argument applies at $x_0^+ = \exp_{x_0}(\varepsilon N^+)$.

We point out, however, that since we have moved the point x_0 a little along the transversal geodesic $\exp_{x_0}(\varepsilon N^-)$, the indicatrix also changes from Σ_{x_0} to $\Sigma_{x_0^-}$. However, we finally take the limit $\varepsilon \rightarrow 0$ so that this small displacement cannot cause much harm.

Having all these done, we can now consider the bounded domain $\tilde{D} \subset M$ with smooth boundary $\partial \tilde{D} = \tilde{\gamma} = \tilde{\gamma}_1 \cup \tilde{\gamma}_2$ and apply to it the same method as in §4.

Indeed, writing our topological lemma for \tilde{D} and taking the limit, we obtain

$$-\int_D X^*(d\Pi) + \lim_{\varepsilon \rightarrow 0} \int_{\tilde{N}(\partial \tilde{D})} \Pi = \mathcal{X}(D)$$

with the obvious notation.

The term concerning the boundary becomes

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\tilde{N}(\partial \tilde{D})} \Pi &= \lim_{\varepsilon \rightarrow 0} \int_{\tilde{\gamma}_1} \tilde{N}^*(\Pi) + \lim_{\varepsilon \rightarrow 0} \int_{\tilde{\gamma}_2} \tilde{N}^*(\Pi) \\ &= \int_{\gamma} N^*(\Pi) + \frac{1}{L(x_0)} \int_{N_{\tau_2}^{\tau_1}} \omega_1^2. \end{aligned}$$

We compute now the second integral in the sum above.

Note that we are now integrating on the segment $N_{\tau_2}^{\tau_1}$, where there is no variation of x ; therefore the integrand reads

$$\omega_1^2 = \frac{\sqrt{g}}{F^2}(y^1 \delta y^2 - y^2 \delta y^1) = \frac{\sqrt{g}}{F^2}(y^1 \dot{y}^2 - y^2 \dot{y}^1) d\tau,$$

where $y^i = y^i(\tau)$ is a unit speed parameterization of the indicatrix Σ_{x_0} and $\{dx^i, \frac{1}{F} \delta y^i\}$ is the dual cobasis of the adapted basis $\{\frac{\delta}{\delta x^i}, F \frac{\partial}{\partial y^i}\}$. Here $\delta y^i = dy^i + N_j^i dx^j$ (see [BCS, p. 96] for details).

Recall that the tangent vector to the indicatrix is given by

$$\lambda = \dot{y}^1 \frac{\partial}{\partial y^1} + \dot{y}^2 \frac{\partial}{\partial y^2}.$$

Therefore, we have

$$\begin{aligned} \int_{N_{\tau_2}^{\tau_1}} \omega_1^2 &= \int_{\tau_1}^{\tau_2} \omega_1^2(\lambda) d\tau = \int_{\tau_1}^{\tau_2} \frac{\sqrt{g}}{F^2}(y^1 \dot{y}^2 - y^2 \dot{y}^1) d\tau \\ (5.7) \qquad &= \int_{\tau_1}^{\tau_2} d\theta = \angle_{x_0}(N^-, N^+). \end{aligned}$$

Putting all these together, we obtain the following main result.

THEOREM 5.1 (THE GAUSS-BONNET THEOREM FOR FINSLER SURFACES WITH REGULAR C^∞ PIECEWISE BOUNDARY)

Let (M, F) be a compact oriented Finslerian surface, and let $D \in M$ be a domain with regular piecewise C^∞ -boundary $\partial D = \gamma$, which consists of the union of k piecewise smooth curves. Let $N : \partial D \rightarrow \Sigma$ be the inward-pointing Finslerian unit normal on ∂D .

Then, we have

$$\begin{aligned} (5.8) \qquad & \int_D \frac{1}{L(x)} [X^*(K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3) - d \log L(x) \wedge X^*(\omega^3)] \\ & + \int_\gamma \frac{1}{L(x)} \frac{k_T^{(N)}(t)}{\sigma(t)} dt + \sum_{s=1}^k \frac{1}{L(x_s)} \angle_{x_s}(N^-, N^+) = \mathcal{X}(D), \end{aligned}$$

where $L(x)$ is the Riemannian length of the indicatrix Σ_x , X is a unit prolongation of N , K is the Gauss curvature, $\angle_{x_s}(N^-, N^+)$ is the Landsberg angle of the unit vectors N^- and N^+ , and $\mathcal{X}(D)$ is the Euler characteristic of D .

REMARKS

(1) If (M, F) is a Riemannian manifold, then the Gauss-Bonnet theorem formulated above reduces to the classical Gauss-Bonnet theorem on Riemannian manifolds. Indeed, it suffices to remark that, in the Riemannian case, the Euclidean angle $\angle_{x_s}(T^-, T^+)$ equals the angle $\angle_{x_s}(N^-, N^+)$ which is also an Euclidean angle. Nevertheless, in the Riemannian case, the sum of interior and exterior angles at a corner equals π , but this is not the case anymore in the Finslerian case, as already discussed in §2.

(2) If D is a domain with regular piecewise C^∞ -boundary $\partial D = \gamma$ on a Landsberg surface (M, F) , then we obtain

$$(5.9) \quad \frac{1}{L} \int_D K \sqrt{g} dx^1 \wedge dx^2 + \frac{1}{L} \int_\gamma \frac{k_T^{(N)}(t)}{\sigma(t)} dt + \frac{1}{L} \sum_{s=1}^k \angle_{x_s}(N^-, N^+) = \mathcal{X}(D)$$

with the obvious notation.

6. A Hadamard-type theorem for N -parallels

We discuss here an application of the Gauss-Bonnet formula (5.9) for Landsberg surfaces.

In Riemannian geometry it is known that the Gauss-Bonnet theorem imposes restrictions on the behavior of geodesics. Namely, the Hadamard theorem states that on a simply connected Riemannian surface of nonpositive Gauss curvature $K \leq 0$, a geodesic cannot have self-intersections.

We prove a similar result for the N -parallels of a Landsberg surface. First, note the following.

LEMMA 6.1

Let x_0 be a point on M , and let us denote by $\Sigma_{x_0} \in T_{x_0}M$ the indicatrix curve of (M, F) at x_0 . Then we have

$$(6.1) \quad \frac{1}{L(x_0)} \angle_{x_s}(N^-, N^+) < 1,$$

where $L(x_0)$ is the Riemannian length of the indicatrix Σ_{x_0} and $\angle_{x_s}(N^-, N^+)$ is the Landsberg angle of the unit length vectors N^-, N^+ .

The proof is trivial. For a positive orientation, the Riemannian length of the indicatrix arc $N^- \widehat{N}^+$ at x_0 is always smaller than the total length of the indicatrix Σ_{x_0} (see Figure 3).

We can now give the following.

THEOREM 6.2

On a simply connected Landsberg surface (M, F) of nonpositive Gauss curvature $K \leq 0$, the N -parallels cannot have self-intersections.

Proof

Let us assume that the N -parallel $\gamma : [a, b] \rightarrow M$ can have self-intersections, and let us denote such a point by x_0 .

This is equivalent to saying that on M we have a domain D with close regular piecewise C^∞ -boundary $\partial D = \gamma$. The boundary curve on M is an N -parallel having a corner at x_0 .

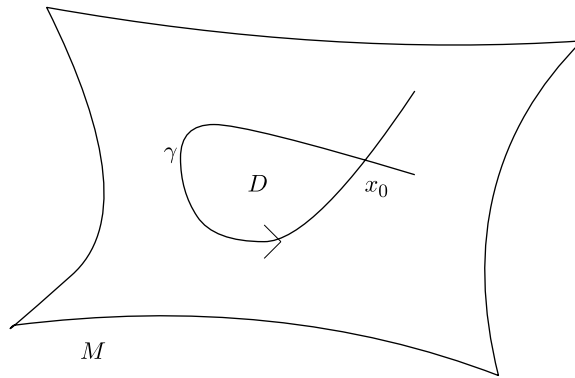


Figure 5. A self-intersecting N -parallel curve

Applying now the Gauss-Bonnet formula (5.9) for the domain D with boundary $\partial D = \gamma$, we obtain

$$(6.2) \quad \frac{1}{L} \int_D K \sqrt{g} dx^1 \wedge dx^2 + \frac{1}{L} \angle_{x_0}(N^-, N^+) = 1,$$

where N^- , N^+ are the left and right normals, respectively, to the boundary in the point x_0 , as before.

One can see now that this formula leads to a contradiction, showing in this way that the assumption is false. Indeed, since $K \leq 0$ is a nonpositive function, the integral in the left-hand side of (6.2) is nonpositive. On the other hand, from Lemma 6.1 we know that the second term in the sum in the left-hand side of (6.2) is less than 1. But this is not possible; therefore we have reached a contradiction.

It follows that the N -parallel curve γ cannot have a self-intersection; in other words, the situation in Figure 5 cannot happen. \square

REMARKS

(1) Recall that *Euler's theorem for polyhedra* states that for any triangulation of a compact surface M , the Euler characteristic of M is given by

$$(1) \quad \mathcal{X}(M) = \# \text{vertices} - \# \text{edges} + \# \text{faces},$$

where the symbol $\#$ means the number of. In particular, if we have a bounded region D on a simply connected surface M as in Figure 5, then D is homeomorphic to a triangle; that is,

$$(2) \quad \mathcal{X}(D) = \# \text{vertices} - \# \text{edges} + \# \text{faces} = 3 - 3 + 1 = 1.$$

This is the reason we have 1 in the right-hand side of (6.2).

(2) There is a second part of the Hadamard theorem that states that on any simply connected Riemannian surface of nonpositive Gauss curvature $K \leq 0$, two distinct geodesics cannot have two points of intersection. This kind of result also extends to the case of N -parallels, but it is a little more complicated and is going

to be discussed in a forthcoming article together with other applications of the Gauss-Bonnet theorem.

Appendix. The existence and unicity of N -parallels

Formula (3.16) is useful for the study of existence and unicity of the N -parallels of a Finsler surface (M, F) .

Indeed, following an idea of M. Matsumoto [M1] from the conditions that define the N -parallels, namely,

$$F(x, N) = 1, \quad g_N(N, T) = 0$$

or, equivalently,

$$\begin{aligned} g_{ij}(x, N) \cdot N^i N^j &= 1, \\ g_{ij}(x, N) \cdot N^i T^j &= 0, \end{aligned}$$

where $i, j = 1, 2$, it follows that

$$[g_{i1}(x, N) \cdot N^i]T^1 + [g_{i2}(x, N) \cdot N^i]T^2 = 0.$$

From here, it follows that there exists a positive scalar k such that

$$-\frac{g_{i1}(x, N) \cdot N^i}{T^2} = \frac{g_{i2}(x, N) \cdot N^i}{T^1} = k > 0$$

(or with opposite signs) and therefore

$$\begin{aligned} g_{i1}(x, N) \cdot N^i &= -k \cdot T^2, \\ g_{i2}(x, N) \cdot N^i &= k \cdot T^1. \end{aligned}$$

Using now the zero-homogeneity of g_{ij} , we obtain the equations

$$(7.1) \quad \begin{aligned} g_{i1}(x, p) \cdot p^i &= -T^2, \\ g_{i2}(x, p) \cdot p^i &= T^1, \end{aligned}$$

where $i = 1, 2$ and p is a vector proportional to N .

Taking into account that the Jacobian of the equation (7.1) is just

$$\det |g_{ij}(x, p)| \neq 0,$$

it follows by the theorem of implicit functions that we can solve these equations with respect to the unknowns p^1, p^2 .

Finally, we can put

$$(7.2) \quad N^i := \frac{p^i}{F(x, p)}, \quad i = 1, 2.$$

One can easily see that this $N = (N^i)$ satisfies condition (3.1).

We point out that the solutions N^1, N^2 of the equation (7.1) depend actually on T .

We can rewrite (3.16) as

$$(7.3) \quad \frac{d^2 \gamma^i}{dt^2} + \Gamma_{jk}^i(\gamma(t), N(t)) \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = \frac{d}{dt} [\log \sigma(t)] \frac{d\gamma^i}{dt},$$

where $N(t) = N(\gamma(t), \dot{\gamma}(t))$ from (7.2).

An initial condition can be given by

$$(7.4) \quad \begin{aligned} \gamma^i(t_0) &= 0, \\ \dot{\gamma}^i(t_0) &= T_0^i, \end{aligned}$$

with $i = 1, 2$ and corresponding to the normal initial condition

$$(7.5) \quad \gamma^i(t_0) = 0, \quad N^i(t_0) = N_0,$$

where N_0 are given as solutions of (7.1) for $T = T_0$.

Then, by an argument similar to that in the case of geodesics, we know from the general theory of ODEs that (7.3) with initial conditions (7.4) have unique solutions.

A detailed study of the N -parallels will be given elsewhere.

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