ON TENSOR PRODUCTS OF BANACH SPACES

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Recently in his interesting paper [3] A. Grothendiek has successfully developed a theory of tensor products on Banach space, which gives a wide-scope to his previous work on topological vector spaces [1]. However, except two fundamental theorems, he has given no demonstrations to his results, some of which deserve our attention and demand non-trivial methods of the proof. Therefore it will not be altogether meaningless to give the proofs of them here (maybe different from the original ones), though there is nothing essentially new to the theory itself, except a slightly better results in respect to Proposition 3 in § 3 of [3], showing that we have $\mathcal{H} \leqq \mathcal{H}'$ in place of $\mathcal{H} \leqq 2\mathcal{H}'$.

We do not refer to some results of [3] which can be easily proved; nor do we refer to any results stated after § 3, n° 4 in the cited paper, because, as Grothendieck himself has remarked in it, they are easily checked according to his directions.

§ 1. Tensor norms.

1. Preliminaries.

For the convenience of the reader, we first sketch the fundamental definitions and notations of the original paper. Let E and F be Banach spaces. A norm α given on the tensor product $E \otimes F$ is called *reasonable* if it satisfies $\alpha(x \otimes y) = \|x\| \cdot \|y\|$, $(x \in E,\ y \in F)$ and $\alpha'(x' \otimes y') = \|x'\| \cdot \|y'\|$, $(x' \in E',\ y' \in F')$, where α' denotes the dual norm on the $E' \otimes F'$; the elements of $E' \otimes F'$ is considered in the dual space of $E \otimes F$. For a given reasonable norm α, $E \overset{\alpha}{\otimes} F$ means by definition the Banach space which is the completion of $E \otimes F$ by the norm α. Then on the $E \otimes F$ there exist the smallest reasonable norm $\vee$ and the greatest one $\wedge$. Specifically, $\vee$ and $\wedge$ are defined by the following:

$$(1) \qquad |u|_{\vee} = \sup_{\substack{\|x'\| \leqq 1 \\ \|y'\| \leqq 1}} |< u, x' \otimes y' >|,^{1)}$$

$$(2) \qquad |u|_{\wedge} = \sup_{\substack{v \in B(E,F) \\ \|v\| \leqq 1}} |< u, v >|.$$

From the definition it follows that the dual space of $E \overset{\wedge}{\otimes} F$ is $B(E, F)$.[2]

A normed space E is called a *numerical normed space* if the underlying

Received September 12, 1957.

1) Notation $\vee$ has the same meaning as $\curlywedge$ which is used in [1].

2) $B(E, F)$ denotes the Banach space consisting of all the continuons bilinear forms on $E \times F$ with the bilinear norm.

vector-space is isomorphic to an R^n or a C^n [3]; the set of numerical normed space is denoted by $\mathfrak{N}$. An object α which is defined on all ordered paires (E, F), $E, F \in \mathfrak{N}$, is called a *tensor norm* (notation: $\otimes$-norm), when it satisfies the following conditions: 1°. α induces a reasonable norm of $E \otimes F$; 2°. Let u_i be linear mappings of E_i into $F_i (i = 1, 2; E_i, F_i \in \mathfrak{N})$. Then the tensor product $u_1 \overset{\alpha}{\otimes} u_2$, regarded as a mapping of $E_1 \overset{\alpha}{\otimes} E_2$ into $F_1 \overset{\alpha}{\otimes} F_2$, fulfills the norm relation $\| u_1 \overset{\alpha}{\otimes} u_2 \| \leqq \| u_1 \| \cdot \| u_2 \|$. In this case we denote by $E \overset{\alpha}{\otimes} F$ the space $E \otimes F$ with the norm α.

For a given $\otimes$-norm α, ${}^t\alpha$ is defined by $E \overset{{}^t\alpha}{\otimes} F = F \overset{\alpha}{\otimes} E$. ${}^t\alpha$ gives rise obviously to a $\otimes$-norm, which is called the *transposed $\otimes$-norm* of α. In case where $\alpha = {}^t\alpha$, α is called *symmetric*; hence the symmetry of α means $E \overset{\alpha}{\otimes} F$ and $F \overset{\alpha}{\otimes} E$ are canonically isomorphic for all E, $F \in \mathfrak{N}$. Now consider $E \otimes F$ as the dual space of $E' \otimes F'$; then the dual norm α' on $E \otimes F$, induced by $E' \overset{\alpha}{\otimes} F'$, gives a new $\otimes$-norm. α' is called the *dual $\otimes$-norm* of α. It is easily seen that ${}^t({}^t\alpha) = \alpha$, $(\alpha')' = \alpha$ and ${}^t(\alpha') = ({}^t\alpha)'$. We put $\check{\alpha} = {}^t(\alpha')$.

For tensor norms α, β and for a positive number λ, $\alpha \leqq \lambda\beta$ is by definition $|u|_\alpha \leqq \lambda |u|_\beta$ for all $u \in E \otimes F \, (E, F \in \mathfrak{N})$. In particular, the relation $\alpha \leqq \beta$ induces an ordered-relation in the set of $\otimes$-norms. The set of all $\otimes$-norms forms a complete lattice. It is easily verified that the reasonable norms $\vee$ and $\wedge$ naturally induce $\otimes$-norms; besides, $\vee$ is the smallest and $\wedge$ is the greatest $\otimes$-norm. Also it is evident that $\vee$ and $\wedge$ are symmetric. We have $(\vee)' = \wedge$ and $(\wedge)' = \vee$.

In the preceding paragraphs the tensor norm has only been considered for numerical spaces. We define it for all Banach spaces. Let α be a $\otimes$-norm; and let E and F be any two Banach spaces. Consider the set $\mathfrak{E}$ and $\mathfrak{F}$ which consist of all finite-dimensional subspaces of E and F, respectively. Then $\{M \otimes N; M \in \mathfrak{E}, N \in \mathfrak{F}\}$, the family of finite-dimensional subspaces of $E \otimes F$, forms a filter by inclusion order and $E \otimes F = \bigcup M \otimes N$. Let u be an element of $E \otimes F$, belonging to an $M \otimes N$. $|u|_{M \overset{\alpha}{\otimes} N}$ denotes the norm of u in $M \overset{\alpha}{\otimes} N$. The condition 2° of the $\otimes$-norm, applied to injection mapping, shows that if $(M, N) \subset (M_1, N_1)$, then $|u|_{M \overset{\alpha}{\otimes} N} \geqq |u|_{M_1 \overset{\alpha}{\otimes} N_1}$. Hence

$$|u|_\alpha = \operatorname{Inf}_{M,N} |u|_{M \overset{\alpha}{\otimes} N}$$

is well-defined; furthermore, $|u|_\alpha$ is actually a reasonable norm of $E \otimes F$. The reasonable norm α, obtained by the above procedure, is called a *$\otimes$-norm* of the Banch spaces E and F. In particular, if α is $\vee$ or $\wedge$, then $|u|_\alpha$ is the same as we have defined by (1) and (2), respectively, so that the notation is compatible.

Let $E_i, F_i \, (i = 1, 2)$ be Banach spaces and u_i be the continuous linear mapping of E_i into F_i. Then it is evident that $u_1 \otimes u_2 \colon E_1 \otimes E_2 \to F_1 \otimes F_2$ induces a continuous linear mapping $u_1 \overset{\alpha}{\otimes} u_2$ of $E_1 \overset{\alpha}{\otimes} E_2 \to F_1 \overset{\alpha}{\otimes} F_2$ and that $\| u_1 \overset{\alpha}{\otimes} u_2 \|$

3) R and C denote the real number field and complex number field, respectively.

$\leqq \|u_1\|\cdot\|u_2\|$.

The dual Banach space of $E\overset{\alpha'}{\otimes}F$ is denoted by $B^\alpha(E,F)$. Since $|u|_\alpha \leqq |u|_\wedge$ and the dual space of $E\overset{\wedge}{\otimes}F$ is $B(E,F)$, the element of $B^\alpha(E,F)$ is in a natural way regarded as an element of $B(E,F)$. A bilinear form A on $E\times F$ is called *type* α if $A\in B^\alpha(E,F)$; the norm of A in $B^\alpha(E,F)$ is denoted by $\|A\|_\alpha$. $L(E;F)$[4] being canonically isomorphic to $B(E,F')$, the correspondig definition to type α is possibly transferred to the elements of $L(E;F)$; the space of all linear mappings of type α, endowed with the norm $\|A\|_\alpha$, is written by $L^\alpha(E;F)$. "Type $\wedge$" is often replaced by the adjective "*integral*".

The following fact is an easy consequence of the definition of $\otimes$-norm. Let $A\in B^\alpha(E,F)$, and let E_1 and F_1 be Banach spaces. Assume that the continuous linear mappings u: $E_1\to E$ and v: $F_1\to F$ are given. We define the form $A\circ(u\otimes v)$ on $E_1\times F_1$ by $A\circ(u\otimes v)(x,y)=A(ux,vy)$. Then we have $A\circ(u\otimes v)\in B^\alpha(E_1,E_1)$ and $\|A\circ(u\otimes v)\|_\alpha \leqq \|A\|_\alpha\cdot\|u\|\cdot\|v\|$. In particular, for any subspaces $E_1\subset E$ and $F_1\subset F$, we have

$$A\,|\,E_1\times F_1\in B^\alpha(E_1,F_1)$$

and

$$\|A\,|\,E_1\times F_1\|_\alpha \leqq \|A\|_\alpha,$$

where $A\,|\,E_1\times F_1$ means the restriction of A to $E_1\times F_1$.

2. Accessible $\otimes$-norms.

Let E and F be Banach spaces. Given a $\otimes$-norm α, we can construct $E\overset{\alpha}{\otimes}F$ and $B^\alpha(E',F')$, both of which contain canonically $E\otimes F$. For $u\in E\otimes F$, the norms of u considered in each space are denoted by $|u|_\alpha$ and $\|u\|_\alpha$, respectively. Then

$\|u\|_\alpha \leqq |u|_\alpha$. *If E and F are metrically accessible,*[5] *then* $\|u\|_\alpha = |u|_\alpha$.

Proof. Let E_1 and F_1 be any finite-dimensional subspaces of E and F such that $u\in E_1\otimes F_1$, and let ι_1 and ι_2 be the injection mappings $E_1\to E$ and $F_1\to F$, respectively. Then ${}^t\iota_1$ is a linear mapping $E'\to E_1'$ with norm one; the same holds for F_1', F' and ${}^t\iota_2$. Consider $u\,|\,E_1'\times F_1'$, the restriction of u to $E_1'\times F_1'$. Then we have $\|u\,|\,E_1'\times F_1'\|_\alpha = |u\,|\,E_1'\times F_1'|_\alpha = |u|_{E_1\overset{\alpha}{\otimes}F_1}$. On the other hand, $\|u\,|\,E_1'\times F_1'\circ({}^t\iota_1\otimes{}^t\iota_2)\|_\alpha \leqq \|u\,|\,E_1'\times F_1'\|_\alpha$, since $\|{}^t\iota_1\otimes{}^t\iota_2\|\leqq 1$. It is clear that the left member is equal to $\|u\|_\alpha$. Hence we have $\|u\|_\alpha \leqq |u|_{E_1\overset{\alpha}{\otimes}F_1}$ which shows $\|u\|_\alpha \leqq |u|_\alpha$ by the definition of $|u|_\alpha$.

Now we assume that E and F are metrically accessible. We wish to show $\|u\|_\alpha = |u|_\alpha$. For that purpose, it is sufficient to prove $\|u\|_\alpha > 1$ whenever $|u|_\alpha > 1$.

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4) $L(E;F)$ denote the Banach space consisting of all the continuous linear mappings from E into F with the usual norm.

5) As to the notion "metrically accesible", see [1], Chap. 1, Def. 10.

Write $u = \sum_{i=1}^n x_i \otimes y_i$. Put $M = \mathrm{Max}_{1 \leq i \leq n}\{\|x_i\|, \|y_i\|\}$, and let ε be any positive number smaller than $|u|_\alpha - 1$. Since E and F are metrically accessible, we can find a linear mapping φ_1 of E into a finite-dimensional subspace E_1 and a φ_2 of F into F_1, such that

$$\|x_i - \varphi_1 x_i\| < \frac{\varepsilon}{2nM},$$
$$\|y_i - \varphi_2 y_i\| < \frac{\varepsilon}{2nM}$$

and that the norm of $\varphi_i\ (i = 1, 2)$ is $\leqq 1$. Without losing generality, we may obviously assume $u \in E_1 \otimes F_1$. We have

$$\begin{aligned} &|u - (\varphi_1 \otimes \varphi_2)\, u|_{E_1 \overset{\alpha}{\otimes} F_1} \\ &= |\textstyle\sum (x_i - \varphi_1 x_i) \otimes y_i + \sum \varphi_1 x_i \otimes (y_i - \varphi_2 y_i)|_{E_1 \overset{\alpha}{\otimes} F_1} \\ &\leqq \textstyle\sum \|x_i - \varphi_1 x_i\| \cdot \|y_i\| + \sum \|\varphi_1 x_i\| \cdot \|y_i - \varphi_2 y_i\| \\ &< \varepsilon < |u|_\alpha - 1. \end{aligned}$$

Besides, we have $|u|_{E_1 \overset{\alpha}{\otimes} F_1} \geqq |u|_\alpha > 1$. Hence we obtain

$$|(\varphi_1 \otimes \varphi_2)\, u|_{E_1 \overset{\alpha}{\otimes} F_1} > 1.$$

Accordingly, there is a $v_1 \in E_1' \otimes F_1'$ such that $|v_1|_{\alpha'} = 1$ and that

$$(3) \qquad |<(\varphi_1 \otimes \varphi_2)\, u, v_1>| > 1.$$

Put $v = ({}^t\varphi_1 \otimes {}^t\varphi_2)\, v_1$. Then $v \in E' \overset{\alpha'}{\otimes} F'$ and $|v|_{\alpha'} \leqq 1$, because $\|{}^t\varphi_1 \otimes {}^t\varphi_2\| \leqq 1$. Moreover, by (3) we have

$$|<u, v>| > 1,$$

which gives $\|u\|_\alpha > 1$. This completes the proof.

A tensor norm α is called *accessible*, if $\|u\|_\alpha = |u|_\alpha$ always holds under the assumption that E or F is finite-dimensional. In case where α is accessible, the above proof remains valid when φ_1 or φ_2 is replaced by the identity operator. Hence we get

If α is an accessible $\otimes$-norm, $\|u\|_\alpha = |u|_\alpha$ holds when E or F is metrically accessible.

If α is accessible, then α' is accessible. This is simply a translation of the following fact to the dual spaces: the accessibility of α means that the canonical mapping $E' \overset{\alpha}{\otimes} F' \to B^\alpha(E, F)$ is an isomorphism onto when E is finite-dimensional. It is trivial that under the same assumption of α, ${}^t\alpha$ and $\check{\alpha}$ are also accessible. Finally we note that it is always valid $\|u\|_\vee = |u|_\vee$. Thus the $\otimes$-norm $\wedge$ is accessible.

3. Canonical prolongation.

$B(E, F)$ is canonically isomorphic to $L(E; F')$. Since F' is in an obvious way imbedded into $(F')''$, $u \in L(E; F')$ is regarded as an element of $L(E; (F')'')$.

This fact can be interpreted in terms of $B(E,F)$; thus $A \in B(E,F)$ corresponds canonically to an element $\tilde{A} \in B(E,F'')$. $\tilde{A}$ is called the *canonical prolongation* of A.

THEOREM 1. *For a given $A \in B(E,F)$, let $\tilde{A}$ be the canonical prolongation of A: $\tilde{A} \in B(E,F'')$. Then in order that A is of type α it is necessary and sufficient that $\tilde{A}$ is of type α. Furthermore we have*

$$\| A \|_\alpha = \| \tilde{A} \|_\alpha .$$

LEMMA. *If E is finite-dimensional, then $L(E;F'') \cong L(E;F)''$ (canonically) for any Banach space F.*

Proof of LEMMA. Since E is finite-dimensional, $L(E;F) = E' \overset{\vee}{\otimes} F$. The $\otimes$-norm $\wedge$ being accessible, we have $L(E;F)' \cong (E' \overset{\vee}{\otimes} F)' = E \overset{\wedge}{\otimes} F'$. Hence we have $L(E;F)'' \cong (E \overset{\wedge}{\otimes} F')' = B(E,F') = L(E;F'')$.

Proof of THEOREM. Sufficiency of the condition and $\| A \|_\alpha \leqq \| \tilde{A} \|_\alpha$ are clear. We must prove the necessity and the converse inequality. Assume that A is of type α and that $\| A \|_\alpha = 1$. We wish to show $\| \tilde{A} \|_\alpha \leqq 1$. Let E_1 and F_1 be any finite-dimensional subspaces of E and F'', whose basis, suitably chosen, are denoted by $\{x_1, \cdots, x_m\}$ and $\{y_1'', \cdots, y_n''\}$, respectively. Now apply Lemma to F_1 and F''. We have $L(F_1;F'') \cong L(F_1;F)''$ (canonically). It follows that the injection operator $\iota\colon F_1 \to F''$ in $L(F_1;F)''$ is weakly approximable by elements belonging to the unit sphere of $L(F_1;F)$, where by the weak topology we mean the one induced by the duality between $F_1 \overset{\wedge}{\otimes} F'$ $(= L(F_1;F)')$ and $L(F_1;F)''$.

We identify $A \in B(E,F)$ with an element of $L(E;F')$ and so $Ax_i (i = 1, \cdots, m)$ are the elements of F'. Consider the $y_j'' \otimes Ax_i (i = 1, \cdots, m; j = 1, \cdots, n)$, which lie in the dual space $F_1 \otimes F'$ of $L(F_1;F)$. Then from the above we find that there is a $u \in L(F_1;F)$ such that $\| u \| \leqq 1$ and that

$$< u, y_j'' \otimes Ax_i > = < \iota, y_j'' \otimes Ax_i > \qquad (i = 1, \cdots, m; j = 1, \cdots, n).$$

It is easily seen that this implies

$$A(x_i, u(y_j'')) = A(x_i, y_j'') \qquad (i = 1, \cdots, m; j = 1, \cdots, n). \tag{4}$$

Let F_2 be the subspace of F spanned by $u(y_j'')$, $j = 1, \cdots, m$. We see that $1 \otimes u$ induces the linear mapping

$$E_1 \overset{\alpha'}{\otimes} F_1 \to E_1 \overset{\alpha'}{\otimes} F_2$$

with the norm $\leqq 1$, and that by (4)

$$\tilde{A} \mid E_1 \times F_1 = A \mid E_1 \times F_2 \circ (1 \otimes u).$$

Therefoe we obtain

$$\| \tilde{A} \mid E_1 \times F_1 \|_\alpha \leqq \| A \mid E_1 \times F_2 \|_\alpha \leqq \| A \|_\alpha = 1,$$

whence we have $\| \tilde{A} \|_\alpha \leqq 1$. This completes the proof.

From the polarization we have

COROLLARY. *The canonical injection $E \overset{\alpha}{\otimes} F \to E \overset{\alpha}{\otimes} F''$ is an isomorphism (into).*

4. Relations between α-mappings and $\check{\alpha}$-mappings.

THEOREM 2. *Let $u \in L^{\alpha}(E; F)$ and $v \in L^{\check{\alpha}}(F; G)$. Suppose that α is an accessible $\otimes$-norm, or that F is metrically accessible. Then $v \circ u$ is an integral operator which satisfies*

$$\| v \circ u \|_{\wedge} \leqq \| v \|_{\check{\alpha}} \| u \|_{\alpha}.$$

Proof. As $\| v \circ u \|_{\wedge}$ (or $\| u \|_{\alpha}$) is the supremum of the $\wedge$-norms of $v \circ u$ (resp. α-norms of u) restricted to finite-dimensional subspaces of E, we may assume E to be finite-dimensional; hence we have $u \in E' \otimes F$. Observe that the assumption on α or on F now gives $\| u \|_{\alpha} = | u |_{\alpha}$. For every $x \in E$ and $z' \in G'$, we have

$$< x \otimes z', v \circ u > = < v \circ u\,(x), z' > = < ux, {}^{t}vz' >$$
$$= < u, x \otimes {}^{t}vz' > = < u, {}^{t}v \circ (x \otimes z') >,$$

where in the last bracket $x \otimes z'$ is regarded as an operator of E' into G'. Hence for any $w \in E \otimes G'$ with $| w |_{\vee} \leqq 1$, we have

$$| < w, v \circ u > | = | < u, {}^{t}v \circ w > | \leqq | u |_{\alpha} \| {}^{t}v \circ w \|_{\alpha'}$$
$$\leqq | u |_{\alpha} \| {}^{t}v \|_{\alpha'},$$

because $| w |_{\vee} \leqq 1$ is nothing but $\| w \| \leqq 1$, if w is considered as an element of $L(E'; G')$. Thus, by $\| u \|_{\alpha} = | u |_{\alpha}$, we obtain

$$| < w, v \circ u > | \leqq \| u \|_{\alpha} \| {}^{t}v \|_{\alpha'}.$$

By the definition of $\| v \circ u \|_{\wedge}$, this yields the desired result.

§ 2. Projective and injective $\otimes$-norms.

1. Banach spaces of class (C) and class (L).

According to Grothendieck [2], we say a Banach space E to be of *class* (L) or *(L)-space*, if, for any Banach space G and its closed subspace F, the canonical injection $E \hat{\otimes} F \to E \hat{\otimes} G$ is an isomorphism; we say a Banach space E to be of *class* (C) or *(C)-space*, if the dual E' is of class (L).

If E is of class (L), then the dual E' is of class (C). If E is isomorphic to an $L^1(\mu)$ for a suitable measure space on a locally compact space, then E is of class (L). Thus, by a result due to Kakutani, we see that the usual Banach space, composed of all continuous functions vanishing at infinity on a locally compact space, is of class (C). These lead to the fundamental fact that any Banach space is on the one hand regarded as a subspace of a (C)-space and on the other hand as a quotient space of an (L)-space. At least in case where the scalar field is R, the notion of class (L) is known to be essentially equivalent to the one of $L^1(\mu)$. In what follows, we mean by the notations L and C Banach spaces of class (L) and (C), respectively. From the definition, the following results are immediately obtained [2]:

1) *Let E be a Banach space and F its closed subspace. Then for any continuous linear mapping $u \in L(F; C)$, there exists a $\bar{u} \in L(E; C'')$ which satisfies: $\bar{u} | F = u$ and $\| \bar{u} \| = \| u \|$.*

2) *Let E and F be as above. Then for any continuous linear mapping $u \in L(L; E/F)$, there exists a $\bar{u} \in L(L; E'')$ which satisfies: $p \circ \bar{u} = u$ and $\|\bar{u}\| = \|u\|$, where p denotes the canonical homomorphism $E'' \to E''/F^{\circ\circ}(\supset E/F)$.*

2. Injective and projective $\otimes$-norms.

A tensor norm α is called *left-injective* (abbreviated: l-injective) if, for any Banach spaces E, G and a closed subspace F of E, the canonical injection

$$F \overset{\alpha}{\otimes} G \to E \overset{\alpha}{\otimes} G$$

is an isomorphism. α is l-injective if and only if the above property holds when E, F and G are numerical normed spaces. α is called *right-injective* (abbreviated: r-injective) if ${}^t\alpha$ is l-injective. A left- and right-injective $\otimes$-norm α is simply called *injective*. α is called *left-projective* (abbreviated: l-projective) if α' is l-injective; in a similar way, the *r-projective* and the *projective $\otimes$-norms* are defined.

It is easily seen that $\vee$ is injective and $\wedge$ is projective.

The supremum of any family of l-injective (resp. r-injective) $\otimes$-norms is l-injective (resp. r-injective). Hence for any $\otimes$-norm α the following definition is meaningful:

$$/\alpha = \sup_{\substack{\beta \leqq \alpha \\ \beta : l\text{-inj.}}} \beta,$$

$$\alpha\backslash = \sup_{\substack{\beta \leqq \alpha \\ \beta : r\text{-inj.}}} \beta.$$

$/\alpha$ is l-injective and $\alpha\backslash$ is r-injective. Correspondingly, we put

$$\backslash\alpha = \inf_{\substack{\beta \geqq \alpha \\ \beta : l\text{-proj.}}} \beta,$$

$$\alpha/ = \inf_{\substack{\beta \geqq \alpha \\ \beta : r\text{-proj.}}} \beta.$$

Then $\backslash\alpha$ is l-projective and $\alpha/$ is r-projective.

THEOREM 3. *For any Banach space E, we have*

$$C \overset{/\alpha}{\otimes} E = C \overset{\alpha}{\otimes} E,$$

$$L \overset{\backslash\alpha}{\otimes} E = L \overset{\alpha}{\otimes} E.$$

Proof. We shall prove $C \overset{/\alpha}{\otimes} E = C \overset{\alpha}{\otimes} E$. For this purpose, we need a lemma:

LEMMA. *Let C be a Banach space of class (C). Suppose that a Banach space G contains the C as a closed subspace. Then for any Banach space E the canonical injection*

$$C \overset{\alpha}{\otimes} E \to G \overset{\alpha}{\otimes} E$$

is an isomorphism.

Proof of LEMMA. Consider the identity mapping $\iota: C \to C$. From the property of C as stated in 1) of the preceding section, it follows that there is a $p \in L(G; C'')$, satisfying $p|C = \iota$ and $\|p\| = 1$. Then $p \otimes 1$ induces a linear mapping

$$G \overset{\alpha}{\otimes} E \to C'' \overset{\alpha}{\otimes} E,$$

whose norm is $\leqq 1$. On the other hand, let ι be the injection $C \to G$; then $\iota \otimes 1$ induces a linear mapping

$$C \overset{\alpha}{\otimes} E \to G \overset{\alpha}{\otimes} E,$$

whose norm is $\leqq 1$. Hence $(p \otimes 1) \circ (\iota \otimes 1)$ gives rise to a linear mapping

$$C \overset{\alpha}{\otimes} E \to C'' \overset{\alpha}{\otimes} E.$$

Besides, it is obvious that $\|(p \otimes 1) \circ (\iota \otimes 1)\| \leqq 1$ and that, if $u \in C \otimes E$, then the image of u is just the canonical image in $C'' \otimes E$. This, combined with Corollary to Theorem 1, yields that $(p \otimes 1) \circ (\iota \otimes 1)$ is norm-preserving, a fortiori $\iota \otimes 1$ has the same property. This completes the proof.

Now we come back to the proof of Theorem. Let E and F be any Banach spaces. Let C be a Banach space of class (C) which imbeds E. Put $E \overset{\tilde{\alpha}}{\otimes} F$ for the closed subspace of $C \overset{\alpha}{\otimes} F$, spanned by $E \otimes F$. We consider $E \overset{\tilde{\alpha}}{\otimes} F$ as a Banach space corresponding to the ordered pair (E, F). By Lemma we know that $E \overset{\tilde{\alpha}}{\otimes} F$ is not dependent on the choice of C and so uniquely determined up to an isomophism by (E, F). We denote by $|u|_{\tilde{\alpha}}$, $u \in E \overset{\tilde{\alpha}}{\otimes} F$, the norm of u in $E \overset{\tilde{\alpha}}{\otimes} F$.

We shall prove that $E \overset{\tilde{\alpha}}{\otimes} F$ actually gives a $\otimes$-norm to E and F. Let (E_1, F_1) be another pair of Banach spaces and suppose that the linear mappings $u_1 \in L(E; E_1)$ and $u_2 \in L(F; F_1)$ be given. We shall first see that $u_1 \otimes u_2$ induces a continuous linear mapping $u_1 \overset{\tilde{\alpha}}{\otimes} u_2 \colon E \overset{\tilde{\alpha}}{\otimes} F \to E_1 \overset{\tilde{\alpha}}{\otimes} F_1$. Let $E \subset C$ and $E_1 \subset C_1$ be the imbeddings into (C)-spaces of E and E_1, respectively. The mapping u_1, being regarded as one of E into C_1, has a norm-preserving extension $\tilde{u}_1$ of C into C_1'': $\|u_1\| = \|\tilde{u}_1\|$. Then the $\tilde{u}_1 \overset{\alpha}{\otimes} u_2$ gives rise to a continuous linear mapping: $C \overset{\alpha}{\otimes} F \to C_1'' \overset{\alpha}{\otimes} F$, which satisfies

$$\|\tilde{u}_1 \overset{\alpha}{\otimes} u_2\| \leqq \|\tilde{u}_1\| \cdot \|u_2\|.$$

Observe that $E_1 \overset{\tilde{\alpha}}{\otimes} F$ is considered as a closed subspace of $C_1'' \overset{\alpha}{\otimes} F$, because C'' is of class (C). Besides, the restriction of $\tilde{u}_1 \otimes u_2$ to $E \otimes F$ is the same mapping as $u_1 \otimes u_2$. Hence the definition of $\tilde{\alpha}$ gives

$$\|u_1 \overset{\tilde{\alpha}}{\otimes} u_2\| \leqq \|u_1\| \cdot \|u_2\|.$$

Now let u be an element of $E \otimes F$ belonging to an $E_\sigma \otimes F_\sigma$, where by E_σ and F_σ we mean finite-dimensional subspaces of E and F, respectively. We wish to show

$$|u|_{\tilde{\alpha}} = \inf \{|u|_{E_\sigma \overset{\tilde{\alpha}}{\otimes} F_\sigma};\ u \in E_\sigma \otimes F_\sigma,\ \ E_\sigma \subset E,\ \ F_\sigma \subset F\}$$

Suppose that E is imbedded into C. By the definition $|u|_{E_\sigma \overset{\tilde{\alpha}}{\otimes} F_\sigma} = |u|_{C \overset{\alpha}{\otimes} F_\sigma}$, whence we have $|u|_{E_\tau \overset{\alpha}{\otimes} F_\sigma} \geqq |u|_{E_\sigma \overset{\tilde{\alpha}}{\otimes} F_\sigma}$, where E_τ is any finite-dimensional subspace of C such that $u \in E_\tau \otimes F_\sigma$. So we have

$$|u|_{\tilde{\alpha}} = \inf_{E_\tau, F_\sigma} |u|_{E_\tau \overset{\alpha}{\otimes} F_\sigma} \geqq \inf_{E_\sigma, F_\sigma} |u|_{E_\sigma \overset{\tilde{\alpha}}{\otimes} F_\sigma}.$$

Since the converse inequality is trivial, we get $|u|_{\tilde{\alpha}} = \inf |u|_{E_\sigma \overset{\alpha}{\otimes} E_\sigma}$. From these properties of $\tilde{\alpha}$ we can easily conclude that $\tilde{\alpha}$ is a $\otimes$-norm.

From the defintion it follows that if E_1 is a closed subspace of E_2, then $E_1 \overset{\tilde{\alpha}}{\otimes} F$ is a closed subspace of $E_2 \overset{\tilde{\alpha}}{\otimes} F$, so that $\tilde{\alpha}$ is l-injective. Since for $E \subset C$ the canonical injection $E \overset{\alpha}{\otimes} F \to C \overset{\alpha}{\otimes} F$ is of norm $\leqq 1$, $\tilde{\alpha} \leqq \alpha$ is clear. We have thus

$$\tilde{\alpha} \leqq /\alpha.$$

Now let β be any l-injective $\otimes$-norm such that $\beta \leqq \alpha$. Then for any Banach spaces E, F and $C (\supset E)$, the canonical injection $E \overset{\beta}{\otimes} F \to C \overset{\beta}{\otimes} F$ is norm-preserving and the canonical injection $C \overset{\alpha}{\otimes} F \to C \overset{\beta}{\otimes} F$ is of norm $\leqq 1$. $E \overset{\tilde{\alpha}}{\otimes} F$ being defined as the closed subspace of $C \overset{\alpha}{\otimes} F$, we have $|u|_\beta \leqq |u|_{\tilde{\alpha}}$ for $u \in E \otimes E$. Hence $\beta \leqq \tilde{\alpha}$ and so

$$/\alpha = \sup \beta \leqq \tilde{\alpha}.$$

This combined with the above yields $\tilde{\alpha} = /\alpha$. Thus by the definition of $\tilde{\alpha}$ we have finally $C \overset{/\alpha}{\otimes} E = C \otimes E$.

The second assertion of Theorem with respect to $\backslash\alpha$ can be proved by the same lines as in $/\alpha$. The corresponding lemma in this case becomes as follows: If L is a quotient space of an E, then the canonical mapping $E \overset{\alpha}{\otimes} F \to L \overset{\alpha}{\otimes} F$ is an onto-homomorphism. This is an alternation of the fundamental property of L stated in 1), n° 1. Put $E \overset{\tilde{\alpha}}{\otimes} F$ for the quotient space of $L \overset{\alpha}{\otimes} F$ induced by the canonical injection $L \otimes F \to E \otimes F$, where E is assumed to be a quotient space of L. Then by the lemma and by the similar disscussions we can prove $\tilde{\alpha} = \backslash\alpha$ and hence $L \overset{\backslash\alpha}{\otimes} E = L \overset{\alpha}{\otimes} E$. This completes the proof.

COROLLARY 1. *$E \overset{/\alpha}{\otimes} F$, $E \overset{\alpha\backslash}{\otimes} F$ and $E \overset{/\alpha\backslash}{\otimes} F$ are identified with the closures of $E \otimes F$ in $C_1 \overset{\alpha}{\otimes} F$, $E \overset{\alpha}{\otimes} C_2$ and $C_1 \overset{\alpha}{\otimes} C_2 (E \subset C_1, F \subset C_2)$, respectively.*

COROLLARY 2. *$E \overset{\backslash\alpha}{\otimes} F$, $E \overset{\alpha/}{\otimes} F$ and $E \overset{\backslash\alpha/}{\otimes} F$ are identified with the quotient spaces of $L_1 \overset{\alpha}{\otimes} F$, $E \overset{\alpha}{\otimes} L_2$ and $L_1 \overset{\alpha}{\otimes} L_2$ by the canonical homomorphisms, respectively, where $E = L_1/K$ and $F = L_2/J$.*

Related to § 1, n° 2, we have

COROLLARY 3. *Notations being as in § 1, n° 2, $\|u\|_\alpha = |u|_\alpha$, if α is injective. If α is projective, then it is accessible.*

Proof. Let α be projective. Express E and F as quotient spaces of (L)-spaces: $E = L_1/J_1$, $F = L_2/J_2$. Then $E \overset{\alpha}{\otimes} F$ is identified with $L_1 \overset{\alpha}{\otimes} L_2 / J_1 \overset{\alpha}{\otimes} J_2$. Let $u \in E' \otimes F'$. u, being in $B^{\alpha\prime}(E, F)$, naturally induces a bilinear form on $L_1 \times L_2$, which we denote by $\tilde{u}$. We have obviously

$$\|\tilde{u}\|_{\alpha'} = \|u\|_{\alpha'}$$

and $\tilde{u} \in L_1' \otimes L_2'$. Put $u = \sum x_i \otimes y_i$. Let φ_1 and φ_2 be the canonical homomorphisms: $\varphi_1: L_1 \to E$, $\varphi_2: L_2 \to F$. Then ${}^t\varphi_1$ and ${}^t\varphi_2$ are the isomorphisms of $E' \to L_1'$ $F' \to L_2'$ respectively, and $\tilde{u} = \sum {}^t\varphi_1 x_i \otimes {}^t\varphi_2 y_i$. Now L_1 and L_2 being metrically accessible, we have

$$(6) \qquad \|\tilde{u}\|_{\alpha'} = |\tilde{u}|_{\alpha'}.$$

On the other hand, as α' is injective, the canonical injection ${}^t\varphi_1 \otimes {}^t\varphi_2$: $E' \overset{\alpha'}{\otimes} F' \to L_1' \overset{\alpha'}{\otimes} L_2'$ is an isomorphism. Hence from $({}^t\varphi_1 \otimes {}^t\varphi_2)\, u = \tilde{u}$ it follows that

$$(7) \qquad |\tilde{u}|_{\alpha'} = |\tilde{u}|_{\alpha'}.$$

(5), (6) and (7) yield $\|u\|_{\alpha'} = |u|_{\alpha'}$. Since any injective $\otimes$-norm can be expressed as α', this proves the first assertion. The second assertion is an immediate consequence.

From the above proof we further find the following

COROLLARY 4. *For any $\otimes$-norm α, the $\otimes$-norms $/\alpha$, $\alpha\backslash$, $\backslash\alpha$ and $\alpha/$ become all accessible.*

§ 3. Tensor-norm related to Hilbert spaces.

1. Hilbertian tensor-norms.

THEOREM 4. *There exists a unique $\otimes$-norm $\mathcal{H}$, which is called the Hilbertian $\otimes$-norm, with the following properties:*

Let E and F be any Banach spaces, and let u be a bilinear form on $E \times F$. Then $\|u\|_{\mathcal{H}} \leqq 1$ if and only if

$$u(x, y) = <\varphi y, \psi y> \qquad \text{for all } x \in E \text{ and } y \in F,$$

where φ is a linear mapping of E into a Hilbert space H with $\|\varphi\| \leqq 1$, and ψ is one of F into H' (the dual space of H) with $\|\psi\| \leqq 1$.

Proof. If such a $\otimes$-norm exists, the uniqueness is almost evident. Hence we shall only prove the existence of the $\otimes$-norm $\mathcal{H}$; the proof will be devided into four steps.

1°. Put U for the subset of $B(E, F)$, consisting of all u with the properties stated in the theorem:

$$u = \{u;\, u(x, y) = <\varphi x, \psi y>,\ \ \varphi \text{ and } \psi \text{ being as above}\}.$$

Then the elements of U are also characterized by the following properties (H): *there are Hilbert spaces H and K, and linear mappings φ and ψ such that*

$$\varphi:\ E \to H, \quad \text{with } \|\varphi\| \leqq 1,$$

$$\psi:\ F \to K, \quad \text{with } \|\psi\| \leqq 1,$$

for which we have

$$|u(x, y)| \leqq \|\varphi(x)\| \cdot \|\psi(y)\|.$$

In fact, the elements of U clearly fulfill the property (H). Conversely assume that $u \in B(E, F)$ have the property (H). Then $|u(x, y)| \leqq \|\varphi(x)\| \cdot \|\psi(y)\|$. We may assume that H and K are spanned by $\{\varphi(x);\, x \in E\}$ and

$\{\psi(y); y \in F\}$, respectively. Put $\tilde{v}(\varphi(x), \psi(z)) = u(x,y)$. It is evident that $\tilde{v}$ induces the unique bilinear form on $H \times K$ by the continuity; besides, $\|\tilde{v}\| \leqq 1$. Hence $\tilde{v}$ is considered as a linear mapping of K into H' with the norm $\leqq 1$. Putting $\psi_1 = \tilde{v} \circ \psi$, we see that $u(x,y) = <\varphi(x), \psi_1(y)>$ with $\|\varphi\| \leqq 1$ and $\|\psi_1\| \leqq 1$, which implies $u \in U$.

A semi-norm on E is called a *Hilbertian norm* if it has the form $\varphi(x,x)^{1/2}$, where $\varphi(x,y)$ means a quasi-inner product on E. Then the condition (H) is interpreted in terms of Hilbertian norms as follows: $u \in B(E,F)$ fulfills the condition (H) if and only if the bilinear-norm of u becomes $\leqq 1$, when E and F are endowed with a suitable Hilbertian norm φ and a ψ, respectively, such that $\varphi(x,x)^{1/2} \leqq \|x\|$ and $\psi(y,y)^{1/2} \leqq \|y\|$. We say a pair of Hilbertian norms $\{\varphi, \psi\}$ to be an *H-attendant* of u, if the above relation holds for u, φ and ψ.

2°. We prove that U is convex (and circular in the complex case) and that it is compact with respect to the simple convergence.

Let u_1 and u_2 be elements in U; let $\{\varphi_i, \psi_i\}$ be the H-attendants of u_i $(i = 1,2)$. Then $\lambda_1 u_1 + \lambda_2 u_2$ $(\lambda_1 \geqq 0.\ \lambda_2 \geqq 0,\ \lambda_1 + \lambda_2 = 1)$ has the H-attendant $\{\lambda_1\varphi_1 + \lambda_2\varphi_2,\ \lambda_1\psi_1 + \lambda_2\psi_2\}$. In fact, applying the Schwarz's inequality, we have

$$|\lambda_1 u_1(x,y) + \lambda_2 u_2(x,y)| \leqq \lambda_1\varphi_1(x,x)^{1/2}\,\psi_1(y,y)^{1/2} + \lambda_2\varphi_2(x,x)^{1/2}\,\psi_2(y,y)^{1/2}$$
$$\leqq \{\lambda_1\varphi_1(x,x) + \lambda_2\varphi_2(x,x)\}^{1/2}\{\lambda_1\psi_1(x,x) + \lambda_2\psi_2(y,y)\}^{1/2};$$

besides,

$$\{\lambda_1\varphi_1(x,x) + \lambda_2\varphi_2(x,x)\}^{1/2} \leqq \sqrt{\lambda_1}\ \varphi_1(x,x)^{1/2} + \sqrt{\lambda_2}\ \varphi_2(x,x)^{1/2} \leqq \|x\|,$$
$$\{\lambda_1\psi_1(y,y) + \lambda_2\psi_2(y,y)\}^{1/2} \leqq \|y\|.$$

If follows that $\lambda_1 u_1 + \lambda_2 u_2 \in U$, whence U is convex. It is clear that U is circular in the complex case.

In order to prove the compactness of U, it is sufficient to show that U is closed in $B_s(E,F)$. Let $u_\lambda \in U$ and assume that u_λ converges to u in $B_s(E,F)$. Let $\{\varphi_\lambda, \psi_\lambda\}$ be an H-attendant of u_λ. Since the totality of inner products on E (or F), with the norm $\leqq 1$, is compact in $B_s(E,E)$ (resp. $B_s(F,F)$), we may assume that $\{\varphi_\lambda, \psi_\lambda\}$ simply converges to a pair of Hilbertian norms $\{\varphi, \psi\}$. Then it is clear that $\{\varphi, \psi\}$ gives rise to an H-attendant of u and so $u \in U$.

3°. For $v \in E \otimes F$, put

$$|v|_{\mathfrak{H}'} = \sup_{\varphi,\psi} |(\varphi \otimes \phi)\, v|_\wedge,$$

where the supremum runs over all such pairs of $\{\varphi, \psi\}$ that φ is a linear mapping of E into an *arbitrary* Hilbert space H and ψ is one of F into K each of whose norm is at most 1, $(\varphi \otimes \psi)v$ being considered in $H \hat{\otimes} K$. Put U° be the polar set of U in $E \otimes F$. Then as is easily seen

$$U^\circ = \{v; |v|_{\mathfrak{H}'} \leqq 1\}.$$

Also it is evident that $|v|_{\mathfrak{H}'}$ is actually a reasonable norm of $E \otimes F$.

Now let $u_i (i = 1,2)$ be the continuous linear mappings of E into E_1 and F into F_1, having the norm 1, respectively. Then, for $v \in E \otimes F$, we have

$$\begin{aligned}|(u_1 \otimes u_2) v|_{\mathfrak{H}'} &= \sup_{\varphi_1,\psi_1} |(\varphi_1 \otimes \psi_1) \circ (u_1 \otimes u_2) v|_{\wedge} \\ &= \sup_{\varphi_1,\psi_1} |(\varphi_1 \circ u_1) \otimes (\psi_1 \circ u_2) v|_{\wedge} \\ &\leqq \sup_{\varphi,\psi} |(\varphi \otimes \psi) v|_{\wedge} = |v|_{\mathfrak{H}'},\end{aligned}$$

where φ_1 means linear mapping of E_1 into a Hibert space of norm $\leqq 1$ and ψ_1 has the same meaning with respect to F_1. Consequently $\| u_1 \overset{\mathfrak{H}'}{\otimes} u_2 \| \leqq \| u_1 \| \cdot \| u_2 \|$.

4°. We wish to show that $|v|_{\mathfrak{H}'}$ is a $\otimes$–norm. For this it remains to prove that for any fixed $v \in E \otimes F$, $|v|_{\mathfrak{H}'}$ is equal to the infimum of $|v|_{E_\sigma \overset{\mathfrak{H}'}{\otimes} F_\sigma}$, where E_σ and F_σ mean finite–dimensional subspaces of E and F, respectively, such that $v \in E_\sigma \otimes F_\sigma (\sigma \in \sum)$. From the result of 3°, it follows $|v| \leqq |v|_{E_\sigma \overset{\mathfrak{H}'}{\otimes} F_\sigma}$. We must prove the converse inequality. For this aim, making the assumption of $\inf |v|_{E_\sigma \overset{\mathfrak{H}'}{\otimes} F_\sigma} > 1$, we shall show that this leads to $|v|_{\mathfrak{H}'} > 1$. For $u \in B(E,F)$, we denote by $\| u \|_{\mathfrak{H}}$ the norm of u induced by the "unit sphere" U if its norm exists. From the above assumption, for all $\sigma \in \sum$ there is a bilinear form u_σ on $E_\sigma \times E_\sigma$ such that

$$\| u_\sigma \|_{\mathfrak{H}} \leqq 1$$

and

$$|< u_\sigma, v >| > 1.$$

Let S and T be the unit spheres of E and F, respectively. We denote by $\mathfrak{F}$ the space of all functions on $S \times T$ with $\sup_{p \in S \times T} |f(p)| \leqq 1$ and assume that $\mathfrak{F}$ have the simple convergence topology. Observe that $\mathfrak{F}$ is compact. For all $\sigma \in \sum$, associate the closed set $\mathfrak{F}_\sigma$ of $\mathfrak{F}$, consisting of all functions f's such that the restriction of f to $(S \times T) \cap (E_\sigma \times F_\sigma)$ is equal to the one of u_σ to $(S \times T) \cap (E_\sigma \times F_\sigma)$. $\mathfrak{F}_\sigma$ is not empty, since it contains a function f_σ defined as follows: $f_\sigma(x,y) = u_\sigma(x,y)$ for $x \in E_\sigma \cap S$, $y \in F_\sigma \cap T$, and $f_\sigma(x,y) = 0$ otherwise. Besides it is clear that $\mathfrak{F}_\sigma$ has the finite intersection property. Hence we can conclude that $\bigcap_{\sigma \in \Sigma} \mathfrak{F}_\sigma$ is not empty. Take a function f_0 from $\bigcap \mathfrak{F}_\sigma$ and put

$$u(x, y) = \| x \| \cdot \| y \| f_0 \left(\frac{x}{\| x \|}, \frac{y}{\| y \|} \right).$$

Then a familiar discussion yields that $u(x, y)$ is a bilinear function on $E \times F$ and that

$$|< u, v >| > 1.$$

Hence $|v|_{\mathfrak{H}'} > 1$ will be proved if we can succeed in obtaining $\| u \|_{\mathfrak{H}} \leqq 1$. From the existince of an H–attendant of u_σ it follows that there are a Hilbertian norm φ_σ on E_σ and a ψ_σ on F_σ such that

$$\varphi_\sigma(x, x) \leqq \| x \|^2, \quad \psi_\sigma(y, y) \leqq \| y \|^2$$

and that

$$|u(x, y)| \leqq \varphi_\sigma(x, x)^{1/2} \psi_\sigma(y, y)^{1/2}$$

for all $x \in E_\sigma$ and $y \in F_\sigma$. Applying the same arguments as above, we find a Hilbertian norm φ on E and a ψ on F, obtained by the "limit" of φ_σ and ψ_σ, respectively, the pair of which serves as an H-attendant of u. Hence we get $\|u\|_{\mathscr{H}} \leqq 1$. Thus, we come to a conclusion that

$$|v|_{\mathscr{H}'} = \inf |v|_{E_\sigma \overset{\mathscr{H}'}{\otimes} E_\sigma},$$

which, together with the preceding results, shows that $\mathscr{H}'$ is a $\otimes$-norm. If follows simultaneously that the dual norm $\mathscr{H}$ of $\mathscr{H}'$ gives the desired Hilbertian $\otimes$-norm. This completes the proof.

REMARK 1. From the definition it results that the $\otimes$-norm $\mathscr{H}'$ is the smallest one in all the $\otimes$-norms α's with the following property: Let E and F be any Banach spaces; let φ and ψ be any linear mappings from E and F into Hilbert spaces H and K, respectively. Then $\varphi \otimes \psi$ induces a linear mapping $E \overset{\alpha}{\otimes} F$ into $H \overset{\wedge}{\otimes} K$ with the norm $\leqq \|\varphi\| \cdot \|\psi\|$.

Hence, by the duality, we find:

REMARK 2. The $\otimes$-norm $\mathscr{H}$ is the greatest one in all the $\otimes$-norms β's with the following property: Let E and F be as above; let φ and ψ be any linear mappings from Hilbert spaces H and K into E and F, respetively. Then $\varphi \otimes \psi$ induces a linear mapping $H \overset{\vee}{\otimes} K$ into $E \overset{\beta}{\otimes} F$ with the norm $\leqq \|\varphi\| \cdot \|\psi\|$.

2. $\mathscr{H}'$-forms on $C_0(M) \times C_0(M)$.

Let us recall some known definitions. Assume that E and F be linear spaces. A form u on $E \times F$ is called *sesquilinear* if $u(x,y)$ is linear with respect to x and anti-linear with respect to y. If we introduce the space $\overline{F}$ which is anti-linearly isomorphic to F in a canonical way, then a sesquilinear form u on $E \times F$ is regarded as a bilinear form on $E \times \overline{F}$. By this correspondence between sesquilinear forms and bilinear ones, the notions on bilinear forms such as type α. α-norm, etc. are naturally inherited to sesquilinear forms. A sesquilinear form u on $E \times E$ is called *Hermitian* if $u(x,y) = \overline{u(y,x)}$, and *positive* if $u(x,x) \geqq 0$. In a usual manner, the order relation is introduced in the family of Hermitian forms on $E \times E$, which is denoted e.g. by $u \gg v$.

For later use, we shall give a characterization of the elements in $E \otimes F$, belonging to the unit sphere in $E \overset{\mathscr{H}}{\otimes} F$. Assume that v be an element in $E \otimes F$ with $|v|_{\mathscr{H}} \leqq 1$. Since $\mathscr{H}$ is injective, by Corollary 3 to Theorem 3 $|v|_{\mathscr{H}} \leqq 1$ is equivalent to $\|v\|_{\mathscr{H}} \leqq 1$, where v is regarded as an element of $B^{\mathscr{H}}(E', F')$. Let an H-attendant of v be $\{\varphi, \psi\}$:

(8) $$\varphi\colon E' \to H, \quad \text{with } \|\varphi\| \leqq 1,$$

(9) $$\psi\colon F' \to H', \quad \text{with } \|\psi\| \leqq 1,$$

and

(10) $$v(x', y') = < \varphi(x'), \psi(y') >.$$

As is easily verified, we may assume without loss of generality that H is finite-dimensional and that φ and ψ are both onto-mappings. Let $e_i (i = 1, \cdots, n)$ be an orthonormal basis in H, and let $e'_i (i = 1, \cdots, n)$ be the dual basis in H'. Since for a fixed y' $v(x', y')$ is $\sigma(E', E)$-continuous in x', it follows that $\varphi(x')$ is weakly continuos. Similary $\psi(y')$ has the same property. Hence there exist $x_i \in E$ and $y_i \in F (i = 1, \cdots, n)$ such that

$$< x', x_i > = < \varphi(x'), e'_i >,$$
$$< y', y_i > = < e_i, \psi(y')>.$$

Thus we have

$$v(x', y') = \textstyle\sum_i < \varphi(x'), e'_i > < e_i, \psi(y')>$$
$$= \textstyle\sum_i < x', x_i > < y', y_i >.$$

Consequently, v can be expressed as follows:

(11) $$v = \textstyle\sum_i x_i \otimes y_i,$$

where we have

(12) $$\textstyle\sum_i | < x', x_i > |^2 \leqq \| x' \|^2,$$

(13) $$\textstyle\sum_i | < y', y_i > |^2 \leqq \| y' \|^2;$$

in fact, for example the first inequality (12) is deduced from

$$\textstyle\sum_i | < x', x_i > |^2 = \sum | < \varphi(x'), e'_i > |^2$$
$$= \| \varphi(x') \|^2 \leqq \| x' \|^2.$$

Conversely, assume that $v \in E \otimes F$ have an expression (11) satisfying the supplementary conditions (12) and (13). Consider $H = l^2(1, \cdots, n)$. Define the linear mappings φ and ψ by

$$\varphi\colon\ x' \to \{< x', x_i >\} \in H,$$
$$\psi\colon\ y' \to \{< y', y_i >\} \in H'.$$

Then it is obvious that (8), (9), (10) can be satisfied and so $\| v \|_{\mathcal{H}} \leqq 1$. In conclusion, the conditions (11), (12) and (13) together give a complete charecterization to elements which belong to the unit sphere of $E \otimes F$ with respect to $\mathcal{H}$-norm.

THEOREM 5. *Let M be any locally compact space. Put $E = C_0(M)$, where $C_0(M)$ donotes the Banch space, consisting of continuous functions on M which vanish at infinity, the norm of functions being defined as its least upper bound. Assume that a sesquilinear $\mathcal{H}'$-form u on $E \times E$ be given. Then there exists a positive measure μ on M which satisfies the following properties*:

(14) $$| u(f, f) | \leqq \int |f|^2 d\mu,$$
$$\| \mu \| \leqq \| u \|_{\mathcal{H}'}.$$

If u is further assumed to be positive, then μ can be taken such that

$$\| \mu \| = \| u \|_{\mathcal{H}'}.$$

Proof. We shall first establish:

(15) $$| \sum_{i=1}^n g_i \otimes h_i |_{\mathcal{H}} \leqq \| \sum_{i=1}^n | g_i |^2 \|^{1/2} \| \sum_{i=1}^n | h_i |^2 \|^{1/2}$$

for any $g_i, h_i \in E (i = 1, \cdots, n)$. Consider the Hilbert space $H = l^2 (1, \cdots, n)$; let φ be a linear mapping of H into E: $\{\lambda_i\} \to \sum_{i=1}^n \lambda_i g_i$, and ψ be one of H'

into E: $\{\lambda_i\} \to \sum_{i=1}^n \lambda_i h_i$. Since we have clearly

$$\sup_{\Sigma|\lambda_i|^2 \leqq 1} \| \sum \lambda_i g_i \| = \| \sum | g_i |^2 \|^{1/2},$$

it follows that $\|\varphi\| = \|\sum |g_i|^2\|^{1/2}$ and $\|\psi\| = \|\sum |h_i|^2\|^{1/2}$. Therefore by Remark 2 in n° 1 we have

$$\|\varphi \otimes \psi\| \leqq \|\sum_{i=1}^n |g_i|^2\|^{1/2} \|\sum_{i=1}^n |h_i|^2\|^{1/2}, \tag{16}$$

$\varphi \otimes \psi$ being regarded as a mapping $l^2 \overset{\vee}{\otimes} l^2 \to E \overset{\mathscr{H}}{\otimes} F$. On the other hand, it is evident that $(\varphi \otimes \psi)\sum e_i \otimes e_i = \sum g_i \otimes h_i$ and that $|\sum_{i=1}^n e_i \otimes e_i|_\vee = 1$, where e_i denotes the elements $\{0, \cdots, 0, \underset{i\text{-}th}{\overset{\frown}{1}}, \cdots, 0\}$ of $H\,(i = 1, \cdots, n)$. Hence (16) shows the validity of (15).

Now let u be a sesquilinear $\mathscr{H}'$-form on $E \times E$. We may assume $\|u\|_{\mathscr{H}'} = 1$. Denote by E_R the real Banach space which consists of all real-valued functions in E, with the same norm as in E. For $f \in E_R$, put

$$P(f) = \inf_{g_i} \{\|f + \sum |g_i|^2\| - \sum |u(g_i, g_i)|\}, \tag{17}$$

where the infimum is taken all over the family of finite elements $\{g_i\}$, $g_i \in E$. We show:

i) $P(\alpha f) = \alpha P(f) \quad (\alpha > 0)$;

ii) $P(f_1 + f_2) \leqq P(f_1) + P(f_2)$;

iii) $P(0) = 0$;

iv) $P(f) \leqq \|f\|$.

Ad i). $P(\alpha f) = \inf_{g_i} \{\|\alpha f + \sum |g_i|^2\| - \sum |u(g_i, g_i)|\}$

$$= \alpha \inf_{g_i} \left\{\left\|f + \sum \left|\frac{1}{\sqrt{\alpha}} g_i\right|^2\right\| - \sum u\left(\frac{1}{\sqrt{\alpha}} g_i, \frac{1}{\sqrt{\alpha}} g_i\right)\right\}$$

$$= \alpha P(f).$$

Ad ii). Clear.

Ad iii). We first prove that $P(0)$ is non-negative. Since

$$P(0) = \inf \{\|\sum |g_i|^2\| - \sum u(g_i, g_i)|\},$$

for this it is sufficient to show

$$\|\sum_{i=1}^n |g_i|^2\| \geqq \sum_{i=1}^n |u(g_i, g_i)|. \tag{18}$$

Choose $\varepsilon_i, |\varepsilon_i| = 1$, such that $u(g_i, \varepsilon_i g_i) = |u(g_i, g_i)|$; put $h_i = \varepsilon_i g_i (i = 1, \cdots, n)$. Then it turns out that (18) becomes an immediate consequence of (15) in view of $\|u\|_{\mathscr{H}'} = 1$. This being established, it is trivial to verify $P(0) = 0$ by taking g_i as 0.

Ad iv). This is also a consequence of (15).

The above mentionned properties i), ii) and iii) mean that $P(f)$ is subadditive. We are now in a position to apply the Hahn-Banach extention theorem. Hence there exists a linear functional μ_1 on E_R, which satisfies

$$\mu_1(f) \leqq P(f). \tag{19}$$

From iv) it follows that for $f \in E_R$

(20) $$-\|f\| \leqq -P(-f) \leqq \mu_1(f) \leqq P(f) \leqq \|f\|.$$

Moreover, by (17) and (19) we have

$$\mu_1(-|f|^2) \leqq \|-|f|^2 + |f|^2\| - |u(f,f)|,$$

for any $f \in E$, so that

(21) $$|u(f,f)| \leqq \mu_1(|f|^2).$$

(20) and (21) imply that μ_1 induces a positive measure μ on M, satisfying

$$\|\mu\| \leqq 1,$$
$$|u(f,f)| \leqq \int |f|^2 \, d\mu.$$

Hence, μ gives a required measure.

Now, we go on the second part of Theorem; we further assume that u is positive. It should be noted that in case $E = C_0(M)$, the conditions (11), (12) and (13) mean that the unit sphere of $E \overset{\mathcal{H}}{\otimes} E$ is the closure of the elements

$$\{\sum f_i \otimes g_i; \sum |f_i|^2 \leqq 1, \sum |g_i|^2 \leqq 1\},$$

the index i running over $\{1, \cdots, n\}$, n arbitrary. Therefore for any $\mathcal{H}'$-form u on $E \times E$ we have

$$\|u\|_{\mathcal{H}'} = \sup \{\sum |u(f_i, g_i)|; \sum |f_i|^2 \leqq 1, \sum |g_i|^2 \leqq 1\}.$$

On the other hand, as u is positive, by the successive application of Schwarz's inequality we obtain

$$\sum_{i=1}^{n} |u(f_i, g_i)| \leqq \sum_{i=1}^{n} u(f_i, f_i)^{1/2} u(g_i, g_i)^{1/2}$$
$$\leqq (\sum_{i=1}^{n} u(f_i, f_i)^{1/2})(\sum_{i=1}^{n} u(g_i, g_i)^{1/2}).$$

From this it follows directly that

(22) $$\|u\|_{\mathcal{H}'} = \sup \{\sum u(f_i, f_i); \sum |f_i|^2 \leqq 1\}.$$

Since μ satisfies

$$u(f,f) \leqq \int |f|^2 \, d\mu,$$

we have

$$\sum u(f_i, f_i) \leqq \sum \mu(|f_i|^2) = \mu(\sum |f_i|^2).$$

Accordingly by (22) we obtain

$$\|u\|_{\mathcal{H}'} \leqq \|\mu\|.$$

This, together with the converse inequality obtained in the first part, yields $\|u\|_{\mathcal{H}'} = \|\mu\|$, which completes the proof.

3. Consequences.

For a positive measure μ on M, we put

(23) $$v_\mu(f, g) = \int f\bar{g} \, d\mu;$$

v_μ is clearly a Hermitian form on $C_0(M) \times C_0(M)$. Theorem 5 shows that for any $\mathcal{H}'$-from u on $C_0(M) \times C_0(M)$, there exists a positive measure μ such that $|u(f,f)| \leqq v_\mu(f,f)$ and $\|\mu\| \leqq \|u\|_{\mathcal{H}'}$. In case where u is Hermitian, this is expressible as follows:

$$-v_\mu \ll u \ll v_\mu, \quad \text{with } \|\mu\| \leqq \|u\|_{\mathcal{H}'}.$$

Further, it holds $\|\mu\| = \|u\|_{\mathcal{H}'}$, when u is positive. Observe that by (23) we can also write v_μ as a weak integral on the unit sphere B of E' such that

$$v_\mu = \int_M \varepsilon_x \otimes \bar{\varepsilon}_x d\mu(x),$$

where ε_x and $\bar{\varepsilon}_x$ denote the Dirac measure at x in $C_0(M)$ and $\overline{C_0(M)}$, respectively, M being regarded as a subset of B. This in turn implies that v_μ is an integral operator [1].

These results can be immediately extended to general cases according to the following considerations. Let E be a Banach space and assume that E is imbedded into a $C_0(M)$. $\mathcal{H}$ being injective, $E \overset{\mathcal{H}}{\otimes} E$ is regarded as a closed subspace of $C_0(M) \overset{\mathcal{H}}{\otimes} C_0(M)$. Let u be any given $\mathcal{H}'$-form on $E \times E$. By the Hahn–Banach extension theorem, u is extended in a norm-preserving way to an $\mathcal{H}'$-form $\tilde{u}$ on $C_0(M) \times C_0(M)$, to which the results mentioned above can be just applied. Then, it is easy to see that the restriction of $\tilde{u}$ to $E \times E$ allows us to formulate its results in terms of u as follows:

THEOREM 6. *Let E be a Banach space and let u be a sesquilinear $\mathcal{H}'$-form on $E \times E$. Then there exists a positive Hermitian integral form v such that*

$$|u(x,x)| \leqq v(x,x).$$

v admits an expression as a weak integral on the unit sphere B of E':

$$v = \int_B x' \otimes \bar{x}' d\mu(x'), \tag{24}$$

where μ is a positive measure on B satisfying

$$\|\mu\| \leqq \|u\|_{\mathcal{H}'}. \tag{25}$$

As a consequence, if u is a Hermitian $\mathcal{H}'$-form on $E \times E$, then there exists a v with the expression (24) *(μ satisfying* (25)), *such that*

$$-v \ll u \ll v.$$

In a special case where u is positive, the equality holds in (25).

We shall again consider Thorem 5. Making use of the notations there, we find that for a Hermitian $\mathcal{H}'$-form u on $C_0(M) \times C_0(M)$ (14) means

$$|u(f,g)| \leqq \left(\int |f|^2 d\mu\right)^{1/2} \left(\int |g|^2 d\mu\right)^{1/2}.$$

Accordingly, by the continuity u is uniquely extensible to a form on $L^2(\mu) \times L^2(\mu)$ with the norm $\leqq 1$.

We proceed to generalize this result to a sesquilinear $\mathcal{H}'$-form on $C_0(M) \times C_0(N)$, where M and N denote locally compact spaces. Put $R = M + N$ (topological union). $M \times N$ are canonically imbedded into $P \times P$, so that $C_0(M) \times C_0(N)$ is regarded as a subspace of $C_0(P) \times C_0(P)$. Define U by

$$U(f+g, f'+g') = u(f,g') + u(f',g) \qquad (f, f' \in C_0(M),\ g,\ g' \in C_0(N)).$$

It is clear that U is a Hermitian form on $C_0(P) \times C_0(P)$ and the restriction of U to $M \times N$ is nothing but u. $\mathcal{H}'$ being projective, we have

$$\| U \|_{\mathfrak{H}'} \leqq 2 \| u \|_{\mathfrak{H}'}.$$

Apply the arguments in the preceding paragraph to U, we know that there exist positive measures μ on M and ν on N such that

$$\| \mu + \nu \| = \| \mu \| + \| \nu \| \leqq \| U \|_{\mathfrak{H}'};$$

furthermore, U is uniquely extended to a form on $L^2(\mu + \nu) \times L^2(\mu + \nu)$ with the norm $\leqq 1$. By the definition of U, these properties however remain true, even when μ is replaced by $\alpha\mu$ and ν by $\nu/\alpha (\alpha > 0)$. Hence, as is easily seen, we may assume that μ and ν are chosen so as to satisfy

$$\| \mu \|, \ \| \nu \| \leqq \| u \|_{\mathfrak{H}}$$

Therefore we obtain

Let u be a sesquilinear $\mathcal{H}'$-form on $C_0(M) \times C_0(N)$. Then there exist positive measures μ on M and ν on N such that u is uniquely extended to a form on $L^2(\mu) \times L^2(\nu)$ with the norm $\leqq 1$; besides

$$\| \mu \|, \ \| \nu \| \leqq \| u \|_{\mathfrak{H}}.$$

This implies that u is a Hilbertian form and $\| u \|_{\mathfrak{H}} \leqq \| u \|_{\mathfrak{H}'}$. In order to state this result in a more general form, we consider Banach spaces E and F. Assume that E is imbedded into $C_0(M)$ and F into $C_0(N)$. Then any $\mathcal{H}'$-form u on $E \times F$ is extended in a norm-preserving way to an $\mathcal{H}'$-form $\tilde{u}$ on $C_0(M) \times C_0(N)$. By the above, we have $\| \tilde{u} \|_{\mathfrak{H}} \leqq \| \tilde{u} \|_{\mathfrak{H}'} = \| u \|_{\mathfrak{H}'}$, which in turn implies that $\| u \|_{\mathfrak{H}} \leqq \| u \|_{\mathfrak{H}'}$.

Thus we finally arrived at the following theorem:

THEOREM 7. *We have* $\mathcal{H} \leqq \mathcal{H}'$.

As stated in Introduction, this result is somewhat better than that due to Grothendieck ([3], § 3, Proposition 3).

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