ON THE JUMP FUNCTIONS

By Kenji Yano

1. Let $f(x)$ be an L-integrable function with period 2π, and its allied Fourier series be

$$(1.1)\qquad \sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

We write

$$(1.2)\qquad \psi(t) = \psi_x(t) = f(x+t) - f(x-t) - l(x),$$

and denote the n-th (C, α) mean of the series (1.1) by $\bar{\sigma}_n^\alpha(x)$ with

$$\bar{s}_n(x) = \bar{\sigma}_n^0(x) \quad \text{and} \quad \bar{\sigma}_n(x) = \bar{\sigma}_n^1(x),$$

that is

$$\bar{\sigma}_n^\alpha(x) = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^\alpha B_\nu(x),$$

where A_n^α is Andersen's notation, $A_n^\alpha = (\alpha+1)(\alpha+2)\cdots(\alpha+n)/n!$.

In this paper we shall consider a particular value of x such that $0 \leqq x < 2\pi$.

O. Szász [1] has showed:

THEOREM S_1. *If $\psi(t)$ satisfies the two conditions*

$$(\text{a})\qquad \int_0^t \psi(u)\,du = o(t),$$

$$(\text{b})\qquad \int_0^t |\psi(u)|\,du = O(t)$$

as $t \to +0$, then

$$(1.3)\qquad \bar{\sigma}_{2n}(x) - \bar{\sigma}_n(x) = \frac{l(x)}{\pi}\log 2 + o(1)$$

as $n \to \infty$.

S. Izumi [3] improved this theorem as following:

THEOREM I. *Theorem* S_1 *is valid even if the condition* (b) *would be replaced by*

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$$\int_{\pi/n}^{\pi} \frac{|\psi(t) - \psi(t+\pi/n)|}{t} dt = O(1) \tag{1.4}$$

as $n \to \infty$.

H. C. Chow [4] has showed:

THEOREM C. *If* $\alpha > 0$ *and* $\psi(t)$ *satisfies the conditions* (a) *and* (b), *then*

$$\bar{\sigma}_{2n}^{\alpha}(x) - \bar{\sigma}_{n}^{\alpha}(x) = \frac{l(x)}{\pi} \log 2 + o(1)$$

as $n \to \infty$.

G. Maruyama [5] has showed:

THEOREM M_1. *Under the conditions* (a) *and* (b), *if* $\mu_n > \lambda_n$ *and* $\lim (\mu_n/\lambda_n) = A$, *then*

$$\bar{\sigma}_{\mu_n}(x) - \bar{\sigma}_{\lambda_n}(x) = \frac{l(x)}{\pi} \log A + o(1)$$

as $n \to \infty$.

Further if $\mu_n/\lambda_n \to \infty$ then

$$[\bar{\sigma}_{\mu_n}(x) - \sigma_{\lambda_n}(x)]/(\log \mu_n - \log \lambda_n) = \frac{l(x)}{\pi} + o(1). \tag{1.5}$$

THEOREM M_2. *If* $f(x)$ *is of bounded variation and* $\mu_n/\lambda_n \to 1$ *then* (1.5) *holds as* $n \to \infty$.

O. Szász [2] showed:

THEOREM S_2. *If* $\psi(t)$ *satisfies the conditions* (a) *and* (b), *then the sequence* $\{nB_n(x)\}$ *is summable* $(C, 2)$ *to the value* $l(x)/\pi$.

Recently, R. Mohanty and M. Nanda [6] proved:

THEOREM M.N. *If* $\psi(t) = o(\log(1/t))^{-1}$ *and* $a_n = O(n^{-\delta})$, $b_n = O(n^{-\delta})$, $0 < \delta < 1$, *then the sequence* $\{nB_n(x)\}$ *is summable* $(C, 1)$ *to the value* $l(x)/\pi$.

In this paper, we shall prove a number of theorems which contain the above theorems as particular cases.

2. We denote the n-th (C, α) conjugate Fejér kernel by $\bar{K}_n^{\alpha}(t)$ with $\bar{D}_n(t) = \bar{K}_n^0(t)$, i. e.

$$\bar{K}_n^{\alpha}(t) = \frac{1}{A_n^{\alpha}} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha} \sin \nu t, \tag{2.1}$$

then we have, from the definition of $\psi(t)$ in (1.2),

$$\bar{\sigma}_n^{\alpha}(x) = \frac{l(x)}{\pi} \int_0^{\pi} K_n^{\alpha}(t)\, dt + \frac{1}{\pi} \int_0^{\pi} \psi(t) \bar{K}_n^{\alpha}(t)\, dt. \tag{2.2}$$

We shall first prove the following

LEMMA 1. *If* $\alpha > -1$ *then*

$$\int_0^\pi \bar{K}_n^\alpha(t)\,dt = \lambda(\alpha, n) + \log 2 + o(1) \tag{2.3}$$

as $n \to \infty$, *where*

$$\lambda(\alpha, n) = \frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \cdots + \frac{1}{\alpha+n}. \tag{2.4}$$

In fact, from (2.1) we have

$$\begin{aligned}\int_0^\pi K_n^\alpha(t)\,dt &= \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^\alpha \frac{1}{\nu} + \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^\alpha \frac{(-1)^{\nu-1}}{\nu} \\ &= P_n + Q_n,\end{aligned}$$

say. Then

$$\begin{aligned}P_n - P_{n-1} &= \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^\alpha \frac{1}{\nu} - \frac{1}{A_{n-1}^\alpha}\sum_{\nu=1}^{n-1} A_{n-1-\nu}^\alpha \frac{1}{\nu} \\ &= \frac{1}{A_n^\alpha}\frac{1}{n} + \sum_{\nu=1}^{n-1}\frac{1}{\nu}\left(\frac{A_{n-\nu}^\alpha}{A_n^\alpha} - \frac{A_{n-1-\nu}^\alpha}{A_{n-1}^\alpha}\right) \\ &= \frac{1}{(\alpha+n)A_{n-1}^\alpha} + \frac{1}{(\alpha+n)A_{n-1}^\alpha}\sum_{\nu=1}^{n-1} A_{n-\nu}^{\alpha-1} \\ &= \frac{1}{(\alpha+n)A_{n-1}^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} = \frac{1}{\alpha+n},\end{aligned}$$

and, since $P_1 = 1/(\alpha+1)$ we have

$$P_n = \frac{1}{\alpha+1} + \frac{1}{\alpha+2} + \cdots + \frac{1}{\alpha+n},$$

which equals to $\lambda(\alpha, n)$. Q_n is the n-th (C, α) mean of the convergent series $0 + 1 - 1/2 + 1/3 - \cdots$, which is summable $(C, -1+\delta)$ for every $\delta > 0$ since the n-th term $(-1)^{n-1}/n$ is $O(1/n)$, and so

$$Q_n = \log 2 + o(1)$$

as $n \to \infty$ for every $\alpha > -1$. Thus we get (2.3) and our lemma is established.

REMARKS. More precisely we can show that

$$\lambda(\alpha, n) = \log\frac{\alpha+n}{\alpha+1} + c_\alpha + O(1/n) \tag{2.4'}$$

if $\alpha > -1$, where c_α is a constant depending on α, and $0 < c_\alpha < 1/(\alpha+1)$. And

$$Q_n = \begin{cases}\log 2 + O(1/n) & (\alpha \geqq 0), \\ \log 2 + O(1/n^{\alpha+1}) & (-1 < \alpha < 0).\end{cases}$$

Thus we get

$$(2.3)' \qquad \int_0^\pi K_n^\alpha(t)\,dt = \log\frac{\alpha+n}{\alpha+1} + c_\alpha + \log 2 + \begin{cases} O(1/n) & (\alpha \geqq 0), \\ O(1/n^{\alpha+1}) & (-1 < \alpha < 0). \end{cases}$$

3. We write

$$(3.1) \qquad \bar{A}(\alpha, n) = \frac{1}{\pi A_n^\alpha}\int_{\pi/n}^\pi \psi(t)\,\frac{\sin(nt + (\alpha+1)(t-\pi)/2)}{(2\sin t/2)^{\alpha+1}}\,dt,$$

then, by Lemma 1, we have the following

LEMMA 2. *If $\alpha > -1$ and $\psi(t)$ satisfies the condition*

$$\text{(a)} \qquad \int_0^t \psi(u)\,du = o(t)$$

s $t \to +0$, then

$$\bar{\sigma}_n^\alpha(x) = \frac{l(x)}{\pi}[\lambda(\alpha, n) + \log 2] + \frac{1}{\pi}\int_{\pi/n}^\pi \frac{\psi(t)}{2\tan t/2}\,dt + \bar{A}(\alpha, n) + o(1)$$

as $n \to \infty$.

Indeed, in the expression (2.2)

$$\int_0^\pi \psi(t)\bar{K}_n^\alpha(t)\,dt = \int_0^{\pi/n} + \int_{\pi/n}^\pi = I_1 + I_2,$$

say. Then from

$$\bar{K}_n^\alpha(t) = O(n) \quad\text{and}\quad \frac{d}{dt}\bar{K}_n^\alpha(t) = O(n^2)$$

we have, integrating by parts, $I_1 = o(1)$. $\bar{K}_n^\alpha(t)$ is the imaginary part of the expression

$$-\frac{1}{2} + \frac{1}{A_n^\alpha}\sum_{\nu=0}^n A_{n-\nu}^\alpha e^{i\nu t},$$

which is written in the form

$$\frac{i}{2\tan(t/2)} - \frac{e^{-it}}{A_n^\alpha}\sum_{p=1}^k \frac{A_n^{\alpha-p}}{(1-e^{-it})^{p+1}}$$
$$+ \frac{e^{int}}{A_n^\alpha(1-e^{-it})^{\alpha+1}} - \frac{e^{int}}{A_n^\alpha(1-e^{-it})^{k+1}}\sum_{\nu=n+1}^\infty A_\nu^{\alpha-k-1}e^{-i\nu t},$$

where k is the positive integer such that $-2 < \alpha - k \leqq -1$. We can easily see that from the last expression, integrating by parts, I_2 equals to the imaginary part of

$$i\int_{\pi/n}^\pi \frac{\psi(t)}{2\tan(t/2)}\,dt + \frac{1}{A_n^\alpha}\int_{\pi/n}^\pi \frac{\psi(t)}{(1-e^{-it})^{\alpha+1}}e^{int}\,dt + o(1)$$

under the condition (a), analogously as A. Zygmund [8]. Thus we get the desired result.

Evidently, if $\beta > \alpha$ and $\bar{A}(\alpha, n) \to 0$ then $\bar{A}(\beta, n) \to 0$.

If $\psi(t)$ satisfies the condition (a) then

$$\int_{\pi/n}^{\pi/m} \frac{\psi(t)}{2\tan(t/2)} dt = o(\log m - \log n) + o(1)$$

as $m \to \infty$ and $n \to \infty$. Particularly if $0 < H < m/n < K$ then $o(\log m - \log n) = o(1)$.

From the above facts and Lemma 2 we have immediately the following

LEMMA 3. *If* $-1 < \alpha \leqq \beta$ *and* $\psi(t)$ *satisfies the conditions* (a) *and* $\bar{A}(\alpha, n) \to 0$, *then*

$$\bar{\sigma}_m^{\beta}(x) - \bar{\sigma}_n^{\alpha}(x) = \frac{l(x)}{\pi}[\lambda(\beta, m) - \lambda(\alpha, n)] + o(1) \tag{3.2}$$

as $n \to \infty$ *for* $0 < H < m/n < K$.

4. THEOREM 1. *If* $\alpha > 0$, $p > 0$ *and* $\psi(t)$ *satisfies the conditions*

$$\text{(a)} \quad \int_0^t \psi(u)\,du = o(t) \qquad \text{and} \qquad \text{(b)} \quad \int_0^t |\psi(u)|\,du = O(t)$$

as $t \to +0$, *then*

$$\bar{\sigma}_{[p n]}^{\alpha}(x) - \bar{\sigma}_n^{\alpha}(x) = \frac{l(x)}{\pi} \log p + o(1),$$

and for each positive integer k

$$\bar{\sigma}_{[pn]}^{\alpha}(x) - \bar{\sigma}_n^{\alpha+k}(x) = \frac{l(x)}{\pi}\left(\log p + \frac{1}{\alpha+1} + \cdots + \frac{1}{\alpha+k}\right) + o(1)$$

as $n \to \infty$.

Indeed, since the condition $A(\alpha, n) \to 0$ follows from (a) and (b) if $\alpha > 0$ we have the desired result by Lemma 3 and the definition of $\lambda(\alpha, n)$ in (2.4).

This theorem contains Theorem C and the first part of Theorem M_1.

THEOREM 2. *If* $0 < \alpha \leqq \beta$ *and* $\psi(t)$ *satisfies the conditions* (a) *and* (b) *then*

$$[\bar{\sigma}_m^{\beta}(x) - \bar{\sigma}_n^{\alpha}(x)]/(\log m - \log n) = \frac{l(x)}{\pi} + o(1)$$

as $n \to \infty$ *for* $m/n \to \infty$.

In fact we have

$$\bar{\sigma}_m^{\beta}(x) - \bar{\sigma}_n^{\alpha}(x) = \frac{l(x)}{\pi}[\lambda(\beta, m) - \lambda(\alpha, n)] + o(\log m - \log n) + o(1),$$

analogously as Lemma 3. And, since $\lambda(\alpha, n) = \log n + O(1)$ the first term of the right hand side equals to $l(x)(\log m - \log n)/\pi + O(1)$. Thus we have the desired result.

This theorem contains the second part of Theorem M_1.

Theorem M_2 is valid even if $\bar{\sigma}_n$ would be replaced by $\bar{s}_n$, i. e.:

THEOREM 3. *If* $f(x)$ *is of bounded variation in* $(0, 2\pi)$ *and* $m/n \to 1$, $m - n$

∞, *then*

$$[\bar{s}_m(x) - \bar{s}_n(x)]/(\log m - \log n) = \frac{l(x)}{\pi} + o(1).$$

Indeed we have, by (2.3)$'$

$$\int_0^\pi \bar{D}_n(t)\,dt = \log n + c_0 + \log 2 + O(1/n),$$

and by (2.2)

$$\pi\bar{s}_n(x) = \int_0^\pi \psi(t)\bar{D}_n(t)\,dt + l(x)\int_0^\pi \bar{D}_n(t)\,dt.$$

Therefore

$$\pi[\bar{s}_m(x) - \bar{s}_n(x)] = \int_0^\pi \psi(t)\,[D_m(t) - \bar{D}_n(t)]dt + \log m - \log n + O(1/n).$$

We write

$$\int_0^\pi \psi(t)\,[\bar{D}_m(t) - \bar{D}_n(t)]dt = \int_0^\delta + \int_\delta^\pi = I_1 + I_2$$

say, where $\delta > 0$ is small. It is evident that $\psi(t) = o(1)$ and

$$V(t) = \int_0^t |d\psi(u)| = o(1)$$

as $t \to +0$, since we may suppose that $\psi(0) = \psi(+0) = 0$. Considering the positive and negative variations of $\psi(t)$, by the second mean value theorem we have

$$\begin{aligned} |I_1| &\leqq V(\delta)\cdot \sup_{0<t\leqq\delta}\left|\int_t^\delta [\sin(n+1)u + \cdots + \sin mu]du\right| \\ &< V(\delta)\cdot 4(m-n)/n, \end{aligned}$$

and

$$|I_2| = \left|\int_\delta^\pi \psi(t)\frac{\cos(m+1/2)t - \cos(n+1/2)t}{2\sin(t/2)}dt\right| < \frac{V(\pi)}{2\sin(\delta/2)}\cdot\frac{4}{n}.$$

Thus we get the desired result since it is evident that

$$\frac{m-n}{n} \sim \log m - \log n \quad \text{and} \quad \frac{1}{n} = o(\log m - \log n)$$

under the restriction concerning m and n.

5. It is easily seen that Theorem 3 is valid even if $\bar{s}_n$ would be replaced by $\bar{\sigma}_n^\alpha$ for $\alpha > 0$. On the contrary, for the negative value of α we have the following

THEOREM 4. *Let* $-1 < \alpha < 0$ *and* $\delta > 0$ *be small. If* $f(x)$ *be of bounded variation over* $(0, 2\pi)$, *and*

$$\int_0^\delta |d(\psi(t) - \psi(t+h))| = O(h^{|\alpha|/(\alpha+1)}) \tag{5.1}$$

as $h \to +0$, *then*

$$[\bar{\sigma}_m^\alpha(x) - \bar{\sigma}_n^\alpha(x)]/(\log m - \log n) = l(x)/\pi + o(1) \tag{5.2}$$

as $n \to \infty$ *for* $(m-n)/n \to 0$ *and* $(m-n)/n^{|\alpha|} \to \infty$.

In the case $\alpha = 0$ this theorem coincides with Theorem 3.

We require a lemma.

LEMMA 4. *If* $-1 < \alpha < 0$, $f(x)$ *is of bounded variation over* $(0, 2\pi)$ *and* $1 < m/n < H$, *then*

$$\begin{aligned}\int_{\pi/n}^{\delta} \psi(t)\,[\bar{K}_m^\alpha(t) - \bar{K}_n^\alpha(t)]\,dt &= o\Big(\frac{m-n}{n}\Big) + O\Big(\frac{1}{n^{\alpha+1}}\Big) \\ &+ \frac{1}{2A_n^\alpha}\int_{\pi/n}^{\delta} \frac{\psi(t) - \psi(t+\pi/n)}{(2\sin t/2)^{\alpha+1}}[\sin(mt+\alpha_t) - \sin(nt+\alpha_t)]\,dt\end{aligned} \tag{5.3}$$

as $n \to \infty$, *where* $\alpha_t = (\alpha+1)(t-\pi)/2$, $\delta > 0$ *is small, and o depends on n and* δ.

This lemma can be proved by an elaboration under the conditions

$$\psi(t) = o(1) \quad \text{and} \quad \int_0^t |d\psi(u)| = o(1) \tag{5.4}$$

as $t \to +0$, which are the consequence of $f \in$ B. V.. The proof is omitted.

The proof of our theorem is as following: under the restriction concerning m and n it is evident that

$$\frac{m-n}{n} \sim \log m - \log n, \quad \frac{1}{n^{\alpha+1}} = o(\log m - \log n),$$

and moreover, we have by an elaboration

$$\int_0^t [\bar{K}_m^\alpha(u) - \bar{K}_n^\alpha(u)]\,du = O\Big(\frac{m-n}{n}\Big), \quad 0 < t \leqq \pi/n,$$

and

$$\int_\delta^t [\bar{K}_m^\alpha(u) - \bar{K}_n^\alpha(u)]\,du = O\Big(\frac{1}{n^{\alpha+1}}\Big), \quad \delta < t \leqq \pi.$$

We write

$$\begin{aligned}\pi[\bar{\sigma}_m^\alpha(x) - \bar{\sigma}_n^\alpha(x)] &= l(x)\int_0^\pi [K_m^\alpha(t) - \bar{K}_n^\alpha(t)]\,dt \\ &\quad + \Big(\int_0^{\pi/n} + \int_{\pi/n}^{\delta} + \int_\delta^\pi\Big)\psi(t)\,[\bar{K}_m^\alpha(t) - \bar{K}_n^\alpha(t)]\,dt \\ &= I + I_1 + I_2 + I_3\end{aligned}$$

say. Then by (2.3)′ we have

$$\begin{aligned}I &= l(x)\,[\log(\alpha+m) - \log(\alpha+n)] + O(1/n^{\alpha+1}) \\ &= l(x)\,(\log m - \log n) + o(\log m - \log n).\end{aligned}$$

Integrating by parts we have $I_1 = o((m-n)/n)$ and $I_3 = O(1/n^{\alpha+1})$ under the conditions (5.4). Therefore it is sufficient to show that $I_2 = o(\log m - \log n)$ under our assumption. From (5.1) it follows

$$\Delta\psi(t) - \Delta\psi(0) = O(h^{|\alpha|(\alpha+1)}) \tag{5.5}$$

as $h \to +0$ in $(0, \delta)$, where $\Delta\psi(t) = \psi(t) - \psi(t+h)$. On the other hand from the fact

$$\begin{aligned}\frac{1}{A_n^\alpha}\int_{\pi/n}^{\delta} \frac{\sin(mt+\alpha_t) - \sin(nt+\alpha_t)}{(2\sin t/2)^{\alpha+1}}dt &= \int_{\pi/n}^{\delta}[\bar{K}_m^\alpha(t) - \bar{K}_n^\alpha(t)]dt + O(\log m - \log n)\\ &= O(\log m - \log n),\end{aligned}$$

we have under the condition (5.4), by Lemma 4

$$\begin{aligned} I_2 &= \frac{1}{2A_n^\alpha}\Big(\int_h^{h^{|\alpha|}/K} + \int_{h^{|\alpha|}/K}^{\delta}\Big)\frac{\Delta\psi(t) - \Delta\psi(0)}{(2\sin t/2)^{\alpha+1}}[\sin(mt+\alpha_t) - \sin(nt+\alpha_t)]dt\\ &\qquad + o(\log m - \log n)\\ &= J_1 + J_2 + o(\log m - \log n),\end{aligned}$$

say, where $h = \pi/n$ and $K > 0$ is arbitrary. Letting

$$\begin{aligned}\chi(t) &= \frac{1}{2\sin(t/2)}[\sin(mt+\alpha_t) - \sin(nt+\alpha_t)]\\ &= \cos((n+1/2)t+\alpha_t) + \cdots + \cos((m-1/2)t+\alpha_t),\end{aligned}$$

we have $\int_0^t \chi(u)\,du = O((m-n)/n)$. Therefore, integrating by parts under the conditions (5.1) and (5.5) we have

$$\begin{aligned} J_1 &= \frac{1}{2A_n^\alpha}\int_h^{h^{|\alpha|}/K}\Big(\sin\frac{t}{2}\Big)^\alpha [\Delta\psi(t) - \Delta\psi(0)]\chi(t)\,dt\\ &= O\Big(\frac{m-n}{n}\Big/K^\alpha\Big).\end{aligned}$$

On the other hand, from

$$\int_0^t [\sin(mu+\alpha_u) - \sin(nu+\alpha_u)]du = O(1/n),$$

again integrating by parts we have $J_2 = O(K/n)^{\alpha+1}$. Thus $I_2 = o(\log m - \log n)$ since $K > 0$ is arbitrary, and our theorem is established.

Further we have the following theorem under another Lipschitz condition cf. [7]).

THEOREM 5. *Theorem 4 is valid even if the condition* (5.1) *is replaced by* $\psi \in \mathrm{Lip}(1, p)$ *in* $(0, \delta)$, *where* $|\alpha| p > 1$.

It is sufficient to show that $I_2 = o(\log m - \log n)$. But by Lemma 4

$$|I_2| \leqq \frac{1}{A_n^\alpha}\int_0^\delta \frac{|\Delta\psi(t)|}{(2\sin t/2)^{\alpha+1}}dt + o(\log m - \log n)$$

where $\Delta\psi(t) = \psi(t) - \psi(t+\pi/n)$. And since it is evident that from $|\alpha| p > 1$ it follows $(\alpha+1)q < 1$ where $1/p + 1/q = 1$, the fisrt term of the right hand side does not exceed

$$\frac{1}{A_n^\alpha}\left(\int_0^\delta |\Delta\psi(t)|^p dt\right)^{1/p}\left(\int_0^\delta \frac{dt}{(2\sin t/2)^{(\alpha+1)q}}\right)^{1/q}$$

$$= \frac{1}{A_n^\alpha}\cdot O\left(\frac{\pi}{n}\right) = O\left(\frac{1}{n^{\alpha+1}}\right),$$

which proves our theorem.

6. Theorem I in the section 1 is trivial under the condition (1.4). The purpose of S. Izumi [3] is, I suppose, to show that the theorem is valid even if $O(1)$, in the right hand side of (1.4), would be replaced by $O(\log n)$. This will be negative. But we can show that (1.4) may be replaced by a weaker condition.

Evidently, from the condition

$$\text{(b)} \qquad \int_0^t |\psi(u)|\,du = O(t) \quad \textit{as} \quad t \to 0$$

it follows

$$\text{(c)} \qquad \int_{t'}^t \frac{|\psi(u)|}{u}du = O\left(\log\frac{2t}{t'}\right)$$

as $0 < t' \leqq t \to 0$. Inversely, if (c) holds then

$$\int_0^t |\psi(u)|\,du = \int_0^t \frac{|\psi(u)|}{u}du\int_0^u dv = \int_0^t dv\int_v^t \frac{|\psi(u)|}{u}du$$

$$= O\left(\int_0^t \log\frac{2t}{v}\,dv\right) = O(t),$$

which is the condition (b). Thus the conditions (b) and (c) are equivalent. Therefore the condition

$$\text{(d)} \qquad \int_{t'}^t \frac{|\psi(u) - \psi(u+t')|}{u}du = O\left(\log\frac{2t}{t'}\right),$$

as $0 < t' \leqq t \to 0$, is weaker than (b), and also evidently than (1.4). Now we have the following

THEOREM 6. *If $\psi(t)$ satisfies the conditions $\int_0^t \psi(u)\,du = o(t)$ and* (d) *then*

$$\bar{\sigma}_{2n}(x) - \bar{\sigma}_n(x) = \frac{l(x)}{\pi}\log 2 + o(1)$$

as $n \to \infty$.

It is sufficient to show, analogously as in the proof of Theorem I, that

$$P_n = \frac{1}{n}\int_{k_n\eta}^{\pi} \frac{|\psi(t) - \psi(t+\eta)|}{t^2} dt = o(1)$$

as $n \to \infty$, where $\eta = \pi/n$ and $\{k_n\}$ is a positive sequence such that $k_n \to \infty$ as $n \to \infty$ sufficiently slowly. We write

$$\Psi(t) = \int_{\eta}^{t} \frac{|\psi(u) - \psi(u+\eta)|}{u} du,$$

then integrating by parts we have

$$\begin{aligned} P_n &= \frac{1}{n}\left[\frac{1}{t}\Psi(t)\right]_{k_n\eta}^{\pi} + \frac{1}{n}\int_{k_n\eta}^{\pi} \frac{1}{t^2}\Psi(t)\,dt \\ &= O\left(\frac{1}{k_n}\log(2k_n)\right) + O\left(\frac{1}{n}\int_{k_n\eta}^{\pi}\frac{1}{t^2}\log\frac{2t}{\eta}dt\right) \\ &= O((\log k_n)/k_n) = o(1) \end{aligned}$$

as $n \to \infty$. Thus we have the desired result.

7. We shall now give an application of Lemma 3 to the sequence $\{nB_n\}$, where $B_n = B_n(x)$ is defined by (1.1).

Let $\bar{\tau}_n^{\alpha} = \bar{\tau}_n^{\alpha}(x)$ be the n-th (C, α) mean of $\{nB_n\}$, i.e.

$$\bar{\tau}_n^{\alpha} = \frac{1}{A_n^{\alpha}}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}(\nu B_\nu).$$

Then we have the well-known identity

$$\bar{\tau}_n^{\alpha+1} = (\alpha+1)(\bar{\sigma}_n^{\alpha} - \bar{\sigma}_n^{\alpha+1}) \tag{7.1}$$

for $\alpha > -1$. On the other hand, from (3.2) with $\beta = \alpha + 1$ and $m = n$ it follows

$$\begin{aligned} \bar{\sigma}_n^{\alpha}(x) - \bar{\sigma}_n^{\alpha+1}(x) &= \frac{l(x)}{\pi}[\lambda(\alpha, n) - \lambda(\alpha+1, n)] + o(1) \\ &= \frac{l(x)}{\pi}\left(\frac{1}{\alpha+1} - \frac{1}{\alpha+1+n}\right) + o(1) = l(x)/\pi(\alpha+1) + o(1) \end{aligned}$$

as $n \to \infty$ if $\psi(t)$ satisfies the conditions (a) and $\bar{\Lambda}(\alpha, n) = o(1)$. Therefore by (7.1) we have the following

LEMMA 5. *If $\alpha > -1$ and $\psi(t)$ satisfies the conditions*

$$\int_0^t \psi(u)\,du = o(t) \qquad \text{as } t \to +0, \tag{a}$$

and $\bar{\Lambda}(\alpha, n) = o(1)$ as $n \to \infty$, then the sequence $\{nB_n(x)\}$ is summable $(C, \alpha + 1)$ to the value $l(x)/\pi$.

From this lemma we have immediately the following

THEOREM 7. *If $\alpha > 0$ and $\psi(t)$ satisfies the conditions* (a) *and*

(b) $$\int_0^t |\psi(u)| \, du = O(t) \qquad as\ t \to 0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(C, \alpha+1)$ *to the value* $l(x)/\pi$.

This theorem with $\alpha = 1$ coincides with Theorem S_2.

The following theorem due to Hardy and Littlewood is well-known.

THEOREM H. L. *The Fourier series of* $f(x)$ *converges at the point* x *to the value* $f(x)$, *if the two conditions be satisfied.*

(i) $f(x+h) - f(x) = o(\log(1/|h|))^{-1}$, *and* (ii) *the coefficients* a_n *and* b_n *are* $O(n^{-\delta})$, $\delta > 0$.

The proof of Theorem M. N. in the section 1 is reduced to the condition $\bar{A}(0, n) = o(1)$, i.e.

$$\int_{\pi/n}^{\pi} \psi(t) \frac{\cos(n+1/2)t}{2\sin(t/2)} dt = o(1),$$

which can be proved analogously as Theorem H. L.. Therefore, Theorem M. N. is derived from Theorem H. L. and Lemma 5 immediately.

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MATHEMATICAL INSTITUTE,
NARA WOMEN'S UNIVERSITY.