WEAK COMPACTNESS IN AN OPERATOR SPACE

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1. Introduction. Many important theorems in measure theory have been extended to W^*-algebras by many authors, especially, Dixmier [2], Dye [3] and Segal [8]. Considered as non-commutative extensions, these extensions are interesting themselves and provide powerful tools in the further investigations of W^*-algebra. In the previous papers ([11] and [12]), we have discussed and extended the concepts of conditional expectations, which have been introduced by Dixmier in the operator theoretical term [2], and martingales in the probability theory into finite and semi-finite W^*-algebras. The concept of the former has been also discussed in a general situation by Nakamura-Turumaru [6].

The purpose of the present paper is to extend certain compactness theorems in L' on measure space to L' on W^*-algebra in the sense of [8] and [2]. Firstly, as a preliminary we shall prove the extension of Vitali-Hahn-Saks's Theorem for any W^*-algebra A with a regular gage μ (cf. § 3) (that is, a regular gage space (A, μ) in the sense of [8]), which implies the equi-absolute continuities of weakly convergent sequence in $L'(A, \mu)$. Secondly, we shall extend the Lebesgue's compactness theorem to W^*-algebra with respect to a finite gage and give a sufficient condition for a subset in $(A_*)^+$ to be weakly compact (A_* being a Banach space in the notation of [2], cf. § 4 as below). The former characterizes the weakly conditional compactness of a subset in $L'(A)$, and the latter is possible to extend a Kakutani's compactness theorem in L' (with respect to measure space) to the present $L'(A)$ with respect to arbitrary gage (cf. § 4). In the last part of § 4, we shall also characterize the weakly sequential compactness of subset in $L'(A, \mu)^+$ by a uniform continuity of the set in the form of Bartle-Dunford-Schwartz [1], and further prove weakly sequential completeness of $L'(A, \mu)$ for A of finite type and any gage μ.

2. Preliminary and notations. Let "$\mathfrak{P}$" be the set of all projections in the W^*-algebra A acting on a Hilbert space H. For any $p \in \mathfrak{P}$ there corresponds uniquely a closed linear subspace $\mathfrak{M}_p \subset H$ such that the projection from H onto $\mathfrak{M}_p$ coincides with p. For any $p, q \in \mathfrak{P}$, the meet $p \wedge q$ and the join $p \vee q$ are uniquely defined as the projections onto $\mathfrak{M}_p \cap \mathfrak{M}_q$ and $\mathfrak{M}_p \oplus \mathfrak{M}_q$ respectively. Whence $\mathfrak{P}$ is a complete lattice with respect to the $\wedge$ and $\vee$.

Let μ be a gage of A in the sense of [8], i.e. non-negative valued, unitary

Received November 1, 1956.

invariant and completely additive function on $\mathfrak{P}$ satisfying that every $p \in \mathfrak{P}$ is l. u. b. of μ-finite projections $q \in \mathfrak{P}$ within $q \leqq p$. Denote by P_μ(or merely P) the set of all μ-finite projections in $\mathfrak{P}$. A gage μ of A is called to be regular if it is faithful (cf. [8]), which coincides with the contraction onto $\mathfrak{P}$ of the "normale, fidèle, éssentielle et maximale" trace in the sense of [2]. Let $L'(A, \mu)$ and $L^2(A, \mu)$ (or merely $L'(A)$ and $L^2(A)$) be the space of all integrable or square integrable operators with respect to a fixed gage μ with the norms $\|x\|_1$ and $\|x\|_2$ respectively. The gage μ is uniquely extended to a positive linear functional on $L'(A)$, which is also denoted by $\mu(x)$ $(x \in L'(A))$.

For any W^*-subalgebra B_1 of A, let $I_0 = \text{l. u. b.}\{p;\ p \in P \cap B_1\}$ which belongs to the center of B_1 and $I_0B_1(= B$, say) is considered as a W^*-algebra on the Hilbert space I_0H. The contracted function of μ onto $\mathfrak{P} \cap B$(denote it by the same notation μ) is also a gage of B and $L'(B, \mu)$ is a subspace of $L'(A, \mu)$, which is uniquely determined by (B_1, μ). We denote it by $L'(B_1, \mu)$. If μ is regular on A, then it is also regular on B.

Denote the set of all non-negative operators in $L'(A)$ by $L'(A)^+$. Any $x \in L'(A)$ is uniquely expressed by $x = x^{(1)} - x^{(2)} + ix^{(3)} - ix^{(4)}$ with $x^{(j)} \in L'(A)^+$. Put $x' = x^{(1)} - x^{(2)}$ and $x'' = x^{(3)} - x^{(4)}$, which are real and imaginary parts of x respectively.

For any $x \in L'(A)$, $W(x)$ denotes the W^*-subalgebra generated by $\{e_\lambda(x'),\ e_\lambda(x'')\}_\lambda$ where $x' = \int \lambda de_\lambda(x')$ and $x'' = \int \lambda de_\lambda(x'')$. Further for any subset S in $L'(A)$, "$W(S)$" denotes the W^*-subalgebra generated by $\{W(x);\ x \in S\}$.

If E is a Banach space, $E^\wedge$ denotes the conjugate space of E. The weak topology in E as point is merely called by weak topology or $\sigma(E, E^\wedge)$-topology, and the weak topology in $E^\wedge$ as functional is called weak* topology or $\sigma(E^\wedge, E)$-topology. The conjugate space of $L'(A)$ is denoted by $L^\infty(A)$.

3. Equi-absolute continuity of a convergent sequence of functionals. Firstly, we give a fundamental definition:

DEFINITION 1. Let A be a W^*-algebra with a gage μ. A set S of linear functionals on A is called to be *equi μ-absolutely continuous,* if for any real $\varepsilon > 0$ there exists a real $\delta > 0$ such that

$$(1) \qquad \mu(p) < \delta \ (p \in \mathfrak{P}) \text{ implies } |f(p)| < \varepsilon \quad \text{for all } f \in S.$$

Similarly if S is a subset of $L'(A, \mu)$ and $\{f_x;\ x \in S\}$ (where $f_x(y) = \mu(xy)$ for all $y \in A$) satisfies (1), then S is called to be *equi μ-absolutely continuous.*

For any given semi-finite W^*-algebra A acting on a Hilbert space H and a regular gage μ of A (it is known by Dixmier that such A has always regular gage), Vitali-Hahn-Saks's Theorem can be extended to this (A, μ):

THEOREM 1. *Let $\{f_n\}$ be a sequence of linear functionals on A which are strongly continuous on the unit sphere of A. If for every projection p in A $\lim_{n\to\infty} f_n(p)$ exists and is finite, then the set $\{f_n\}$ is equi μ-absolutely continuous.*

LEMMA 1.1. *For any pair* $p, q \in P$, *putting* $\rho(p, q) = \sqrt{\mu(|p-q|^2)}$, ρ *satisfies the metric conditions and* (P, ρ) *is a complete metric space.*

Proof. It follows from [2] or [8] that ρ satisfies the metric conditions. Now let us prove the completeness of (P, ρ). Taking $\{p_n\} \subset P$ such that $\rho(p_m, p_n) \to 0$ (as $m, n \to \infty$), by the completeness of $L^2(A, \mu)$, there exists an $x \in L^2(A, \mu)$ such that $\mu(|x - p_n|^2) \to 0$ (as $n \to \infty$). Since $0 \leqq p_n \leqq 1$, p_n converges strongly to x on the Hilbert space H and $0 \leqq x \leqq 1$ on H. Hence for any $\xi, \eta \in H$

$$(x\xi, \eta) = \lim_{n\to\infty} (p_n\xi, \eta) = \lim_{n\to\infty} (p_n\xi, p_n\eta) = (x\xi, x\eta) = (x^2\xi, \eta).$$

Since $x^* = x$ in $L^2(A, \mu)$, we get $\mu(x) < \infty$ and $x \in P$.

LEMMA 1.2. *For any* $\delta > 0$ *and* $p_0 \in P$, *putting* $U_\delta(p_0) = \{p \in P; \rho(p_0, p) < \sqrt{\delta}\}$ *and* $V_\delta(p_0) = \{p \in P; \mu(p) < \mu(p_0) + \delta, \mu(pp_0) > \mu(p_0) - \delta\}$, *then* $V_\delta(p_0) \subset U_{3\delta}(p_0)$.

Proof. If $p \in V_\delta(p_0)$, then

$$\rho(p_0, p)^2 = \mu(|p_0 - p|^2) = \mu((p_0 - p)^2) = \mu(p_0) + \mu(p) - 2\mu(pp_0)$$
$$< 2\mu(p_0) + \delta - 2\mu(p_0) + 2\delta = 3\delta.$$

Therefore $p \in U_{3\delta}(p_0)$.

Proof of Theorem 1. Since each f_n is a continuous function on (P, ρ), for any fixed integer $n_0 > 0$ and any fixed $\varepsilon > 0$ putting

$$E_{n_0} = \{p \in P; \sup_{m,n \geqq n_0} |f_m(p) - f_n(p)| \leqq \varepsilon/4\},$$

each E_{n_0} is closed in (P, ρ) and $\bigcup_{n_0=1}^{\infty} E_{n_0} = P$ by the assumption of $\{f_n\}$. By Lemma 1.1 and Baire's category theorem, for some n_0 E_{n_0} has a non-empty interior in (P, ρ). Therefore there exist p $(\neq 0) \in P$ and $\delta > 0$ such that $U_{3\delta}(p)$ is non-empty and contained in E_{n_0}. Let q be any fixed projection in P with $\mu(q) < \delta_1 = \min(\delta, \mu(p))$. Putting $r = p \vee q$, we have

$$\mu(r - p) \leqq \mu(p) + \mu(q) - \mu(p) = \mu(q) < \delta$$

and $\mu(rp) = \mu(p) > \mu(p) - \delta$, i. e. $r \in V_\delta(p)$. Furthermore, since $\mu(q) < \mu(p)$, we get $r > q$, $\mu(r - q) \leqq \mu(p) < \mu(p) + \delta$ and

$$\mu((r - q)p) = \mu(p) - \mu(qp) > \mu(p) - \delta,$$

i.e. $r - q \in V_\delta(p)$. Hence we deduce that $r, r - q \in E_{n_0}$. Since $q = r - (r - q)$,

$$|f_m(q) - f_n(q)| \leqq |f_m(r) - f_n(r)| + |f_m(r - q) - f_n(r - q)|$$
$$< \varepsilon/2 \qquad \text{for all } m, n \geqq n_0.$$

For $n = 1, 2, \cdots, n_0$, we can find a $\delta_2 > 0$ $(\delta_2 < \delta_1)$ such that

(2) $$|f_n(q)| < \varepsilon/2 \qquad \text{for any } q \in P \text{ with } \mu(q) < \delta_2$$

and $n = 1, 2, \cdots, n_0$. Consequently we obtain that $\mu(q) < \delta_2$ implies

(3) $$|f_n(q)| \leqq |f_n(q) - f_{n_0}(q)| + |f_{n_0}(q)| < \varepsilon \quad \text{for all} \quad n \geqq n_0.$$

(2) and (3) imply the equi μ-absolute continuity of $\{f_n\}$.

REMARK 1. The above proof has done under the Lemmas 1.1 and 1.2 by a similar proof of classical Vitali-Hahn-Saks's Theorem[1], in which the metric ρ is defined (denote it ρ_1 as the following) by the L'-norm, i.e. $\rho_1(p, q) = \mu(|p - q|)$ for $p, q \in P$. If the gage μ is finite, then the metrics ρ and ρ_1 are equivalent, and the neighborhood topologies in P defined by $\{U_\delta(p)\}$ and $\{V_\delta(p)\}$(cf. Lemma 1.2) are also equivalent to the metric topology.

4. Weak compactness of subset in $L'(A, \mu)$. A subset K of a Banach space E is called to be weakly (or equally $\sigma(E, E^\wedge)$-)conditionally compact, if the weak ($\sigma(E, E^\wedge)$-)closure of K is weakly ($\sigma(E, E^\wedge)$-)compact subset of E.[2] Firstly we shall extend Lebesgue's compactness theorem to a W^*-algebra A.

THEOREM 2. *Let μ be a gage of A with $\mu(I) < \infty$. Then, for a subset K of $L'(A, \mu)$ to be weakly conditionally compact it is necessary and sufficient that K is equi μ-absolutely continuous and K', K'' are bounded in the L'-norm where $K' = \{x'; x \in K\}$ and $K'' = \{x''; x \in K\}$.*

LEMMA 2.1. *For any equi μ-absolutely continuous subset K of $L'(A, \mu)$, K_j $(j = 1, \cdots, 4)$ are also equi μ-absolutely continuous, where $K_j = \{x^{(j)}; x \in K\}$.*

Proof.[3] Since for every projection p

$$|\mu(xp)|^2 = |\mu(x'p)|^2 + |\mu(x''p)|^2,$$

K' and K'' are equi μ-absolutely continuous. For fixed x', there exists $q \in \mathfrak{P}$ such that $x^{(1)} = qx' = x'q$ and $x^{(2)} = (1 - q)x' = x'(1 - q)$. For any $\varepsilon > 0$ and K', take $\delta > 0$ as in (1). Since $\mu(p) < \delta$ $(p \in \mathfrak{P})$ implies $\mu(qpq) < \delta$ and $\mu((1 - q)p(1 - q)) < \delta$,

$$0 \leqq \mu(px^{(1)}) = \mu(pqx') = \mu(qpqx') < \varepsilon$$

and similarly $0 \leqq \mu(px^{(2)}) < \varepsilon$. Hence K_1 and K_2 are equi μ-absolutely continuous, and also similarly for K_3 and K_4.

Proof of Theorem 2. (Sufficiency). Let $\overline{K}_j (j = 1, \cdots, 4)$ be $\sigma(L^\infty, L^{\infty\wedge})$-closures of K_j respectively which are $\sigma(L^\infty, L^{\infty\wedge})$-compact in $L^{\infty\wedge}$. For any $p \in \mathfrak{P}$ with $\mu(p) < \delta$ and for any fixed $f \in \overline{K}_1$ there exists $x \in K$ such that

$$|f(p) - \mu(x^{(1)}p)| < \varepsilon.$$

1) See Saks [7] for finite measure space and also see e.g. Sunouchi [10] for σ-finite measure space.

2) Further, a subset K of E is called to be *weakly* (or equally $\sigma(E, E^\wedge)$-) *sequentially conditionally compact*, if any countable subset C of K contains always a sequence $\{x_n\}$ which converges weakly to some $x \in E$.

3) This proof also holds for any gage without finiteness $\mu(I) < \infty$.

Since $0 \leqq \mu(x^{(1)}p) < \varepsilon$ for every such p,

$$0 \leqq f(p) \leqq |f(p) - \mu(x^{(1)}p)| + \mu(x^{(1)}p) < 2\varepsilon,$$

i.e. $0 \leqq f(p) < 2\varepsilon$ for every $p \in \mathfrak{P}$ with $\mu(p) < \delta$. Therefore by Radon-Nikodym's Theorem of Dye [3] there exists $z \in L'(A)$ such that $f(y) = \mu(zy)$ for every $y \in A$. This means that K_1 is weakly conditionally compact in $L'(A, \mu)$. Similarly we get $K_j (j = 2, 3, 4)$. Consequently K is weakly conditionally compact in $L'(A, \mu)$.

(Necessity). For this purpose we can assume μ to be regular without loss of generality, and hence A is countably decomposable and of finite type, because $\mu(I) < \infty$. Since K' and K'' are weakly sequentially conditionally compact (cf. [9]), they are bounded in the L'-norm. Assuming the contrary of the equi μ-absolute continuity of K, there exist $\{p_n\} \subset \mathfrak{P}$ and a weakly convergent sequence $\{x_n\} \subset K$ such that

$$\mu(p_n) < \frac{1}{n} \quad \text{and} \quad |\mu(x_n p_n)| > \varepsilon \tag{4}$$

for some $\varepsilon > 0$ and for all $n = 1, 2, \cdots$. Putting $f_n(y) = \mu(x_n y)$ for $y \in A$, $\lim_{n\to\infty} f_n(y)$ exists for every $y \in A$ which contradicts (4) by Theorem 1.

In a general situation, we can give a sufficient condition for weak compactness: Let A be a W^*-algebra and A_* be the Banach space of all linear functionals on A which are strongly continuous on the unit sphere of A. Then $(A_*)^\wedge = A$ (cf. [2]). Denote the set of all non-negative functionals in A_* by $(A_*)^+$, then

COROLLARY 2.1. *If a subset K of $(A_*)^+$ is bounded in the norm of A_* and satisfies*

(5) *for any decreasing directed set $\{p_\alpha\}$ of projections in A with $p_\alpha \downarrow 0$, $f(p_\alpha)$ converges to 0 uniformly for every $f \in K$,*

then K is $\sigma(A_*, A)$-conditionally compact.

Proof. Since any completely additive positive linear functional on A belongs to A_* by Dixmier (cf. Théorème 1 and footnote 6 of [2]), the proof will be obtained by the method almost similar with the proof of suficiency of Theorem 2, that is, let $\bar{K}$ being $\sigma(A, A^\wedge)$-closure of K, then every $f \in \bar{K}$ is non-negative linear functional on A, and by (5) f is completely additive. Hence by the theorem of Dixmier f belongs to $(A_*)^+$, and K is $\sigma(A_*, A)$-conditionally compact.

By Corollary 2.1, Kakutani's compactness Theorem (cf. Theorem 10 of [5]) will be extended to the following:

COROLLARY 2.2. *Let A be a W^*-algebra with gage μ. Let $x_1, x_2 \in L'(A, \mu)^+$ with $x_1 < x_2$. Then $\{x; x_1 \leqq x \leqq x_2\}$ is weakly conditionally compact in $L'(A, \mu)^+$.*

Uuder the same notation of the above Corollary 2.2, we prove the following:

THEOREM 3. *For a subset $K \subset L^1(A, \mu)^+$ to be weakly sequentially conditionally compact, it is necessary and sufficient that K is bounded in L^1-norm and satisfies*

(5′) *for any sequence of projections $\{p_n\}$ in A with $p_n \downarrow 0$, $\mu(xp_n)$ converges to 0 uniformly for every $x \in K$.*

Proof of sufficiency. In this case we can also assume μ to be regular without loss of generality. Let $\{x_n\} \subset K$. Putting $B_1 = W(\{x_n\})$ and $B =$ weak closure of $B_1 \cap L^1(A, \mu)$, B is a countably decomposable W^*-algebra on a closed linear subspace of $L^2(A, \mu)$. Further $\{x_n\}$ is contained in $L^1(B_1, \mu)$ and satisfies (5′) on (B, μ). Therefore by Corollary 2.1, $\{x_n\}$ is weakly conditionally compact in $L^1(B_1, \mu)$, and there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges weakly to an $x \in L^1(B_1, \mu)$, i.e. $\mu(xy) = \lim_k \mu(x_{n_k}y)$ for all $y \in B$. Let y^e be the conditional expectation of $y \in A$ relative to B_1,[4] then $\mu(zy^e) = \mu(zy)$ for all $z \in B_1 \cap L^1(A, \mu)$ (cf. [2] or [12]). Since $L^1(B_1, \mu)$ coincides with the $L^1(\mu)$-closure of $B_1 \cap L^1(A, \mu)$, $\mu(zy^e) = \mu(zy)$ for all $z \in L^1(B_1, \mu)$. Moreover x and $x_n (n=1,2,\cdots)$ belong to $L^1(B_1, \mu)$ and therefore for every $y \in A$

$$(6) \qquad \mu(xy) = \mu(xy^e) = \lim_{k\to\infty} \mu(x_{n_k}y^e) = \lim_{k\to\infty} \mu(x_{n_k}y),$$

that is, x_{n_k} converges weakly to x in $L^1(A, \mu)$ and K is weakly sequentially conditionally compact.

Proof of necessity. The boundedness of K in the L^1-norm is obvious. Assuming the contrary of (5′), there exist $\varepsilon_1 > 0$, $\{p_n\} \subset \mathfrak{P}$ and weakly convergent sequence $\{x_n\} \subset K$ such that

$$(7) \qquad p_n \downarrow 0 \quad \text{and} \quad \mu(x_n p_n) > \varepsilon_1 \quad \text{for all} \quad n = 1, 2, \cdots.$$

Putting $f_n(y) = \mu(x_n y)$ $(n = 1, 2, \cdots)$ and $\nu(y) = \sum_{n=1}^{\infty} f_n(y)/c \cdot 2^n$ $(c = \sup\{\|x\|_1; x \in K\})$, f_n are absolutely continuous with respect to ν. Let C be W^*-subalgebra generated by $\{p_n\}$ which is commutative. Hence by Vitali-Hahn-Saks's Theorem on commutative case of Theorem 1 or on usual measure space, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_n(p)| < \varepsilon$ for every $p \in \mathfrak{P} \cap C$ with $\nu(p) < \delta$. Since $\nu(p_n) \to 0$ (as $n \to \infty$), $\mu(x_n p_n) = f_n(p_n) \to 0$, (7) yields a contradiction.

REMARK 2. This theorem has been proved by Bartle-Dunford-Schwartz (cf. Theorem 1 of [1]) for subset of space of measures on abstract set and the proof of necessity is done by a method similar with that. If A is commutative, then we get a similar fact with [1], i.e. taking a gage μ of A, for subset K of $L^1(A, \mu)$ without the restriction that $K \subset L^1(A, \mu)^+$, Theorem 3 will be obtained by our proof, because any countably additive linear functional on the W^*-algebra B (cf. proof of Theorem 3) is strongly continuous on its unit sphere. We have also the same fact for subset K in A_*, because A_* is isometrically isomorphic to $L^1(A, \mu)$ with respect to a regular gage μ on A.

4) The notion of the conditional expectation refers to [12].

REMARK 3. In Theorem 3, applying the Eberlein's Theorem (cf. [4]), if K is weakly closed, then the condition is necessary and sufficient for K to be weakly compact.

Applying Theorem 2 and the proof of Theorem 3, we have

COROLLARY 3.1. *Let A be a finite W^*-algebra and let μ be any fixed gage. Then $L'(A, \mu)$ is weakly sequentially complete.*

Proof. Again we can assume μ to be regular. Let $\{x_n\} \subset L'(A, \mu)$ be a sequence with finite $\lim \mu(x_n y)$ for all $y \in A$. For this $\{x_n\}$, we take the W^*-algebras B_1 and B as in the proof of Theorem 3. Then B has a finite regular gage τ. Putting $f_n(y) = \mu(x_n y)$ for $y \in A$, f_n are strongly continuous on the unit sphere of B and there exists $z_n \in L'(B, \tau)$ such that $f_n(y) = \tau(z_n y)$ for all $y \in B$ and $n = 1, 2, \cdots$. Since $\lim f_n(y) (= f(y)$ say) exists and is finite for every $y \in A$, $\{z_n'\}$ and $\{z_n''\}$ are bounded in $L'(\tau)$-norm and by Theorem 1 $\{z_n\}$ is equi τ-absolutely continuous, and by Theorem 2 $\{z_n\}$ is weakly conditionally compact in $L'(B, \tau)$. Consequently $f(y)$ is strongly continuons on the unit sphere of B, and there exists $x \in L'(B_1, \mu)$ such that $f(y) = \mu(xy)$ for all $y \in B$. Let y^e be the conditional expectation of $y \in A$ relative to B_1, then by the same computation of the proof of Theorem 3, we get the equation (6) for $\{x_n\}$ in the place of $\{x_{n_k}\}$.

REFERENCES

[1] R. G. BARTLE, N. DUNFORD AND J. SCHWARTZ, Weak compactness and vector measures. Canadian Journ. Math. 8 (1955), 289—305.

[2] J. DIXMIER, Formes linéaires sur un anneau d'opérateurs. Bull. Soc. Math. France **81** (1953), 6—39.

[3] H. A. DYE, The Radon-Nikodym theorem for finite rings of operator. Trans. Amer. Math. Soc. **72** (1952), 234—280.

[4] W. F. EBERLEIN, Weak compactness in Banach space. Proc. Nat. Acad. Sci. U. S. A. **33** (1947), 51—53.

[5] S. KAKUTANI, Concrete representation of abstract (L)-spaces and the mean ergodic theorem. Ann. of Math. **42** (1941), 523—537.

[6] M. NAKAMURA and T. TURUMARU, Expectations in an operator algebra. Tôhoku Math. Journ. **6** (1954), 182—188.

[7] S. SAKS, Addition to the note on some functionals. Trans. Amer. Math. Soc. **35** (1931), 965—970.

[8] I. E. SEGAL, A non-commutative extension of abstract integration. Ann. of Math. **57** (1953), 401—457.

[9] V. ŠUMLIAN, Über lineare topologische Räume. Recueil Math. **49** (1940).

[10] G. SUNOUCHI, On the sequence of additive set functions. Journ. Math. Soc. Japan **3** (1951), 290—295.

[11] H. UMEGAKI, Conditional expectation in an operator algebra. Tôhoku Math. Journ. **6** (1954), 177—181.

[12] H. UMEGAKI, Conditional expectation in an operator algebra II. Tôhoku Math. Journ. 8 (1956), 86—100.

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