A SUPPLEMENT TO "NOTES ON CONFORMAL MAPPINGS OF A RIEMANN SURFACE ONTO ITSELF"

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We prove here the following equality in order to supplement the results in our paper "Notes on conformal mappings of a Riemann surface onto istelf" (these Seminar Reports 8 (1956), 23—30):

For $2g+k-1\geqq 2$ $(g\geqq 0,\ k\geqq 1)$,

$$N(g,k)=N'(g,k).$$

It suffices to show the inequality $N(g,k)\leqq N'(g,k)$, since the opposite direction has been known already (Theorem 1).

Let W be a bordered Riemann surface with genus g, having k boundary curves $C_1, C_2, \cdots, C_k$, and the order of the group $\mathfrak{G}$, which consists of all conformal mappings of W onto itself, is equal to $N(g,k)$. In the following lines we shall show that W can be imbedded in a closed Riemann surface W^* with the same genus g in such a way that any element of $\mathfrak{G}$ is continuable to a conformal mapping of W^* onto itself.

Since the doubled Riemann surface $\hat{W}$ of W is a closed Riemann surface with genus $2g+k-1\geqq 2$, we can introduce the non-Euclidean metric on $\hat{W}$ in the well-known manner: It is defined by the projection of the non-Euclidean metric

$$ds=\frac{|dt|}{1-|t|^2}$$

of the universal covering surface $|t|<1$ of $\hat{W}$. The non-Euclidean distance of two points $p,q\in\hat{W}$ is, of course, the greatest lower bound of the lengths $\int ds$ of curves connecting p and q. We know that any conformal mapping of $\hat{W}$ onto itself does not change this metric.

On $\hat{W}$, we consider the sets

$$D_\nu=\{p;\ 0<\text{dist.}(p,C_\nu)<r,\ p\in W\},\qquad \nu=1,2,\cdots,k.$$

It is not difficult to see that, if r is taken sufficiently small, they are doubly connected subregions of W and mutually disjoint. For any $\varphi\in\mathfrak{G}$, we have $\varphi(D_\nu)=D_\mu$ provided that $\varphi(C_\nu)=C_\mu$, because φ can be considered, by reflec-

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tion, as a conformal mapping of $\hat{W}$ onto itself, which preserves the non-Euclidean distance on $\hat{W}$. We map D_ν conformally onto the concentric annulus

$$A_\nu : \qquad 1 < |z_\nu| < \rho_\nu$$

on z_ν-plane in such a way that C_ν corresponds to $|z_\nu| = 1$ ($\nu = 1, 2, \cdots, k$). Then, any $\varphi \in \mathfrak{G}$ satisfying a condition that $\varphi(C_\nu) = C_\mu$ can be considered as a conformal mapping of A_ν onto A_μ (hence $\rho_\nu = \rho_\mu$) such that $|z_\nu| = 1$ corresponds to $|z_\mu| = 1$. As is known, it must be of the form

$$z_\mu = e^{i\theta} z_\nu,$$

which is evidently continuable to a conformal mapping of $|z_\nu| < \rho_\nu$ onto $|z_\mu| < \rho_\mu$, satisfying a condition that $z_\nu = 0$ corresponds to $z_\mu = 0$.

The Riemann surface W^* that we want to construct is the union of sets W and $|z_\nu| \leqq 1$ ($\nu = 1, \cdots, k$), where corresponding points on $|z_\nu| = 1$ and C_ν are identified; local parameters are, as in usual, taken z_ν in the region $D_\nu \cup (|z_\nu| \leqq 1) = (|z_\nu| < \rho_\nu)$ ($\nu = 1, \cdots, k$) and original parameters in W. It is not dif ficult to see that W^* is a closed Riemann surface with genus g and contains W. Furthermore, denoting by p_ν the point on W^* corresponding to $z_\nu = 0$ ($\nu = 1, \cdots, k$), we see immediately that any $\varphi \in \mathfrak{G}$ is continuable to a conformal mapping ofl the region $W^* - \{p_1, \cdots, p_k\}$ onto itself. This fact implies that $N(g, k) \leqq N'(g, k)$.

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