A NOTE ON THE DIFFERENT OF THE COMPOSED FIELD

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Let K_1, K_2 be finite algebraic extensions of an algebraic number field of finite degree K_0, and K_3 be a composed field of K_1 and K_2.

In this note we will investigate a relation between the relative different of K_3 and those of K_1 and K_2.

We denote the relative differents of K_1, K_2 and K_3 with respect to K_0 by $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$ respectively. Then we obtain the following

Theorem.

$$[\mathfrak{D}_1, \mathfrak{D}_2] \,/\, \mathfrak{D}_3 \,/\, \mathfrak{D}_1\mathfrak{D}_2 ,$$

where the bracket means the least common multiple.

Proof

By chain relation of differents

$$\mathfrak{D}_3 = \mathfrak{D}_{3,2}\mathfrak{D}_2 ,$$

where $\mathfrak{D}_{3,2}$ denotes the relative different between K_3 and K_2. By the definition of differents, $\mathfrak{D}_1$ is an ideal generated by all the differents of integers of K_1 with respect to K_0. Therefore it is contained in the $\mathfrak{D}_{3,2}$, which is generated by all the differents of integers of K_3 with respect to K_2. That is

$$\mathfrak{D}_{3,2} \,/\, \mathfrak{D}_1 .$$

Hence

$$\mathfrak{D}_3 = \mathfrak{D}_{3,2}\mathfrak{D}_2 \,/\, \mathfrak{D}_1\mathfrak{D}_2 .$$

On the other hand, it is evident that

$$\mathfrak{D}_1 / \mathfrak{D}_3 , \quad \mathfrak{D}_2 / \mathfrak{D}_3 .$$

Finally we obtain

$$[\mathfrak{D}_1, \mathfrak{D}_2] \,/\, \mathfrak{D}_3 , \qquad \text{q.e.d.}$$

Corollary. If, $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are relatively prime to each other, then

$$\mathfrak{D}_3 = \mathfrak{D}_1\mathfrak{D}_2 .$$

Proof. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are relatively prime to each other, then

$$[\mathfrak{D}_1, \mathfrak{D}_2] = \mathfrak{D}_1\mathfrak{D}_2 .$$

so

$$\mathfrak{D}_3 = \mathfrak{D}_1\mathfrak{D}_2$$

Corollary.(1) If relative discriminants $\mathfrak{D}_1$, $\mathfrak{D}_2$ of K_1 and K_2 are relatively prime, then

$$D_3 = D_1^m D_2^n$$

where, m, n are relative degrees of K_3 with respect to K_1 and K_0.

Proof.

$$\begin{aligned} D_3 &= N_{3,0}(\mathfrak{D}_3) = N_{3,0}(\mathfrak{D}_1\mathfrak{D}_2) \\ &= N_{1,0}\{N_{3,1}(\mathfrak{D}_1)\}\, N_{2,0}\{N_{3,2}(\mathfrak{D}_2)\} \\ &= N_{1,0}(\mathfrak{D}_1^m)\, N_{2,0}(\mathfrak{D}_2^n) = D_1^m D_2^n . \end{aligned}$$

We shall give an example of a case when $[\mathfrak{D}_1, \mathfrak{D}_2]$, $\mathfrak{D}_3$, $\mathfrak{D}_1\mathfrak{D}_2$ do not coincide with each other.

Let $K_0 = \Gamma$ be the rational number field, and

$$K_1 = \Gamma(\sqrt{-7})$$

$$K_2 = \Gamma(\sqrt{2})$$

$$K_3 = \Gamma(e^{\frac{2\pi i}{8}})$$

Then

$$\mathfrak{D}_1 = (1+\sqrt{-7})^2$$

$$\mathfrak{D}_2 = (\sqrt{2})^3$$

In K_3, where $\zeta = e^{\frac{2\pi i}{8}}$

$$(2) = (1-\zeta)^4 = \mathfrak{L}^4 ,$$

so $\mathfrak{D}_1 = \mathfrak{L}^4$, $\mathfrak{D}_2 = \mathfrak{L}^6$.
The discriminant of K_3 is

$$D_3 = \pm 2^8 \text{ (2)} .$$

$$\mathfrak{D}_3 = \mathfrak{L}^8$$

therefore

$$[\mathfrak{D}_1, \mathfrak{D}_2] = \mathfrak{L}^6, \ \mathfrak{D}_3 = \mathfrak{L}^8, \ \mathfrak{D}_1 \mathfrak{D}_2 = \mathfrak{L}^{10}.$$

REFERENCES

(1) Hilbert, Zahlbericht, Satz, 88.
(2) Zahlbericht, Satz, 121.

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