A THEOREM ON FOURIER TRANSFORM

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1. We have defined in my paper(1) on Wiener Prediction theory a class of function $\mathfrak{K}_F$. Let $F(x)$ be a non-decreasing bounded function in $(-\infty, \infty)$. If there exists a function $K(\theta)$ of bounded variation in every finite interval such that

$$\lim_{A\to\infty}\int_{-\infty}^{\infty}\left| k(x) - \int_{-A}^{A} e^{-ix\theta} dK(\theta)\right|^2 dF(x) = 0$$

then $k(x)$ is said to belong to $\boldsymbol{K}(-\infty,\infty)$. Further if there exists a sequence of functions of $\boldsymbol{K}(-\infty,\infty)$, $\{k_n(x)\}$, such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} | k(x) - k_n(x)|^2 dF(x) = 0$$

then, $k(x)$ is called a function of $\mathfrak{K}_F$.

In connection with this, we shall show that $L_2(F)$ -continuous function can be always approximated by Fourier-Stietjes integral as closely as we please.

If

$$(1.1)\quad \int_{-\infty}^{\infty} | f(x)|^2 dF(x) < \infty,$$

then $f(x)$ is said to belong to $L_2(F)$, and if $f(x+u)\in L_2(F)$ for every u, and

$$(1.2)\quad \lim_{u\to 0}\int_{-\infty}^{\infty} |f(x+u) - f(x)|^2 dF(x) = 0,$$

then $f(x)$ is said to be $L_2(F)$ - continuous.

2. Let $f_R(x) = f(x)$, for $|x|\leqq R$,
$\qquad\qquad = 0$, for $|x| > R$.

We shall prove, under the condition (1.2), that for any given $\varepsilon > 0$, there exist a δ and an R_0 such that, for $|u| < \delta$, $R > R_0$,

$$(2.1)\quad \int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 dF(x) < \varepsilon.$$

Let $u \geqq 0$. The case $u<0$ can be similarly treated.

We have

$$\int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 dF(x)$$
$$= \int_{-R-u}^{-R} |f(x+u)|^2 dF(x) + \int_{-R}^{R-u} |f(x+u) - f(x)|^2 dF(x)$$
$$+ \int_{R-u}^{R} |f(x)|^2 dF(x)$$

$$(2.2)\quad = I_1 + I_2 + I_3,$$

say. Then

$$(2.3)\quad I_2 \leqq \int_{-\infty}^{\infty} |f(x+u) - f(x)|^2 dF(x) \to 0,\ u\to 0.$$

If $0\leqq u < \delta$, then for any $\delta > 0$,

$$(2.4)\quad I_3 \leqq \int_{R-\delta}^{R} |f(x)|^2 dF(x)$$

which tends to zero as $R\to\infty$, since $f(x)\in L_2(F)$.

Next,

$$I_1 = \int_{-R-u}^{-R} |f(x+u) - f(x) + f(x)|^2 dF(x)$$
$$\leqq 2\int_{-R-u}^{-R} |f(x+u) - f(x)|^2 dF(x)$$
$$+ 2\int_{-R-u}^{-R} |f(x)|^2 dF(x)$$
$$\leqq 2\int_{-\infty}^{\infty} |f(x+u) - f(x)|^2 dF(x) + 2\int_{-R-\delta}^{-R} |f(x)|^2 dF(x).$$

Hence

$$(2.5)\quad \lim_{R\to\infty}\limsup_{u\to 0} I_1 = 0.$$

(2.3), (2.4) and (2.5) give (2.1).

3. We shall prove

<u>Theorem 1. Let $f(x)\in L_2(F)$ be $L_2(F)$- continuous and be squarely integrable in ordinary Lebesgue sense in every finite interval. Then for any given positive $\varepsilon > 0$, there exist an A and a function $K(\theta) = K_A(\theta)$ of bounded variation in $[-A, A]$, such that</u>

$$(3.1)\quad \int_{-\infty}^{\infty} \Big| f(x) - \int_{-A}^{A} e^{-ix\theta} dK(\theta) \Big|^2 dF(x) < \varepsilon.$$

Put

$$(3.2)\quad f_R(x, A) = \frac{1}{\pi} \int_{-\infty}^{\infty} f_R(y) \frac{\sin^2 A(y-x)}{A(y-x)^2} dy.$$

Then

$$f_R(x,A) - f(x)$$
$$= f_R(x,A) - f_R(x) + f_R(x) - f(x)$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \{f_R(y) - f_R(x)\} \frac{\sin^2 A(y-x)}{A(y-x)^2} dy$$
$$+ f_R(x) - f(x)$$

Hence we have

$$\int_{-\infty}^{\infty} |f_R(x,A) - f(x)|^2 dF(x)$$
$$\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \Big| \int_{-\infty}^{\infty} \{f_R(y) - f_R(x)\} \frac{\sin^2 A(y-x)}{A(y-x)^2} dy \Big|^2 dF(x)$$
$$+ \int_{-\infty}^{\infty} |f_R(x) - f(x)|^2 dF(x)$$
$$(3.3)\quad = J_1 + J_2,$$

say. We have

$$(3.4)\quad J_2 \leqq \int_{|x|>R} |f(x)|^2 dF(x)$$

which tends to zero as $R \to \infty$

By Schwarz inequality

$$|J_1| \leqq \int_{-\infty}^{\infty} \Big\{ \frac{1}{\pi} \int_{-\infty}^{\infty} |f_R(y) - f_R(x)|^2 \frac{\sin^2 A(y-x)}{A(y-x)^2} dy \Big\} dF(x)$$
$$= \int_{-\infty}^{\infty} dF(x) \frac{1}{\pi} \int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 \frac{\sin^2 Au}{Au^2} du$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 Au}{Au^2} du \int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 dF(x)$$
$$= \frac{1}{\pi}\int_{|u|<\delta} + \frac{1}{\pi}\int_{|u|\geqq\delta}$$
$$= J_{1,1} + J_{1,2},$$

say. By (2.1)

$$(3.5)\quad |J_{1,1}| \leqq \frac{\varepsilon}{\pi} \int_{|u|<\delta} \frac{\sin^2 Au}{Au^2} du$$
$$\leqq \varepsilon \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 Au}{Au^2} du = \varepsilon.$$

We shall, here, prove that for every finite R, the integral

$$(3.6)\quad \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} |f_R(x+u)|^2 dF(x)$$

is finite.

In fact,

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} |f_R(x+u)|^2 dF(x)$$
$$= \int_{-\infty}^{\infty} du \int_{-R-u}^{R-u} |f(x)|^2 dF(x)$$
$$= \int_{-\infty}^{\infty} dF(x) \int_{-R-x}^{R-x} |f(x+u)|^2 du$$
$$= \int_{-\infty}^{\infty} dF(x) \int_{-R}^{R} |f(y)|^2 dy < \infty.$$

This also shows

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 dF(x) < \infty,$$

for every finite $R > 0$. Now

$$J_{1,2} \leqq \frac{1}{A\pi} \int_{|u|>\delta} \frac{du}{u^2} \int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 dF(x)$$
$$\leqq \frac{1}{A\delta^2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} |f_R(x+u) - f_R(x)|^2 dF(x).$$

Hence it is evident that there exists an $A_0 = A_0(\delta, R)$ such that for $A \geqq A_0$,

$$(3.7)\quad J_{1,2} \leqq \varepsilon$$

(3.5) and (3.7) show that

$$(3.8)\quad |J_1| \leqq 2\varepsilon.$$

By (3.4) and (3.8), for given ε, there exists an R_0 such that for $R \geqq R_0$ and $A = A(R)$,

$$\int_{-\infty}^{\infty} |f_R(x,A) - f(x)|^2 dF(x) < \varepsilon$$

Since

$$f_R(x,A) = \frac{1}{\pi}\int_{-\infty}^{\infty} f_R(y) \frac{\sin^2 A(y-x)}{A(y-x)^2} dy$$
$$= \frac{1}{2\pi}\int_{-A}^{A} \Big(1 - \frac{|\theta|}{A}\Big) e^{-ix\theta} d\theta \int_{-\infty}^{\infty} f_R(t) e^{i\theta t} dt$$

We can write

$$f_R(x,A) = \int_{-A}^{A} e^{ix\theta} dK(\theta),$$

$K(\theta)$ being

$$= 0, \quad |\theta| \geq A$$

$$= \frac{1}{2\pi}\int_{-A}^{\theta}\left(1-\frac{|u|}{A}\right)du\int_{-\infty}^{\infty} f_R(t)e^{iut}dt, \quad |\theta| < A.$$

Thus the theorem is proved.

Reference

(1) T. Kawata, On Wiener's prediction theory, Rep. Stat. Appl. Res. J.U.S.E. 2 (1953).

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