ON A LEMMA OF SUNOUCHI AND YANO

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Consider a jumper on the circle with length one in which an interval J forms a snare. The jumper jumps width c at one jump where c is a certain fixed irrational number. It is clear, he will fall into the snare after some jumps wherever he starts. The number m of jumps by which he fades away will depend on his starting point a.

Recently, G. Sunouchi and S. Yano [1] proved in their note on the absolute convergence of a certain function series, to generalizing a theorem of Szász, a lemma which can be formulated in our terms as follows:

PROPOSITION. There exists a positive integer k such that the jumper fades away into the snare after at most k jumps wherever he starts.

In this note, it will be shown that the proposition remains true even if the circle is replaced by certain abstract spaces.

A group G is called monothetic if G is compact abelian and if G contains an element c whose iterations nc are dense in G. Clearly, the torus is a monothetic group with an irrational number as generator. In this case, an open set N plays a role of the snare and the transformation f which maps x to $s+c$ can be considered as the jump in G. If the proposition is proved in this case, it is a slight generalization of Sunouchi-Yano's lemma.

Since G is monothetic and f is topological, the set A of all elements with the form $a+nc$ is also dense in G, for an arbitrary starting point a. Using this fact, we shall generalize the situation.

Let S be a compact Hausdorff space, and let T be a homeomorphism on S. T will be called a jump if the set of all $T^n(a)$'s (with $n = 1, 2, 3, \cdots$) is dense in S whenever a is a point of S which will be called starting point. An open set N of S will be called a snare. A jumper starting from a is called that he fades away after n jumps provided n is the least positive integer of the solution of the following implication:

$$T^n(a) \in N.$$

We shall prove the proposition under these circumstances. Clearly, our former monothetic case are contained in it.

PROOF: A point x will be called a point of the (first) scoring position N_1 if $T(x)$ is contained in N. Since T is topological, the scoring position N_1 is an open set. The second scoring position N_2 is defined as the set of all points which can reach the first scoring position after a jump. The n-th scoring position N_n will be defined by the induction. It is clear that N_n is open too. (For the sake of the simplicity, we may assume that a point can belong to two or more scoring positions, i. e., the jumper continues imaginary jumps after he fell into the snare.)

It is obvious that the jumper starting from a fades away after at most n jumps if and only if a belongs to the set union $N_1 \cup N_2 \cup \ldots \cup N_n$. Hence, to prove the proposition, it suffices to show that the finite number of scoring positions covers the space:

$$S = \bigcup_{i=1}^{k} N_i .$$

The maximum of the indices of such scoring positions becomes our k in the proposition.

By the compactness of the space S, it suffices to insure the above statement to prove that the space S is covered by all scoring positions, i. e.,

$$S = \bigcup_{i=1}^{\infty} N_i \ .$$

Suppose the contrary. Then there exists a starting point a which is not contained in all scoring positions. This is equivalent to that all $T^n(a)$'s are not contained in N. On the other hand, we have assumed that T is a jump, that is, $T^n(a)$'s are dense in S. This shows that one of $T^n(a)$'s belongs to N. This contradiction proves our statement.

REMARK: Since Theorem 1 of Sunouchi and Yano [1] is based on the above proposition, it is not hard to trace their proof under somewhat weaker hypothesis, which will give us an another formulation of the absolute convergence of function series on a certain abstract space. However, we do not enter in these problems.

We shall only point out a few modifications of our proposition. It will be natural to extend it in the case when the transformation group depends on continuous parameter t. Even in this case, our proof remins valid with a few verbal changes: N_t will be defined as the set of points which are contained after t-time, i. e., $N = T_t(N_t)$. Using the denseness of $\{T_t(a)\}$, we have also

$$S = \bigcup_t N_t$$

Again, the maximum of indices of finite subcovering becomes our desired k.

The proof remins valid too if T^n has a certain inhomogeneous properties. For an example, let V_n be a topological transformation of S onto S, and let T_n be the successive product of V_n:

$$T_n = V_1 V_2 \cdots\cdots V_n \ .$$

Since the homogenity of jumps are not necessary to prove the above proposition, it is still true if the set of all $\{T_n(a)\}$ is dense in S whenever it starts. However, it is to be noted that this generalization is not applicable to the theorems of Szász-Sunouchi-Yano.

(*) Received Oct. 5, 1953.

REFERENCE

1 G. Sunouchi and S. Yano, Notes on Fourier series, XXX: On the absolute convergence of certain series of functions, Proc. Amer. Math. Soc., 2 (1951), 380-389.

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