ON THE CONDITIONS OF UNIVALENCY OF CONFORMAL MAPPING

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Let D be a planar schlicht n-ply connected domain with a boundary Γ consisting of analytic curves Γ_ν ($\nu = 1, \ldots, n$). For simplicity's sake, we shall assume that D contains the origin. Let $F_m(z; \zeta, \alpha)$ be a regular function in D with an exception $z = \zeta$, where $F_m(z; \zeta, \alpha) - 1/(z-\zeta)^m$ is regular and all the images of Γ_ν ($\nu = 1, \ldots, n$) by F_m be the segments with inclination α to the real axis. Then we put

$$\varphi_m(z;\zeta) = \frac{1}{2}\left[F_m(z;\zeta,0) - F_m(z;\zeta,\frac{\pi}{2})\right],$$

$$(\varphi_0(z;\zeta) \equiv 0)$$

and

$$\phi_m(z;\zeta) = \frac{1}{2}\left[F_m(z;\zeta,0) + F_m(z;\zeta,\frac{\pi}{2})\right],$$

$$(\phi_0(z;\zeta) \equiv 1).$$

Therefore both functions have the expansions

$$\sum_{\nu=1}^{\infty} S_{m\nu}(\zeta)\,(z-\zeta)^\nu$$

and

$$\frac{1}{(z-\zeta)^m} + \sum_{\nu=1}^{\infty} B_{m\nu}(\zeta)\,(z-\zeta)^\nu$$

around the point $z = \zeta$, respectively. moreover we put $S_{m\nu}(0) = S_{m\nu}$ and $B_{m\nu}(0) = B_{m\nu}$.

Let $f(z)$ be a single-valued regular function in D excepting at $z=0$, where $f(z) - \frac{1}{z} = \sum_{\nu=0}^{\infty} a_\nu z^\nu$. Let $p_m(w)$ be a polynomial with degree m with respect to w or m-th Faber polynomial of $f(z)$ such that

$$p_m(f(z)) - \frac{1}{z^m}$$

is a regular function in D and is equal to zero at $z=0$. And let

$$p_m(f(z)) = \frac{1}{z^m} + \sum_{\nu=1}^{\infty} A_{m\nu} z^\nu.$$

Then we have the following theorem due to H. Grunsky [1].

A necessary and sufficient condition in order that $f(z)$ is univalent in D is that

$$\left|\sum_{\mu,\nu=1}^{N} \nu(A_{\mu\nu} - B_{\mu\nu}) x_\mu x_\nu\right| \le \sum_{\mu,\nu=1}^{N} \nu\, S_{\mu\nu} x_\mu \bar{x}_\nu$$

holds for all N and all complex numbers x_μ.

For brevity's sake we call this G-condition.

Let $L_s^2(D)$ be a class of single-valued regular functions $u(z)$ having the finite norm $\|u\| = \left(\iint |u(z)|^2 d\sigma_z\right)^{1/2}$ and the single-valued indefinite integral in D. Let $K_s(z, \bar{\zeta})$ and $L_s(z, \zeta)$ be the so-called reproducing Kernel function and the corresponding L-kernel of $L_s^2(D)$, that is, a meromorphic function in D having a pole $z = \zeta$, where $L_s(z, \zeta) - \frac{1}{\pi(z-\zeta)^2}$ is regular, and having a boundary relation

$$L_s(z,\zeta)\,dz = -\overline{K_s(z,\bar{\zeta})\,dz}$$

on Γ. Let $K_s(z,\bar{\zeta}) = \sum_{\mu,\nu=0}^{\infty} k_{\mu\nu} z^\mu \bar{\zeta}^\nu$,

$$L_s(z,\zeta) = \frac{1}{\pi(z-\zeta)^2} - \sum_{\mu,\nu=0}^{\infty} \ell_{\mu\nu} z^\mu \zeta^\nu$$

and

$$\frac{1}{\pi}\frac{\partial^2}{\partial z \partial \zeta} \log \frac{f(z)-f(\zeta)}{z-\zeta} = \sum_{\mu,\nu=0}^{\infty} c_{\mu\nu} z^\mu \zeta^\nu.$$

Then we have a theorem due to S.Bergman-M.Schiffer [1][1].

A necessary and sufficient condition in order that $f(z)$ is univerlent in D is that

$$\left|\sum_{\mu,\nu=0}^{N} (c_{\mu\nu} + \ell_{\mu\nu}) x_\mu x_\nu\right| \le \sum_{\mu,\nu=0}^{N} k_{\mu\nu} x_\mu \bar{x}_\nu$$

holds for all N and all complex numbers x_μ.

For brevity's sake we call this BS-conditions.

As a whole the equivalency between G-cond. and BS-cond. is evident. In the present note we give the simple intrinsic relations among them.

The desired results are now stated in the following manner:

(a) $k_{\mu\nu} = \frac{1}{\pi}(\mu+1)\; S_{\nu+1,\,\mu+1}$.

(b) $\ell_{\mu\nu} = \frac{1}{\pi}(\mu+1)\; B_{\nu+1,\,\mu+1}$.

(c) $c_{\mu\nu} = -\frac{1}{\pi}(\mu+1)\; A_{\nu+1,\,\mu+1}$.

(a) was already proved in the previous paper. See M. Ozawa [1].

Proof of (b). We can proceed in the purely formal viewpoints, since in the first place we may restrict our discussions in the small neighborhood of $z=0$, $\zeta=0$ and the validity domains of the convergence and next we can apply the continuability theorem due to Bergman-Schiffer [1], page 240.

On Γ, we have

$$
\begin{aligned}
L_s(z,\zeta)\,dz &= -\overline{K_s(z,\bar\zeta)\,dz} \\
&= -\sum_{\mu,\nu=1}^{\infty} \bar{b}_{\mu\nu}\,\overline{\varphi'_\mu(z)}\,\varphi'_\nu(\zeta)\,\overline{dz} \\
&= -\sum_{\mu,\nu=1}^{\infty} \bar{b}_{\mu\nu}\,\phi'_\mu(z)\,\varphi'_\nu(\zeta)\,dz,
\end{aligned}
$$

and hence on Γ

$$
L_s(z,\zeta) = -\sum_{\mu,\nu=1}^{\infty} b_{\nu\mu}\,\phi'_\mu(z)\,\varphi'_\nu(\zeta),
$$

where

$$
K_s(z,\bar\zeta) = \sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}\,\varphi'_\mu(z)\,\varphi'_\nu(\zeta),
$$

$$
b_{\mu\nu} = \bar{b}_{\nu\mu} \quad {}^{2)}
$$

and

$$
\varphi'_n(z) = \varphi'_n(z;0), \quad \phi'_n(z) = \phi'_n(z;0)
$$

and, on Γ,

$$
\overline{\varphi'_\mu(z)\,dz} = \phi'_\mu(z)\,dz.
$$

These were already stated in OZAWA [1].

Purely formal discussion leads us to the following relation:

$$
\begin{aligned}
&-\sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}\,\varphi'_\mu(z)\,\varphi'_\nu(\zeta) \\
&= -\sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}\Big(\frac{-\mu}{z^{\mu+1}} + \sum_{\sigma=1}^{\infty}\sigma B_{\mu\sigma} z^{\sigma-1}\Big)\Big(\sum_{\tau=1}^{\infty}\tau \bar{S}_{\nu\tau}\zeta^{\tau-1}\Big) \\
&= \sum_{\mu,\tau=1}^{\infty}\mu\sum_{\nu=1}^{\infty} b_{\mu\nu}\,\nu\,\bar{S}_{\tau\nu}\frac{\zeta^{\tau-1}}{z^{\mu+1}} - \sum_{\sigma,\tau,\mu=1}^{\infty}\sum_{\nu=1}^{\infty}\nu\, b_{\mu\nu}\,\bar{S}_{\tau\nu}\,\sigma B_{\mu\sigma} z^{\sigma-1}\zeta^{\tau-1} \\
&= \sum_{\mu,\tau=1}^{\infty}\mu\,\delta_{\mu\tau}\frac{\zeta^{\tau-1}}{z^{\mu+1}} - \sum_{\sigma,\tau,\mu=1}^{\infty}\delta_{\mu\tau}\,\sigma B_{\mu\sigma} z^{\sigma-1}\zeta^{\tau-1} \\
&= \sum_{\mu=1}^{\infty}\mu\frac{\zeta^{\mu-1}}{z^{\mu+1}} - \sum_{\mu,\sigma=1}^{\infty}\sigma B_{\mu\sigma} z^{\sigma-1}\zeta^{\mu-1} \\
&= \frac{1}{(z-\zeta)^2} - \sum_{\mu,\sigma=1}^{\infty}\sigma B_{\mu\sigma} z^{\sigma-1}\zeta^{\mu-1},
\end{aligned}
$$

here we should refer to the relation

$$
\sum_{\nu=1}^{\infty} b_{\mu\nu}\,\nu\,\bar{S}_{\tau\nu} = \delta_{\mu\tau}
$$

which was already proved in OZAWA [1] implicitly. Thus in a suitable validity domain of convergence $L_s(z,\zeta)$ and $-\sum_{\mu,\nu=1}^{\infty} b_{\nu\mu}\phi'_\mu(z)\varphi'_\nu(\zeta)$ have the same singularity $1/(z-\zeta)^2$, therefore we can infer that

$$
\ell_{\mu\nu} = \frac{1}{\pi}(\mu+1)\,B_{\nu+1,\mu+1}.
$$

But by virtue of a continuability theorem in Bergman-Schiffer [1] we can infer that (b) holds throughout D.

Proof of (c). M. Schiffer [1] introduced the generating function of Faber polynomials and proved that

$$
\log\frac{f(z)-t}{\frac{1}{z}} = -\sum_{m=1}^{\infty}\frac{1}{m}\,p_m(t)\,z^m.
$$

Therefore we have

$$
\begin{aligned}
&\sum_{\mu,\nu=0}^{\infty} c_{\mu\nu} z^\mu \zeta^\nu \\
&= \frac{1}{\pi}\frac{\partial^2}{\partial z\partial\zeta}\log\frac{f(z)-f(\zeta)}{z-\zeta} \\
&= \frac{1}{\pi}\frac{\partial^2}{\partial z\partial\zeta}\log\frac{f(z)-f(\zeta)}{\frac{1}{z}} - \frac{1}{\pi}\frac{\partial^2}{\partial z\partial\zeta}\log z(z-\zeta) \\
&= \frac{1}{\pi}\frac{\partial^2}{\partial z\partial\zeta}\Big[-\sum_{m=1}^{\infty}\frac{1}{m}\Big[\frac{1}{\zeta^m} + \sum_{n=1}^{\infty}A_{mn}\zeta^n\Big]z^m\Big] \\
&\qquad - \frac{1}{\pi}\frac{1}{(z-\zeta)^2} \\
&= \frac{1}{\pi}\sum_{m=1}^{\infty} m\frac{z^{m-1}}{\zeta^{m+1}} - \frac{1}{\pi}\sum_{m,n=1}^{\infty} n A_{mn} z^{m-1}\zeta^{n-1} \\
&\qquad - \frac{1}{\pi}\frac{1}{(z-\zeta)^2} \\
&= -\frac{1}{\pi}\sum_{\mu,\nu=0}^{\infty}(\nu+1)\,A_{\mu+1,\nu+1}\,z^\mu\zeta^\nu.
\end{aligned}
$$

And hence we can infer that (c) holds, here we should notice the similar fact as stated at the end of the proof of (b).

Bergman, S.-M. Schiffer [1]: Kernel functions and conformal mapping, Compositio Mathematica, vol.8(1951), pp. 205-249.

Grunsky, H. [1]: Koefficientenbedingungen für schlicht abbildende meromorphe Funktionen, Math. Zeitschrift, vol.45(1939), pp. 29-61.

Ozawa, M. [1] : On functions of bounded Dirichlet integral, Kôdai Math. Sem. Rep., 1952 No. 4. pp. 95-98.

Schiffer, M. [1] : Faber polynomials in the theory of univalent functions, Bull. Amer. Math. Soc., vol. 54(1948), pp. 503-517

NOTES

1) Bergman-Schiffer gave a more wider result than we stated above, that is, a corresponding result for Λ -space in their terminology. But as they stated all the situations are similar for Λ_s -space in their terminology which corresponds to our statement.

2) Let Δ_n and D_{nj} be the following two determinants

$$\Delta_n = \left| \nu S_{\mu\nu} \right|_{\mu,\nu = 1,\cdots,n}$$

and

$$D_{nj} = \begin{vmatrix} S_{11} & \cdots & S_{j-1,1} & 0 & S_{j+1,1} & \cdots & S_{n1} \\ 2S_{12} & \cdots & 2S_{j-1,2} & 0 & 2S_{j+1,2} & \cdots & 2S_{n2} \\ & \cdot & & \cdot & \cdot & & \\ (n-1)S_{1,n-1} & \cdots & (n-1)S_{j-1,n-1} & 0 & (n-1)S_{j+1,n-1} & \cdots & (n-1)S_{n,n-1} \\ nS_{1,n} & \cdots & nS_{j-1,n} & \frac{1}{\pi} & nS_{j+1,n} & \cdots & nS_{n,n} \end{vmatrix},$$

then we have

$$\Delta_n \gtrless 0 \quad \text{and} \quad q_{\mu\nu} = \pi \sum_{n=1}^{\infty} \frac{D_{n\mu} \overline{D}_{n\nu}}{\Delta_{n-1} \Delta_n}.$$

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