ON THE ESTIMATION OF THE COEFFICIENT OF VARIATION BY THE RATIO OF TWO QUANTITLES IN LARGE SAMPLES

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1. INTRODUCTION. It is well known that the coefficient of variation of a distribution, defined as its standard deviation σ divided by the mean m , is one of the most useful statistical measure --- especially in situation where the distribution is normal. When the population distribution is such that the variable x takes only positive values and has at least the fourth moment, we can show that the sample coefficient of variation, defined usually as sample standard deviation s divided by the sample mean $\bar{x}$, is a consistent estimate of the population coefficient of variation V , and its mean and variance are respectively as follows[1] :

$$
(1)\qquad
\begin{aligned}
E\left(\frac{s}{\bar{x}}\right) &= \frac{\sigma}{m} + O\left(\frac{1}{n}\right) \\
D^2\left(\frac{s}{\bar{x}}\right) &= \frac{m^2(\mu_4-\mu_2^2) - 4m\mu_2\mu_3 + 4\mu_2^3}{4m^4\mu_2 n} + O\left(\frac{1}{n^{3/2}}\right)
\end{aligned}
$$

where μ_i denotes the i-th central moment of the distribution and n denotes the sample size. A normal distribution does not satisfy the condition that the variable takes only positive values, therefore we cannot admit these arguments in this case. But, practically, we may consider a normal distribution with positive mean truncated at $x=0$ and when $V\,(=\sigma/m)$ is fairly small, the central moments of such a distribution will be approximately equal to the corresponding moments of a complete normal distribution. In this case the approximate expressions for the mean and the variance of the sample coefficient of variation $s/\bar{x}$ are respectively as follows:

$$
(2)\qquad
\begin{aligned}
E\left(\frac{s}{\bar{x}}\right) &= V \\
D^2(s/\bar{x}) &= V^2(1+2V^2)/2n
\end{aligned}
$$

In this paper we shall propose another new method of estimating the coefficient of variation of a normal distribution in large samples which is constructed by the ratio of two appropriately chosen quantiles and set up the confidence interval corresponding to a given confidence coefficient. Optimum spacing of the quantiles and its efficiency are also discussed.

Although it is not efficient, this method promises to furnish a simple and effective method for estimating the coefficient of variation of a normal distribution --- especially in situation where large samples are easily available.

2. JOINT DISTRIBUTION OF TWO QUANTILES. Consider a random sample of size n from a one-dimensional distribution of the continuous type, with the distribution function $F(x)$ and the probability density function $f(x) = F'(x)$. Let ζ_1 and ζ_2 are the quantiles of order p_1 and p_2 of the distribution respectively (we assume as $0 < p_1 < p_2 < 1$), i.e. the roots (assumed to be unique respectively) of the equations:

$$
(3)\qquad F(\zeta_i) = p_i \,, \quad (i=1,2)
$$

We shall suppose that $f(\zeta_i) \neq 0 \quad (i=1,2)$ and that in the neighbourhood of $x=\zeta_i\,(i=1,2)$, $f(x)$ is continuous and has a continuous derivative $f'(x)$. We denote by $z_i\,(i=1,2)$ the corresponding quantiles of the sample, that is, if we arrange the sample values in ascending order of magnitude:

$$
(4)\qquad x(1) < x(2) < \cdots < x(n)
$$

(we have assumed no ties, which is a consequence, with probability one, of the continuous distribution of x), we define

$$
(5)\qquad z_1 \equiv x([np_1]+1)\,,\quad z_2 \equiv x([np_2]+1)
$$

where $[np]$ denotes the greatest integer not exceeding np . Now we quote the following theorem[2] .

The joint distribution of two quantiles z_1 and z_2 is asymptotically normal. The means of the limiting distribution are the corresponding quantiles ζ_1 , and ζ_2 of the population, while the asymptotic expressions of the second order moments μ_{20} , μ_{11} , μ_{02} are respectively

$$
(6)\qquad \frac{p_1 q_1}{n f^2(\zeta_1)}\,,\quad \frac{p_1 q_2}{n f(\zeta_1) f(\zeta_2)}\,,\quad \frac{p_2 q_2}{n f^2(\zeta_2)}
$$

where $q_i = 1 - p_i \quad (i=1,2)$

We omit the proof here.

For the case of a normal population, with mean m and standard deviation σ, if we denote

$$(7) \quad \xi_i \equiv m + \sigma u_i, \quad (i=1,2)$$

and

$$(8) \quad g_i \equiv \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u_i^2} \quad (i=1,2)$$

we have

$$(9) \quad p_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u_i} e^{-\frac{t^2}{2}}\,dt$$

$$f(\xi_i) = \frac{1}{\sigma} g_i, \quad (i=1,2)$$

Hence the joint distribution of two sample quantiles z_1 and z_2 is asymptotically normal and its probability density function is

$$(10) \quad \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{z_1-\xi_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{z_1-\xi_1}{\sigma_1}\right)\left(\frac{z_2-\xi_2}{\sigma_2}\right)+\left(\frac{z_2-\xi_2}{\sigma_2}\right)^2\right\}\right]$$

where

$$(11) \quad \sigma_1^2 = \frac{p_1 q_1}{n g_1^2}\sigma^2, \quad \rho\sigma_1\sigma_2 = \frac{p_1 q_2}{n g_1 g_2}\sigma^2, \quad \sigma_2^2 = \frac{p_2 q_2}{n g_2^2}\sigma^2$$

3. DISTRIBUTION OF THE RATIO AND ITS APPROXIMATION. When the joint probability density function of variables z_1 and z_2 is (10), the distribution of the ratio ζ of two joint normally distributed variables z_1 and z_2, namely

$$(12) \quad \zeta = z_2/z_1$$

is well known as the distribution of the "Index", to which several contributions have been made [3]. The author also obtained a new formula of its distribution function as a mixture of distribution [4] and made some contributions concerning it [5]. We shall quote them here briefly and details will be omitted.

The distribution function of the variable ζ is in the form of mixture of distribution:

$$(13) \quad F(\zeta) = \sum_{\nu=0}^{\infty} e^{-\frac{k^2}{2}} \frac{\left(\frac{k^2}{2}\right)^{\nu}}{\nu!} F_\nu(\zeta)$$

where

$$k^2 = \frac{1}{1-\rho^2}\left\{\left(\frac{\xi_1}{\sigma_1}\right)^2 - 2\rho\frac{\xi_1\xi_2}{\sigma_1\sigma_2} + \left(\frac{\xi_2}{\sigma_2}\right)^2\right\}$$

and $F_\nu(\zeta)$ is also a distribution function of the form:

$$F_\nu(\zeta) = \frac{1}{B\left(\frac{1}{2}, \nu+\frac{1}{2}\right)}\int_{-\pi/2}^{\Theta} \sin^{2\nu}(\theta+\alpha)\,d\theta$$

and

$$\Theta = \tan^{-1}\frac{\sigma_1\zeta - \rho\sigma_2}{\sigma_2\sqrt{1-\rho^2}}, \quad \sin\alpha = \frac{\xi_1}{k\sigma_1}$$

$$\cos\alpha = \frac{1}{k\sqrt{1-\rho^2}}\left(\frac{\xi_2}{\sigma_2} - \rho\frac{\xi_1}{\sigma_1}\right)$$

The probability density function of ζ is obtained by differentiation, namely,

$$(14) \quad \frac{1}{\pi}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\xi_1}{\sigma_1}\right)^2 - 2\rho\frac{\xi_1\xi_2}{\sigma_1\sigma_2} + \left(\frac{\xi_2}{\sigma_2}\right)^2\right\}\right] \cdot \frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2} \cdot \sum_{\nu=0}^{\infty}\frac{2^\nu \nu!}{(2\nu)!}\left\{\frac{\sigma_1(\xi_2\sigma_1 - \xi_1\rho\sigma_2)\zeta + \sigma_2(\xi_1\sigma_2 - \xi_2\rho\sigma_1)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2}}\right\}^{2\nu}$$

or in the form due to Fieller

$$(15) \quad \frac{1}{\pi}\frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{\xi_1}{\sigma_1}\right)^2 - 2\rho\frac{\xi_1\xi_2}{\sigma_1\sigma_2} + \left(\frac{\xi_2}{\sigma_2}\right)^2\right\}\right] + \frac{\sigma_1(\xi_2\sigma_1 - \xi_1\rho\sigma_2)\zeta + \sigma_2(\xi_1\sigma_2 - \xi_2\rho\sigma_1)}{\pi(\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2)^{3/2}} \cdot \exp\left[-\frac{1}{2}\frac{\xi_1\zeta - \xi_2}{\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2}\right] \cdot \int_0^{\frac{\sigma_1(\xi_2\sigma_1 - \xi_1\rho\sigma_2)\zeta + \sigma_2(\xi_1\sigma_2 - \xi_2\rho\sigma_1)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2}}} e^{-\frac{t^2}{2}}\,dt$$

The exact distribution of the ratio ζ cited above is very complicated and momentless, so we cannot treat it well. But when k^2 is large, the distribution of the variable

$$(16) \quad \eta = \frac{\xi_1\zeta - \xi_2}{\sqrt{\sigma_1^2\zeta^2 - 2\rho\sigma_1\sigma_2\zeta + \sigma_2^2}}$$

is approximately normal with zero mean and unit variance. This will be shown as follows. Let

$$(17)\qquad k=\frac{|\sigma_1(\zeta_2\sigma_1-\zeta_1\rho\sigma_2)\xi+\sigma_2(\zeta_1\sigma_2-\zeta_2\rho\sigma_1)|}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2\xi^2-2\rho\sigma_1\sigma_2\xi+\sigma_2^2}},$$

then

$$(18)\qquad \eta^2=h^2-k^2,$$

and

$$(19)\qquad \frac{d\eta}{d\xi}=\frac{\sigma_1(\zeta_2\sigma_1-\zeta_1\rho\sigma_2)\xi+\sigma_2(\zeta_1\sigma_2-\zeta_2\rho\sigma_1)}{\sigma_1\sigma_2\sqrt{1-\rho^2}\sqrt{\sigma_1^2\xi^2-2\rho\sigma_1\sigma_2\xi+\sigma_2^2}}$$

so we can reduce the probability density function of ξ in the form

$$(20)\qquad \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\eta^2}\left|\frac{d\eta}{d\xi}\right|\cdot\sqrt{\frac{2}{\pi}}\left(\frac{1}{k}e^{-\frac{1}{2}k^2}+\int_0^k e^{-\frac{t^2}{2}}dt\right)$$

For any fixed η, $k\to\infty$ as $h^2\to\infty$, therefore

$$(21)\qquad \frac{1}{k}e^{-\frac{1}{2}k^2}\to 0,\quad \int_0^k e^{-\frac{t^2}{2}}dt\to\sqrt{\frac{\pi}{2}}$$

Hence we have the approximate formula for the probability density function of ξ,

$$(22)\qquad \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\eta^2}\left|\frac{d\eta}{d\xi}\right|$$

4. ESTIMATION OF THE COEFFICIENT OF VARIATION. In 2 we have seen that, when the population distribution is normal with mean m and standard deviation σ, and if we denote two sample quantiles of orders p_1 and p_2 of ordered sample (4) by (5) and the corresponding population quantiles by (7), the asymptotic expression of the probability density function of the joint distribution of variables z_1 and z_2 is normal (10).

As far as the coefficient of variation $V\ (=\sigma/m)$ is concerned, it frequently occurs that the mean m is positive and the coefficient of variation V is at most about 30%, so we can suppose $\zeta_1>0$ and therefore $\zeta_2>0$. While σ_1^2 and σ_2^2 are of order n^{-1} and ρ is positive and less than one. Hence h^2 tends to infinity as n tends to infinity and we can use the approximation for the distribution of $\xi\ (=z_2/z_1)$ —— the criterion shown in § 3 —— in large samples. Under these circumstances

$$\eta=\frac{\zeta_1\xi-\zeta_2}{\sqrt{\frac{\sigma^2}{n}\left(\frac{p_1q_1}{g_1^2}\xi^2-2\frac{p_1q_2}{g_1g_2}+\frac{p_2q_2}{g_2^2}\right)}}$$

or

$$(23)\qquad \eta=\frac{\sqrt{n}\left\{(\xi-1)/V-(u_2-u_1\xi)\right\}}{\sqrt{\frac{p_1q_1}{g_1^2}\xi^2-2\frac{p_1q_2}{g_1g_2}\xi+\frac{p_2q_2}{g_2^2}}}$$

is approximately normal with zero mean and unit variance. For given spacing --- spacing means the choice of the orders p_1 and p_2 of two quantiles --- g_1, g_2, u_1 and u_2 are all known constants. Hence it is a remarkable fact that η involves in its expression only one unknown parameter V. Accordingly we can test the statistical hypothesis or estimate V by using η as follows.

As a point estimate of V we may take

$$(24)\qquad \hat{V}=\frac{\xi-1}{u_2-u_1\xi}\quad\text{or}\quad=\frac{z_2-z_1}{u_2z_1-u_1z_2}$$

for which η vanishes. For testing statistical hypothesis $V=V_0$ we propose as the critical region of size α:

$$(25)\qquad |\eta|\geq t_\alpha$$

where t_α is the $100\alpha\%$ point of the standard normal distribution. For setting up the confidence interval for V, solving the inequality

$$(26)\qquad |\eta|\leq t_\alpha$$

we get after some easy calculations the required confidence intervals with confidence coefficient $1-\alpha$:

$$(27)\qquad \underline{V}\leq V\leq\overline{V}$$

where

$$(28)\qquad \underline{V}=\frac{\xi-1}{u_2-u_1\xi+\frac{t_\alpha}{\sqrt{n}}\Delta},\quad \overline{V}=\frac{\xi-1}{u_2-u_1\xi-\frac{t_\alpha}{\sqrt{n}}\Delta}$$

and

$$(29)\qquad \Delta=\sqrt{\frac{p_1q_1}{g_1^2}\xi^2-2\frac{p_1q_2}{g_1g_2}\xi+\frac{p_1q_2}{g^2}}$$

5. OPTIMUM SPACING AND ITS EFFICIENCY. Now we proceed to determine the optimum spacing of the quantiles and evaluate its efficiency in a certain sense considered below. According to (28), as the length of random interval (27) is

$$(30)\qquad \overline{V}-\underline{V}=\frac{2\frac{t_\alpha}{\sqrt{n}}\frac{\Delta}{\xi-1}}{\left(\frac{u_2-u_1\xi}{\xi-1}\right)^2-\frac{t_\alpha^2}{n}\left(\frac{\Delta}{\xi-1}\right)^2}$$

it is sufficient to determine the spacing of two quantiles to minimize the length of interval in the average under the following sense. That is, under the hypothesis $V = V_0$, when the variable ξ takes its median ξ_0, which render the value $\eta = 0$, namely

$$(31) \qquad \xi_0 = \frac{1+u_2 V_0}{1+u_1 V_0} \quad \text{or} \quad V_0 = \frac{\xi_0 - 1}{u_2 - u_1 \xi_0}$$

it may be adequate to determine the orders p_1 and p_2 of quantiles to minimize the length of interval (30) as the optimum spacing. For this purpose it is sufficient to find the values p_1 and p_2 which minimize the function:

$$(32) \qquad \Psi(p_1, p_2, V_0) = \left(\frac{\Delta_0}{\xi_0 - 1}\right)^2$$

$$= \frac{\frac{p_1 q_1}{g_1^2}(1+u_2 V_0)^2 - \frac{2 p_1 q_2}{g_1 g_2}(1+u_2 V_0)(1+u_1 V_0) + \frac{p_2 q_2}{g_2^2}(1+u_1 V_0)^2}{(u_2 - u_1)^2 V_0^2}$$

The values of p_1 and p_2 which minimize (32) essentially depend upon the value V_0. Unfortunately the writer cannot obtain the values in general. Owing to the symmetric property of the functional form of (32), it may be adequate to assume the symmetricity of spacing, that is,

$$(33) \qquad p_1 + p_2 = 1 \quad \text{or} \quad u_1 + u_2 = 0$$

Under such assumption we can solve the problem numerically and determine the spacing as follows.

Let

$$(34) \qquad u \equiv u_2 = -u_1, \quad p \equiv p_2 = 1 - p_1, \quad g \equiv g_1 = g_2,$$

then (32) reduced to

$$(35) \qquad \Psi(p, V_0) = \frac{1}{2V_0^2}\Psi_1(p) + \frac{1}{2}\Psi_2(p)$$

where

$$(36) \qquad \Psi_1(p) = \frac{1}{u^2}\cdot\frac{1-p}{g}\cdot\frac{2p-1}{g}, \quad \Psi_2(p) = \frac{1}{g}\cdot\frac{1-p}{g}$$

The curves of $\Psi_1(p)$ and $\Psi_2(p)$ for $\frac{1}{2} < p < 1$ are illustrated in Fig.1. By numerical computation we find that the values of $\Psi_1(p)$ become minimum when

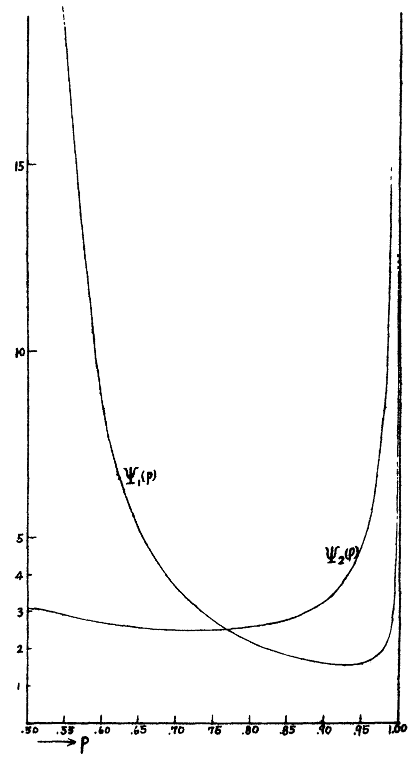

Fig. 1

$$(37) \qquad p = 0.930, \quad u = 1.4758$$

and fortunately for all $V_0 \leq 0.1$ the effect of $\Psi_2(p)$ in $\Psi(p, V_0)$ is negligibly small and the value of p which minimizes the value $\Psi(p, V_0)$ is included in the interval (0.9295, 0.9305), although it depends on the value V_0. Hence we may adopt (37) as optimum spacing which minimizes the value $\Psi(p, V_0)$ for all V_0 not exceeding 0.1. When V_0 moves up to 0.2 the spacing which minimizes the value $\Psi(p, V_0)$ moves up to

$$(38) \qquad p = 0.922, \quad u = 1.419,$$

owing to the effect of $\Psi_2(p)$. However the effect of increase in $\Psi(p, V_0)$ is rather small when we adopt the spacing (37) as optimum symmetric spacing.

In usual practical cases, as far as I know, the coefficient of variation of the population is less than 0.1. Hence it is reasonable to adopt as optimum symmetric spacing the following,

$$(39)\quad \begin{array}{ll} p_1 = 0.070, & u_1 = -1.4758 \\ p_2 = 0.930, & u_2 = 1.4758 \end{array}$$

and in this case the confidence limits (28) reduced to

$$(40)\quad \underline{V} = \frac{\xi - 1}{1.4758(\xi+1) + \frac{t_\alpha}{\sqrt{n}}\Delta}, \quad \overline{V} = \frac{\xi - 1}{1.4758(\xi+1) - \frac{t_\alpha}{\sqrt{n}}\Delta},$$

where

$$(41)\quad \Delta = \sqrt{3.6111(\xi^2+1) - 0.54363\xi}$$

It may be convenient to provide the table of Δ for various values of ξ or the chart of confidence belt for various sample size and for some given confidence coefficient as illustrated in Table 1 and in Fig. 2 under the spacing (39).

Finally we want to compare the efficiency of our method with ordinary method cited in § 1 in the following manner. If we construct as usual the confidence interval in large samples by using standard error (2) of $s/\bar{x}$ and $100\alpha\%$ point t_α of standard normal distribution, we obtain the following as average length of confidence intervals:

$$(42)\quad 2t_\alpha D\left(\frac{s}{\bar{x}}\right) = 2t_\alpha \frac{V}{\sqrt{2n}}\sqrt{1+2V^2}$$

Hence we define the efficiency of our method compared with ordinary method by the ratio of the reciprocals of average lengths of intervals as follows:

$$(43)\quad e = \lim_{n\to\infty} 2t_\alpha \frac{V_0}{\sqrt{2n}}\sqrt{1+2V_0^2} \Bigg/ \frac{2\frac{t_\alpha}{\sqrt{n}}\sqrt{\Psi(p,V_0)}}{\frac{1}{V_0^2} - \frac{t_\alpha}{\sqrt{n}}\Psi(p,V_0)},$$

For $V \leq 0.1$, neglecting $\Psi_2(p)$, we obtain

$$(44)\quad e = \frac{1}{\sqrt{\Psi_1(p)}}\sqrt{1+2V_0^2}$$

Although this efficiency depends on V_0, it is almost equal to 0.80.

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TABLE I

ξ	Δ	ξ	Δ	ξ	Δ	ξ	Δ
1.01	2.5972	1.21	2.8706	1.41	3.1660	1.61	3.4780
1.02	2.6103	1.22	2.8849	1.42	3.1813	1.62	3.4939
1.03	2.6234	1.23	2.8992	1.43	3.1966	1.63	3.5099
1.04	2.6366	1.24	2.9137	1.44	3.2119	1.64	3.5259
1.05	2.6498	1.25	2.9281	1.45	3.2272	1.65	3.5419
1.06	2.6631	1.26	2.9426	1.46	3.2427	1.66	3.5580
1.07	2.6765	1.27	2.9572	1.47	3.2581	1.67	3.5741
1.08	2.6900	1.28	2.9718	1.48	3.2736	1.68	3.5902
1.09	2.7035	1.29	2.9865	1.49	3.2891	1.69	3.6064
1.10	2.7171	1.30	3.0012	1.50	3.3046	1.70	3.6226
1.11	2.7307	1.31	3.0159	1.51	3.3202	1.71	3.6388
1.12	2.7444	1.32	3.0308	1.52	3.3358	1.72	3.6550
1.13	2.7582	1.33	3.0456	1.53	3.3515	1.73	3.6713
1.14	2.7721	1.34	3.0605	1.54	3.3672	1.74	3.6876
1.15	2.7860	1.35	3.0755	1.55	3.3829	1.75	3.7039
1.16	2.7999	1.36	3.0904	1.56	3.3987	1.76	3.7202
1.17	2.8139	1.37	3.1055	1.57	3.4145	1.77	3.7366
1.18	2.8280	1.38	3.1206	1.58	3.4303	1.78	3.7530
1.19	2.8422	1.39	3.1357	1.59	3.4461	1.79	3.7694
1.20	2.8563	1.40	3.1508	1.60	3.4620	1.80	3.7858

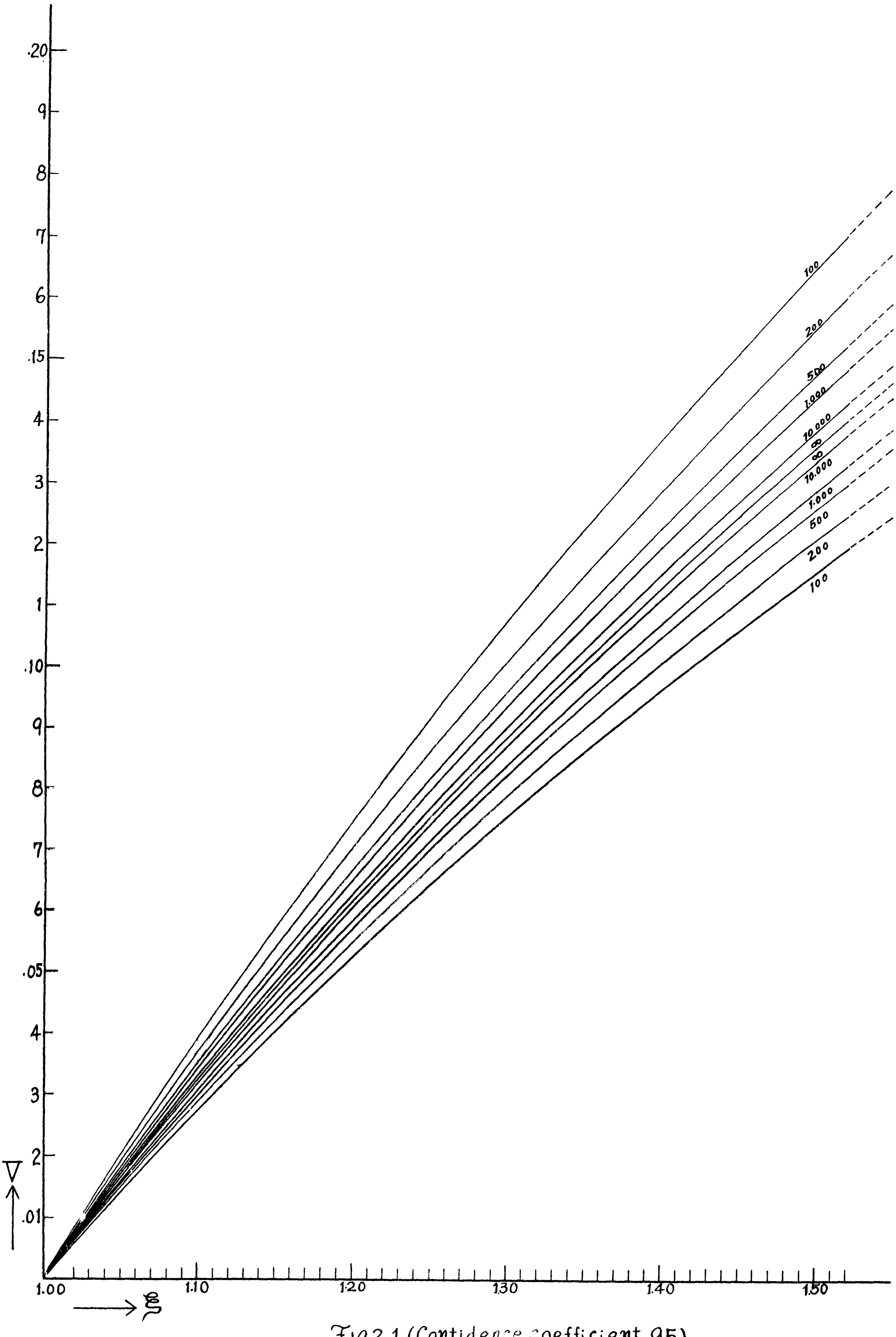

Fig.2.1 (Confidence coefficient .95)

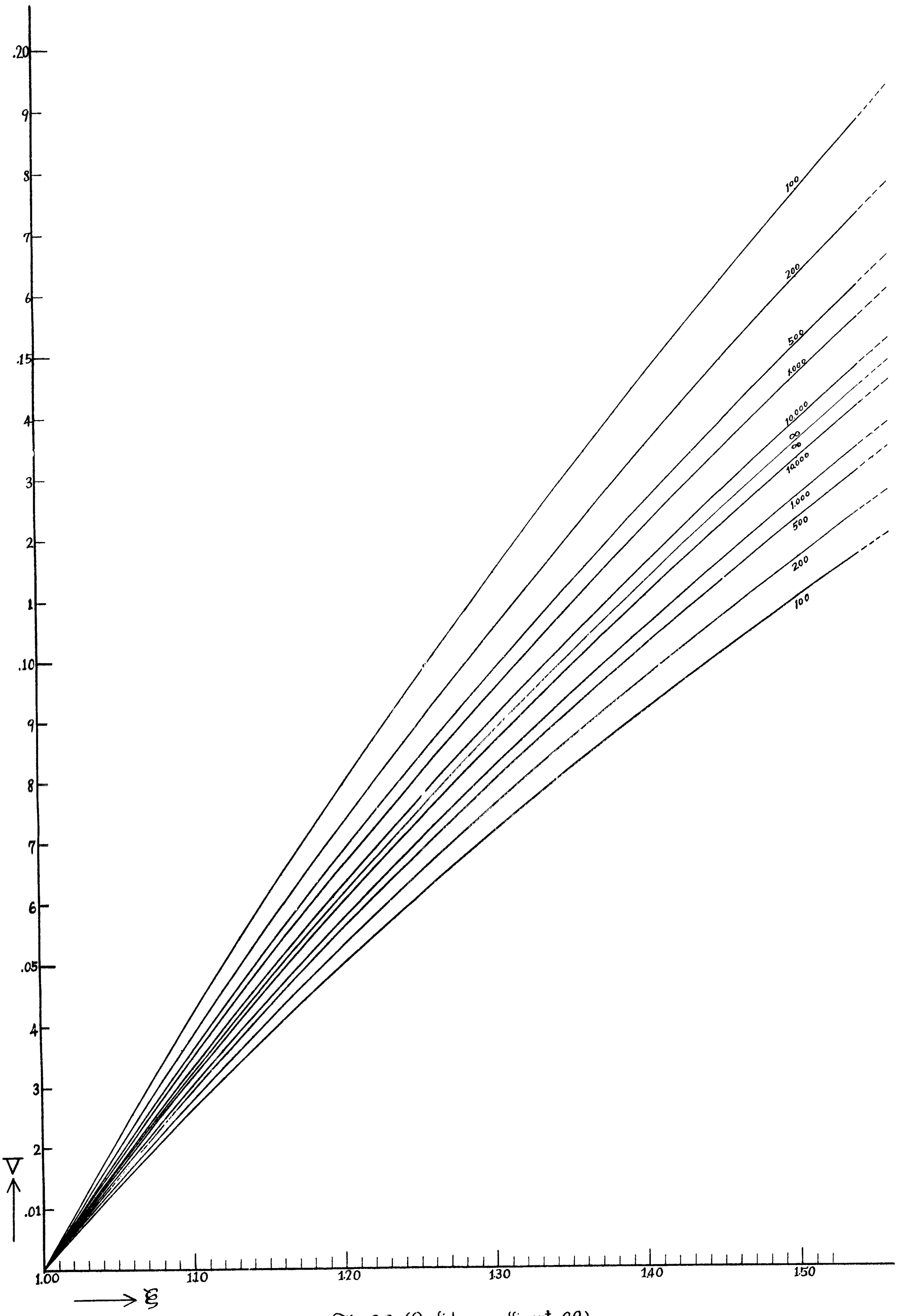

Fig. 2.2. (Confidence coefficient .99)

(*) Received May 30, 1952.

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