ON SOME DETERMINANT EQUATION

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In his theory of communication, C.E.Shannon[1] determined the channel capacity of a disrete noiseless system by means of a determinant equation of the following type:

$$|E - A(z)| = 0,$$

where $A(Z)$ is a square matrix dependent on a complex variable z.

In this note I will prove the existence of a real positive root of the smallest absolute value, which is assumed in Shannon's theory.

Theorem.

Let $A(z)$ be a matrix subject to the following conditions:

(1) $A(z)$ is a square matrix of order n.

(2) Every matrix element $A_{ik}(z)$ of $A(z)$ is an entire function of z

$$A_{ik}(z) = \sum_{m=0}^{\infty} A_{ikm} z^m$$

and

$$A_{ik}(0) = 0$$

(3) Every coefficient $A_{ik,m}$ of $A_{ik}(z)$ is non-negative

$$A_{ik,m} \geqq 0.$$

(4) At least one coefficient of the characteristic polynomial $|\lambda E - A(z)|$ is not constant.

Then the determinant equation

$$|E - A(z)| = 0$$

has a real positive root of the smallest[2] absolute value.

Lemma 1. If all of the traces $Tr(A(z)), Tr(A^2(z)) \cdots Tr(A^n(z))$ of $A^k(z)$ are constant, then the characteristic polynomial has constant coefficients.

Proof. Let $\lambda_1, \lambda_2 \cdots\cdots \lambda_n$ be eigenvalues of $A(z)$, then

$$Tr(A(z)) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = const$$

$$Tr(A^2(z)) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = const$$

$$\cdots\cdots\cdots\cdots$$

$$Tr(A^n(z)) = \lambda_1^n + \lambda_2^n + \cdots + \lambda_n^n = const$$

From these equations follows

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = const$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n = const.$$

$$\cdots\cdots\cdots\cdots$$

$$\lambda_1\lambda_2\lambda_3 \cdots \lambda_n = const$$

by the theorem of symmetric functions.

Lemma 2. The matrix $(E - A(z))^{-1}$ is not an entire function.

Proof. By Lemma 1, at least one $Tr(A^k(z))$ is not constant for $1 \leqq k \leqq n$.

Hence at least one diagonal element of $A^k(z)$ is not all constant.

$$A_{jj}^{k}(z) = \sum_{m=0}^{\infty} A_{jj\,m}^{k} z^m$$

Let $A_{jj,p}^{k}$ be a non-zero coefficient of the smallest order in the above expansion. Because all the coefficients is non-negative, the following inequality holds:

$$A_{jj,sp}^{ks} \geqq (A_{jj,p}^{k})^s$$

In the expansion

$$\sum_{\ell=0}^{\infty} A(z)^{\ell}$$

j-th diagonal element has a subsequence

$$\sum_{s=0}^{\infty} A_{jj}^{ks}(z),$$

where

$$\sum_{s=0}^{\infty} A_{jj;sp}^{ks} z^{ks} \geqq \sum_{s=0}^{\infty} (A_{jj;p}^{k})^{s} z^{ks}$$

The last series has a convergence radius $(A_{jj;p}^{k})^{\frac{1}{p}}$.

Hence $\sum_{\ell=0}^{\infty} A(z)^{\ell}$ is not an entire function, q.e.d.

Because the matrix $(E - A(z))^{-1}$ is singular, when and only when $|E - A(z)|$ is zero, it suffices to examine the singularity of $(E - A(z))^{-1}$.

Instead of $(E - A(z))^{-1}$ we consider the infinite series:

$$\sum_{\ell=0}^{\infty} A(z)^{\ell} = B(z).$$

Every element $B_{ik}(z)$ of B(z) has non-negative coefficients. By the well-known theorem of Pringsheim, such a function has a singularity r_{ik} if r_{ik} is the convergence radius of $B_{ik}(z)$.

Hence

$$r = Min(r_{11}, r_{12} \cdots r_{nn})$$

is the real positive root of the smallest absolute value, q.e.d.

<u>Remark.</u> Especially when $A_{ik,m} = 0 \; (m \neq 1)$, our theorem reduces to the theorem of Frobenius.[3]

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(1) C.E.Shannon and W.Weaver, The mathematical theory of communication, 1949, The University of Illinois Press, Urbana, p.9.

(2) Shannon used $1/z$ instead of z.

(3) Frobenius, Berliner Sitzungsberichte, 1908.

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