ON NUMBERS OF POSITIVE SUMS OF INDEPENDENT RANDOM VARIABLES

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The following theorem on numbers of positive sums of independent random variables has been proved by P. Erdos and M. Kac.

Theorem A. Let $X_1, X_2, \dots$ be independent random variables each having mean 0 and variance 1 and such that the central limit theorem is applicable. Put $S_n = X_1 + X_2 + \dots + X_n$ and let N_n denote the numbers of S_k 's, $1 \leq k \leq n$, which are positive. Then

$$\lim_{n \rightarrow \infty} \text{Prob} \left\{ \frac{N_n}{n} < \alpha \right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}},$$

$$0 \leq \alpha \leq 1$$

In the proof of P. Erdos and M. Kac, the existence of the mean $E[X_n]$ and variance $E[X_n^2]$, $n=1, 2, \dots$, are presumed. We shall extend this result when the mean and variance does not necessarily exist. The result is following.

Theorem 1. Let $X_1, X_2, \dots$ be independent identically distributed random variables and suppose that there exists positive sequence $\{A_n\}$, increasing to infinity such that the distribution of S_n/A_n tends to normal distribution $\tilde{F}(x)$. Then

$$\lim_{n \rightarrow \infty} \text{Prob} \left\{ \frac{N_n}{n} < \alpha \right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}}$$

$$0 \leq \alpha \leq 1$$

The key points of the proof are as the same as in the P. Erdos and M. Kac. That is, we shall prove that we can take a particular sequence of independent random variables $G_1, G_2, \dots$, each having normal distribution $\tilde{F}(x)$, instead of $X_1, X_2, \dots$. To prove this, we use some known theorems.

Let $-X'$ be independent random variable and has same distribution function $F(x)$ with X , then the distribution function $\tilde{F}(x)$ of $\tilde{X} = X + X'$ is $(1 - F(-x)) * F(x)$, $\tilde{F}(x)$ and $\tilde{X}$ are said, respectively, the symmetrized distribution of $F(x)$ and the symmetrized random variable X .

$$\text{Let } \Phi_F(h) = \int_{-\infty}^{\infty} \frac{h^2}{x^2 + h^2} dF(x)$$

$$\Psi_{\tilde{F}}(h) = \int_{-\infty}^{\infty} \frac{h^2}{x^2 + h^2} d\tilde{F}(x)$$

which are introduced by K. Kunisawa, and he called typical function and mean concentration function of $F(x)$. In our proof of the theorem, the following Kunisawa's fundamental inequality²⁾ are used.

Lemma 1. For any $h > 0$

$$(1) \quad 1 - \Psi_{F_1 * \dots * F_n}(h) \leq \sum_{k=1}^n (1 - \Psi_{F_k}(h)),$$

where $F_1, F_2, \dots, F_n$ are any distribution functions.

Lemma 2. Let α be a positive number ($0 < \alpha < \frac{1}{2}$) and let $F(x)$ be a distribution function satisfying

$$(2) \quad F(+0) \geq \lambda > 0, \quad F(-0) \leq 1 - \lambda,$$

$$\text{and} \quad (0 < \lambda < 1)$$

and

$$(3) \quad 1 - F(h) + F(-h) < \alpha,$$

then we have

$$(4) \quad 1 - \Psi_{\tilde{F}}(h) \geq K(\alpha, \lambda) (1 - \Psi_F(h)),$$

where $K(\alpha, \lambda)$ is a positive constant depending on α and λ .

Lemma 3. Under the same assumptions of Theorem 1,

$$\lim_{n \rightarrow \infty} n \int \alpha \tilde{F}(x) = 0$$

$$|x| > \varepsilon A_n$$

$$\lim_{n \rightarrow \infty} \frac{n}{2A_n} \int x^2 \alpha \tilde{F}(x) = 1$$

$$|x| < \varepsilon A_n$$

for every $\varepsilon > 0$, where $\tilde{F}(x)$ is the symmetrized distribution of $F(x)$ which is distribution of X'_k 's.

Proof. This follows immediately from Feller's central limit theorem³⁾, for the distribution of $\tilde{S}_n/\sqrt{n} A_n$ tends to the normal distribution $\Phi(x)$, having mean 0 and variance 1, where $\tilde{S}_n = \tilde{X}_1 + \tilde{X}_2 + \dots + \tilde{X}_n$.

Proof of Theorem 1. Let $\psi(s) = 0$ for $s \leq 0$, $= 1$ for $s > 0$, then

$$N_n = \sum_{r=1}^n \psi(s_r)$$

For fixed k , let

$$n_j = [j \frac{n}{k}] \quad j = 0, 1, \dots, k$$

and

$$D_n = \frac{1}{n} \sum_{j=1}^k \sum_{r=n_{j-1}+1}^{n_j} [\psi(s_{n_j}) - \psi(s_r)]$$

then

$$(5) \quad E[|D_n|] \leq \frac{1}{n} \sum_{j=1}^k \sum_{r=n_{j-1}+1}^{n_j} E[|\psi(s_{n_j}) - \psi(s_r)|]$$

$$(6) \quad E[|\psi(s_{n_j}) - \psi(s_r)|] = \text{Prob.} \{s_{n_j} > 0, s_r \leq 0\} + \text{Prob.} \{s_{n_j} \leq 0, s_r > 0\}$$

For any $\varepsilon > 0$

$$(7) \quad \begin{aligned} & \text{Prob.} \{s_{n_j} > 0, s_r \leq 0\} \\ &= \text{Prob.} \{s_{n_j} \geq \varepsilon A_{n_j}, s_r \leq 0\} \\ &+ \text{Prob.} \{0 < s_{n_j} < \varepsilon A_{n_j}, s_r \leq 0\} \\ &\leq \text{Prob.} \{s_{n_j} - s_r \geq \varepsilon A_{n_j}\} + \text{Prob.} \{0 < s_{n_j} < \varepsilon A_{n_j}\} \end{aligned}$$

Similarly

$$(8) \quad \begin{aligned} & \text{Prob.} \{s_{n_j} \leq 0, s_r > 0\} \leq \text{Prob.} \{s_{n_j} - s_r \leq -\varepsilon A_{n_j}\} \\ &+ \text{Prob.} \{-\varepsilon A_{n_j} < s_{n_j} \leq 0\} \end{aligned}$$

Thus from (6), (7), (8), we have

$$(9) \quad E[|\psi(s_{n_j}) - \psi(s_r)|] \leq \text{Prob.} \{|s_{n_j} - s_r| \geq \varepsilon A_{n_j}\} + \text{Prob.} \{-\varepsilon A_{n_j} < s_{n_j} < \varepsilon A_{n_j}\}$$

Now

$$(10) \quad \text{Prob.} \{|s_{n_j} - s_r| \geq \varepsilon A_{n_j}\} = \int_{|x| \geq \varepsilon A_{n_j}} dF_{r, n_j}(x)$$

where $F_{r, n_j}(x)$ is distribution of $s_{n_j} - s_r = X_{r+1} + \dots + X_{n_j}$. Clearly

$$\begin{aligned} \int_{|x| \geq \varepsilon A_{n_j}} dF_{r, n_j}(x) &\leq 2 \int_{-\infty}^{\infty} \frac{x^2}{\varepsilon^2 A_{n_j}^2 + x^2} dF_{r, n_j}(x) \\ (11) \quad &= 2 \int_{-\infty}^{\infty} (1 - \frac{\varepsilon^2 A_{n_j}^2}{\varepsilon^2 A_{n_j}^2 + x^2}) dF_{r, n_j}(x) \\ &= 2 \left(1 - \Phi_{r, n_j}(\varepsilon A_{n_j}) \right) \end{aligned}$$

Now, by our assumptions, $X_1, X_2, \dots$ has the same distribution function $F(x)$ and the distribution of S_n/A_n tends to $\Phi(x)$, so there exists a constant $\lambda > 0$ satisfying the conditions (2) in Lemma 2, that is

$$F(+0) \geq \lambda > 0, \quad F(-0) \leq 1 - \lambda \quad (0 < \lambda < 1)$$

In this case, more strongly, we can take $\lambda > 0$ satisfying the condition

$$F_{r, n_j}(+0) \geq \lambda > 0, \quad F_{r, n_j}(-0) \leq 1 - \lambda \quad (0 < \lambda < 1)$$

for every r and n_j .

On the other hand, by a simple consideration

$$\lim_{n \rightarrow \infty} A_{n_j}/A_n = \sqrt{\lambda}$$

So for a given $\varepsilon > 0$, we may take k such that

$$\frac{A_{n_j}}{A_n} > \frac{1}{\varepsilon^2} \quad (j \geq k_0)$$

are satisfied for sufficiently large n .

So that we may take k such that

$$(12) \quad k_0/k < \varepsilon$$

and

$$(1 - F_{r, n_j}(\varepsilon A_{n_j}) + F_{r, n_j}(-\varepsilon A_{n_j})) < \alpha < \lambda/2 \quad (j > k_0)$$

are satisfied. Thus $F_{r, n_j}(x)$ satisfies the conditions of Lemma 2 for $k = \varepsilon A_{n_j}$ when $j > k_0$. By Lemma 2 and Lemma 1,

$$1 - \Phi_{r, n_j}(\varepsilon A_{n_j}) \leq \frac{1}{K(\alpha, \lambda)} (1 - \Psi_{r, n_j}(\varepsilon A_{n_j}))$$

$$\begin{aligned}
(13) \quad & \leq \frac{1}{K(\alpha, \lambda)} \sum_{i=1}^{n_i} (1 - \psi_{\tilde{F}}(\varepsilon A_{n_i})) \\
& \leq \frac{1}{K(\alpha, \lambda)} (n_i - r) (1 - \psi_{\tilde{F}}(\varepsilon A_{n_i})), \\
& \quad (\lambda' > k_0)
\end{aligned}$$

By (11) and (13), we have

$$\begin{aligned}
(14) \quad & \int_{|x| \geq \varepsilon A_{n_i}} \alpha \tilde{F}_{r, n_i}(x) \leq \frac{2}{K(\alpha, \lambda)} (n_i - r) (1 - \psi_{\tilde{F}}(\varepsilon A_{n_i})) \\
& = \frac{2}{K(\alpha, \lambda)} (n_i - r) \int_{-\infty}^{\infty} \frac{x^2}{\varepsilon^2 A_{n_i}^2 + x^2} \alpha \tilde{F}(x) \\
& \quad (\lambda' > k_0)
\end{aligned}$$

But

$$\begin{aligned}
(15) \quad & \int_{-\infty}^{\infty} \frac{x^2}{\varepsilon^2 A_{n_i}^2 + x^2} \alpha \tilde{F}(x) = \int_{|x| < \varepsilon A_{n_i}} \frac{x^2}{\varepsilon^2 A_{n_i}^2 + x^2} \alpha \tilde{F}(x) + \int_{|x| \geq \varepsilon A_{n_i}} \frac{x^2}{\varepsilon^2 A_{n_i}^2 + x^2} \alpha \tilde{F}(x) \\
& \leq \frac{1}{\varepsilon^2 A_{n_i}^2} \left(\int_{|x| < \varepsilon A_{n_i}} x^2 \alpha \tilde{F}(x) + \int_{|x| \geq \varepsilon A_{n_i}} \alpha \tilde{F}(x) \right)
\end{aligned}$$

Thus by (9), (10), (14), (15),
for $\lambda' \geq k_0$, we have

$$\begin{aligned}
E[|\psi(S_{n_i}) - \psi(\tilde{S}_r)|] & \leq \frac{2}{K(\alpha, \lambda)} \frac{(n_i - r)}{\varepsilon^2 A_{n_i}^2} \int_{|x| < \varepsilon A_{n_i}} x^2 \alpha \tilde{F}(x) \\
& + \frac{2}{K(\alpha, \lambda)} (n_i - r) \int_{|x| \geq \varepsilon A_{n_i}} \alpha \tilde{F}(x) + \text{Prob.} \left\{ -\varepsilon A_{n_i} < S_{n_i} < \varepsilon A_{n_i} \right\}
\end{aligned}$$

From (5)

$$\begin{aligned}
E[|D_n|] & \leq \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} E[|\psi(S_{n_i}) - \psi(\tilde{S}_r)|] \\
E[|D_n|] & \leq \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} E[|\psi(S_{n_i}) - \psi(\tilde{S}_r)|] \\
& = \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} E[|\psi(S_{n_i}) - \psi(\tilde{S}_r)|] \\
& + \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} E[|\psi(S_{n_i}) - \psi(\tilde{S}_r)|] \\
& \leq \frac{k_0}{k} + \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} \left[K \frac{(n_i - r)}{\varepsilon^2 A_{n_i}^2} \int_{|x| < \varepsilon A_{n_i}} x^2 \alpha \tilde{F}(x) \right. \\
& \quad \left. + K(n_i - r) \int_{|x| \geq \varepsilon A_{n_i}} \alpha \tilde{F}(x) + \text{Prob.} \left\{ -\varepsilon A_{n_i} < S_{n_i} < \varepsilon A_{n_i} \right\} \right]
\end{aligned}$$

where $K = 2/K(\alpha, \lambda)$. Hence

$$E[|D_n|] \leq \frac{k_0}{k} + \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} \frac{(n_i - r)}{\varepsilon^2 A_{n_i}^2} \int_{|x| < \varepsilon A_{n_i}} x^2 \alpha \tilde{F}(x)$$

$$\begin{aligned}
(17) \quad & + \frac{K}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} (n_i - r) \int_{|x| \geq \varepsilon A_{n_i}} \alpha \tilde{F}(x) \\
& + \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} \text{Prob.} \left\{ -\varepsilon A_{n_i} < S_{n_i} < \varepsilon A_{n_i} \right\} \\
& \equiv \frac{k_0}{k} + J_1 + J_2 + J_3
\end{aligned}$$

say.

From (11)

$$(18) \quad \frac{k_0}{k} < \varepsilon$$

$$\begin{aligned}
(19) \quad J_3 & = \frac{1}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} \text{Prob.} \left\{ -\varepsilon A_{n_i} < S_{n_i} < \varepsilon A_{n_i} \right\} \\
& \leq C_1 \varepsilon \quad (n > n_0)
\end{aligned}$$

where C_1 is an absolute constant.

$$J_2 = \frac{K}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} \frac{n_i - r}{\varepsilon^2 A_{n_i}^2} \cdot n_i \int_{|x| \geq \varepsilon A_{n_i}} \alpha \tilde{F}(x)$$

Hence from Lemma 3

$$(20) \quad J_2 \leq C_2 \varepsilon \quad (n > n_0)$$

Similary

$$\begin{aligned}
J_1 & = \frac{1}{\varepsilon^2} \frac{K}{n} \sum_{i=1}^k \sum_{r=n_{i-1}+1}^{n_i} \frac{n_i - r}{\varepsilon^2 A_{n_i}^2} \cdot \frac{n_i}{A_{n_i}^2} \int_{|x| < \varepsilon A_{n_i}} x^2 \alpha \tilde{F}(x) \\
(21) \quad & \leq C_3 \frac{(1 + \log k)}{\varepsilon^2 k} \quad (n > n_0)
\end{aligned}$$

Thus from (17), (18), (19), (20), (21), we have

$$\begin{aligned}
E[|D_n|] & \leq C_3 \frac{(1 + \log k)}{\varepsilon^2 k} + C_4 \varepsilon \\
(22) \quad & \equiv R(n, \varepsilon, k) \quad (n > n_0)
\end{aligned}$$

Hence, for every $\delta > 0$,

$$\text{Prob.} \{ |D_n| > \delta \} \leq R(n, \varepsilon, k) / \delta$$

Thus as in the proof of Erdo's and Kac,

$$\begin{aligned}
(23) \quad & \text{Prob.} \left\{ \frac{1}{n} \sum_{r=1}^n \psi(S_r) < \alpha \right\} = \text{Prob.} \left\{ \frac{1}{n} \sum_{r=1}^n \psi(S_r) < \alpha, |D_n| < \delta \right\} \\
& + \text{Prob.} \left\{ \frac{1}{n} \sum_{r=1}^n \psi(S_r) < \alpha, |D_n| \geq \delta \right\} \\
& \leq \text{Prob.} \left\{ \frac{1}{n} \sum_{r=1}^n (n_i - n_{i-1}) \psi(S_{n_i}) < \alpha + \delta \right\} \\
& + \frac{R(n, \varepsilon, k)}{\delta}
\end{aligned}$$

Similarly

$$(24) \quad \text{Prob.} \left\{ \frac{1}{n} \sum_{i=1}^k \psi(q_i) < d \right\} \geq \text{Prob.} \left\{ \frac{1}{n} \sum_{i=1}^k (n_i - n_{i-1}) \psi(s_{n_i}) < d - \delta \right\} \\ = \frac{R(n, \delta, k)}{\delta}$$

From (23), (24)

$$(25) \quad \text{Prob.} \left\{ \frac{1}{n} \sum_{i=1}^k (n_i - n_{i-1}) \psi(s_{n_i}) < d - \delta \right\} = \frac{R(n, \delta, k)}{\delta} \\ \leq \text{Prob.} \left\{ \frac{\sqrt{N_n}}{n} < \alpha \right\} \\ \leq \text{Prob.} \left\{ \frac{1}{n} \sum_{i=1}^k (n_i - n_{i-1}) \psi(s_{n_i}) < d + \delta \right\} + \frac{R(n, \delta, k)}{\delta}$$

By a simple consideration, the multidimensional distribution of random vector

$$\left(\frac{S_{n_1}}{A_{n_1}}, \frac{S_{n_2}}{A_{n_2}}, \dots, \frac{S_{n_k}}{A_{n_k}} \right)$$

tends to the distribution of

$$(q_1, q_1 + q_2, \dots, q_1 + q_2 + \dots + q_k)$$

where $q_1, q_2, \dots, q_k$ are independent normally distributed random variables each having mean and variance 1. Hence

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{1}{n} \sum_{i=1}^k (n_i - n_{i-1}) \psi(s_{n_i}) < \beta \right\}$$

$$= \lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{1}{n} \sum_{i=1}^k (n_i - n_{i-1}) \psi\left(\frac{s_{n_i}}{A_{n_i}}\right) < \beta \right\}$$

$$= \text{Prob.} \left\{ \frac{1}{k} \sum_{i=1}^k \psi(R_i) < \beta \right\}$$

where $R_i = q_1 + q_2 + \dots + q_i$, $i = 1, 2, \dots, k$. Thus as in the same way with Erdős and Kac's paper¹¹⁾, we get the result. So we shall omit the rest of the proof.

Next we shall give a following extension of Theorem 1.

Theorem 2. Let $X_1, \dots, X_k, \dots$ be independent identically distributed random variables and suppose that there exists a sequence of positive numbers $\{A_n\}$ increasing to infinity such that the distribution of S_n/A_n tends to some symmetric stable distribution. Then

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{\sqrt{N_n}}{n} < \alpha \right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}}$$

$$0 \leq \alpha \leq 1$$

To prove this, without loss of generality, we may assume that

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{S_n}{A_n} < x \right\} = G(x), \\ (26) \quad \int_{-\infty}^{\infty} e^{itx} dG(x) = e^{-|t|^\alpha} \\ 0 < \alpha < 2$$

Now we shall state some known result which we shall use to prove the theorem.

Lemma 4. Under the same assumptions of Theorem 2,

$$\lim_{n \rightarrow \infty} n \int_{|x| > \varepsilon A_n} d\tilde{F}(x) = C \varepsilon^{-\alpha}$$

where C is constant.

This is Doeblin's result¹²⁾.

Lemma 5. Under the hypothesis of Theorem 2, the multidimensional distribution of random vector

$$\left(\frac{S_{n_1}}{A_{n_1}}, \frac{S_{n_2}}{A_{n_2}}, \dots, \frac{S_{n_k}}{A_{n_k}} \right)$$

tends to the distribution of

$$(z_1, z_1 + z_2, \dots, z_1 + z_2 + \dots + z_k)$$

where $z_1, z_2, \dots, z_k$ are independent random variables each having the same distribution $G(x)$ which is indicated in (26)

Proof. Putting

$$U = t_1 \frac{S_{n_1}}{A_{n_1}} + t_2 \frac{S_{n_2}}{A_{n_2}} + \dots + t_k \frac{S_{n_k}}{A_{n_k}}$$

$t_1, t_2, \dots, t_k$ being any real numbers, then

$$U = (t_1 + t_2 + \dots + t_k) \frac{S_{n_1}}{A_{n_1}} + (t_2 + \dots + t_k) \frac{(S_{n_2} - S_{n_1})}{A_{n_1}} \\ + \dots + t_k \frac{(S_{n_k} - S_{n_{k-1}})}{A_{n_1}}$$

Hence

$$E[e^{itxU}] = E[e^{itx(t_1 + t_2 + \dots + t_k) \frac{S_{n_1}}{A_{n_1}}}] \\ = E[e^{itx(t_1 + t_2 + \dots + t_k) \frac{(S_{n_2} - S_{n_1})}{A_{n_1}}}] \dots \\ \dots E[e^{itx t_k \frac{(S_{n_k} - S_{n_{k-1}})}{A_{n_1}}}]$$

Since by the assumption of the theorem

$$\lim_{n \rightarrow \infty} E[e^{itx \frac{(S_{n_k} - S_{n_{k-1}})}{A_{n_1}}}] = e^{-|t|^\alpha}$$

we have

$$\lim_{n \rightarrow \infty} E[e^{i\lambda V}] = e^{-|\lambda|^{\alpha} (|t_1 + t_2 + \dots + t_k|^{\alpha} + |t_2 + \dots + t_k|^{\alpha} + \dots + |t_k|^{\alpha})}$$

On the other hand, if we put

$$V = t_1 z_1 + t_2 (z_1 + z_2) + \dots + t_k (z_1 + z_2 + \dots + z_k)$$

then

$$E[e^{i\lambda V}] = e^{-|\lambda|^{\alpha} (|t_1 + t_2 + \dots + t_k|^{\alpha} + |t_2 + \dots + t_k|^{\alpha} + \dots + |t_k|^{\alpha})}$$

Thus by the theorem of Cramér-Hold⁽⁶⁾, we get the result of Lemma 5.

Lemma 6. Let $X(t)$ ($X(0) = 0$) be a homogeneous process with independent increments obeying a symmetric stable law with exponent α .

Then

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{1}{n} \sum_{k=1}^n \psi \left(X \left(\frac{k}{n} \right) \right) < \alpha \right\} = \text{Prob.} \left\{ \int_0^1 \psi(X(t)) dt < \alpha \right\}$$

Proof. It has been shown by Levy and Doob^(7,8) that almost every $X(t)$ has at most a denumerable number of discontinuities, all of the first kind. Thus $\psi(X(t))$ is Riemann integrable in $[0, 1]$ for almost every $X(t)$, that is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi \left(X \left(\frac{k}{n} \right) \right) = \int_0^1 \psi(X(t)) dt$$

for almost all X , from which it follows that

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{1}{n} \sum_{k=1}^n \psi \left(X \left(\frac{k}{n} \right) \right) < \alpha \right\} = \text{Prob.} \left\{ \int_0^1 \psi(X(t)) dt < \alpha \right\}$$

Lemma 7. Under the hypotheses of Theorem 2.

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{1}{n} \sum_{k=1}^k (n_k - n_{k-1}) \psi(S_{n_k}) < \alpha \right\} = \text{Prob.} \left\{ \frac{1}{k} \sum_{j=1}^k \psi \left(X \left(\frac{j}{k} \right) \right) < \alpha \right\}$$

Proof. Since

$$\text{Prob.} \left\{ \frac{1}{n} \sum_{k=1}^k (n_k - n_{k-1}) \psi(S_{n_k}) < \alpha \right\} = \text{Prob.} \left\{ \frac{1}{n} \sum_{k=1}^k (n_k - n_{k-1}) \psi \left(\frac{r_{n_k}}{A_{n_k}} \right) < \alpha \right\}$$

we have by Lemma 5

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \sum_{k=1}^k (n_k - n_{k-1}) \psi(S_{n_k}) < \alpha \right\} = \text{Prob.} \left\{ \frac{1}{k} \sum_{j=1}^k \psi(T_j) < \alpha \right\},$$

where $T_j = z_1 + \dots + z_j$, z_i ($i=1, 2, \dots$) being those as in Lemma 5.

Thus

$$\lim_{n \rightarrow \infty} \text{Prob.} \left\{ \frac{1}{n} \sum_{k=1}^k (n_k - n_{k-1}) \psi(S_{n_k}) < \alpha \right\} = \text{Prob.} \left\{ \frac{1}{k} \sum_{j=1}^k \psi \left(X \left(\frac{j}{k} \right) \right) < \alpha \right\}$$

Lemma 8.

$$\text{Prob.} \left\{ \int_0^1 \psi(X(t)) dt < \alpha \right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{\alpha}}$$

$$0 \leq \alpha \leq 1$$

This is a result of due to M. Kac.⁹⁾

Proof of Theorem 2.

As in the proof of Theorem 1, we have the inequalities

$$\begin{aligned} E[|Q_n|] &\leq \frac{K}{k} + \frac{1}{n} \sum_{k=1}^k \sum_{r=n_{k-1}+1}^{n_k} K \frac{n_k-r}{A_{n_k}^{\alpha}} \left(x^2 d \widetilde{H}(x) \right)_{|x| \leq \epsilon A_{n_k}} \\ &\quad + \frac{1}{n} \sum_{k=1}^k \sum_{r=n_{k-1}+1}^{n_k} K(n_k-r) \left(d \widetilde{H}(x) \right)_{|x| \geq \epsilon A_{n_k}} \\ &\quad + \frac{1}{n} \sum_{k=1}^k \sum_{r=n_{k-1}+1}^{n_k} \text{Prob.} \left\{ -\epsilon A_{n_k} < S_{n_k} < \epsilon A_{n_k} \right\} \\ &= \frac{K}{k} + J_1 + J_2 + J_3 \end{aligned}$$

$$(19') \quad J_3 = \frac{1}{k} \sum_{k=1}^k \text{Prob.} \left\{ -\epsilon A_{n_k} < S_{n_k} < \epsilon A_{n_k} \right\} \leq C_1 \epsilon \quad (n > n_0)$$

$$(20') \quad J_1 = \frac{K}{n} \sum_{k=1}^k \sum_{r=n_{k-1}+1}^{n_k} \frac{n_k-r}{n_k^{\alpha}} n_k \left(d \widetilde{H}(x) \right)_{|x| \geq \epsilon A_{n_k}}$$

$$\leq C_2 \frac{1 + \log k}{\epsilon^{\alpha} k}$$

(by Lemma 4)

$$J_1 \leq \frac{K}{n \epsilon^{\alpha}} \sum_{k=1}^k \sum_{r=n_{k-1}+1}^{n_k} \frac{n_k-r}{n_k^{\alpha}} n_k \int_{-\epsilon}^{\epsilon} x^2 d \widetilde{H}(A_{n_k} x)$$

But as a special case of Gnedenko's result,⁽⁹⁾ we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} n_{\varepsilon} \int x^2 d\tilde{F}(A n_{\varepsilon}, x) = 0$$

Hence for given $\varepsilon > 0$,

$$n_{\varepsilon} \int x^2 d\tilde{F}(A n_{\varepsilon}, x)$$

are bounded. Thus we have

$$(21)' \quad J_{\varepsilon} \leq C_3 \frac{1 + \log h}{\varepsilon^2 \log h}$$

From (19'), (20'), (21'), we have

$$E[|D_n|] \leq C_2 \frac{1 + \log h}{\varepsilon^2 \log h} + C_3 \frac{1 + \log h}{\varepsilon^2 h} + C_4 \varepsilon$$

$$\equiv R(n, \varepsilon, h) \quad (n \rightarrow \infty)$$

Thus as in the proof of Theorem 1 by using Lemma 5, Lemma 6, Lemma 7 and Lemma 8, we have

$$\lim_{n \rightarrow \infty} \text{Prob} \left\{ \frac{N_n}{n} < \alpha \right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}}$$

$$0 \leq \alpha \leq 1$$

(*) Received Dec. 20, 1951.

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(1) We shall interpret here that the Lindeberg's conditions are satisfied, that is, the distribution function of $S_n/\sqrt{n}$ tends to the normal distribution function $\Phi(x)$ with mean 0 and variance 1 when $n \rightarrow \infty$. In this connection, the following fact may be mentioned: we can select independent random variables $X_1, X_2, \dots$, each having mean 0 and variance 1 and distribution function of $S_n/\sqrt{\log n}$ tends to the normal distribution function $\Phi(x)$. In this case, recently G. Maruyama has proved that $\text{prob.} \left\{ \frac{N_n}{n} < \alpha \right\}$ does not converge to arc sin law.