A THEOREM ON CONTINUOUS FUNCTIONS

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In the present paper we call an "α-function" a function $f(x)$, continuous in $0 \leqq x \leqq 1$, such that

$$f(0) = 0, \quad f(1) = 1$$

and

$$0 \leqq f(x) \leqq 1 \qquad (0 < x < 1).$$

Definition 1. A function is called nowhere constant, if there exists no interval on which it remains constant.

We now begin with the following

Theorem 1. Let $f(x)$ and $g(x)$ be nowhere constant α-functions, then there exist α-functions $\varphi(x)$ and $\psi(x)$ such that $f(\varphi(x)) = g(\psi(x))$.

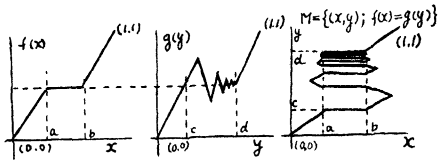
If we consider a subset M of the euclidean plane, defined by $M = \{(x, y);\ f(x) = g(y),\ 0 \leqq x, y \leqq 1\}$, then Theorem 1 is equivalent to the existence of a continuous curve in M connecting the points $(0, 0)$ and $(1, 1)$.

T.Minagawa showed me an example (see the figure below), illustrating the Theorem 1 is not always true if the given functions are not nowhere constant.

Definition 2. We say that a closed interval $[a, b]$ is an "α-interval of $f(x)$", if

$$\sup_{x \in [a,b]} f(x) - \inf_{x \in [a,b]} f(x) = |f(a) - f(b)|$$

We shall first prove the following lemmas A), B), C), concerning nowhere constant α-functions. In these lemmas, $f(x)$ denote a nowhere constant α-function.



Lemma A. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|a - b| < \varepsilon$ provided $\sup_{x \in [a,b]} f(x) - \inf_{x \in [a,b]} f(x) < \delta$.

Lemma B. Let $[p_0, p]$ be an arbitrary closed interval of $I = [0, 1]$. Let p_1 be a maximum point of $f(x)$ in $[p_0, p]$, let p_2 be a minimum point of $f(x)$ in $[p_1, p]$, let p_3 be a maximum point in $[p_2, p]$, and so on. Then the sequence $\{p_i\}$ converges to p.

Lemma C. For any $\varepsilon > 0$, there exist a finite number of points a_i $(i = 1, 2, \quad , n-1)$ such that $0 \leqq a_i - a_{i-1} < \varepsilon$ $(i = 1, \cdots, n,\ a_0 = 0,\ a_n = 1)$ and that every closed interval $[a_{i-1}, a_i]$ is an α-interval of $f(x)$.

Lemmas A and B can easily be proved, so we shall give a proof of Lemma C. First we prove it for $\varepsilon = \frac{2}{3}$. By Lemma B, we can construct two convergent sequences:

$$0 = p_0 \leqq p_1 \leqq p_2 \leqq \cdots \quad \to \quad \frac{1}{2},$$

$$1 = q_0 \geqq q_1 \geqq q_2 \geqq \cdots \quad \to \quad \frac{1}{2},$$

where all $[p_{i-1}, p_i]$, $[q_{i-1}, q_i]$ are α-intervals of $f(x)$. If there exist positive integers m, n such that $p_m = q_n = \frac{1}{2}$ then the points, $0 = p_0, p_1, p_2, \cdot, p_m = \frac{1}{2} = q_n, \cdots, q_2, q_1, q_0 = 1$, are the required ones. Otherwise, suppose $p_m < \frac{1}{2}$ for $m = 1, 2, \cdots$, then there exists a positive integer m such that $\frac{1}{3} < p_m < \frac{1}{2}$. From Lemma B, we can again construct another convergent sequence: $1 = q_0' \geqq q_1' \geqq \cdots \to p_m$, where every closed interval $[q_{i-1}', q_i']$ is an α-interval of $f(x)$. Now we can readily see that there exists a positive integer n such that $q_n' = p_m$, since p_m is a maximum or minimum point of $f(x)$ in $[p_m, \frac{1}{2}]$. Then the points: $0 = p_0, p_1, \cdots, p_m = q_n', \ldots, q_1', q_0' = 1$ are the required points. Repeating the above process in each interval, we obtain the desired points for $\varepsilon = (\frac{2}{3})^2$, and thus we can, by induction, complete the proof of Lemma C.

Lemma 1. Under the condition that $f(x)$ and $g(x)$ are both polygonalized - linear on $I = [0,1]$ except a finite number of points, Theorem 1 is true.

Lemma 2. If $f(x)$ is polygonalized, then Theorem 1 is true.

The proof of Lemma 2 may be omitted, since we shall later derive Theorem 1 by means of Lemma 2 and this lemma itself can be quite similarly derived from Lemma 1.

Proof of Lemma 1. At any point p of $M = \{(x,y)\ ;\ f(x) = g(y),\ 0 \leqq x, y \leqq 1\}$ except at two points $(0,0)$ and $(1,1)$, M has a neighborhood of p which is homeomorphic with an isolated point, or an open line-segment, or a cross point of two open line-segments. At the points $(0,0)$ and $(1,1)$, M has a neighborhood which is homeomorphic with a half- open linesegment (i.e. $\{x\ ;\ a \leqq x < b\}$) and the point $(0,0)$ or $(1,1)$ corresponds to the closed endpoint a of the half-open line-segment. Moreover M consists of a finite number of line-segments. Hence we can conclude by the unicursal principle that there exists a polygon in M connecting $(0,0)$ and $(1,1)$. We express the polygon by $(\varphi(t), \psi(t))$; $0 \leqq t \leqq 1$, $\varphi(0) = \psi(0) = 0$, $\varphi(1) = \psi(1) = 1$. Then $\varphi(x)$ and $\psi(x)$, $0 \leqq x \leqq 1$, are parametric functions which are required in Lemma 1.

Proof of Theorem 1. We shall construct, for each positive integer n, a set of α-functions $\varphi_n(x)$, $\psi_n(x)$ and countable disjoint open intervals $\{I_{ni} = (\alpha_{ni}, \alpha_{ni}')\}$ of $I = [0,1]$ satisfying the following relations:

(n.1) $f(\varphi_n(x)) = g(\psi_n(x))$ $x \in I - (\cup I_{ni}) = A_n$,

(n.2) $\varphi_n(x) = \varphi_{n-1}(x)$, $\psi_n(x) = \psi_{n-1}(x)$ $x \in A_{n-1}$;

(n.3) $A_n \supset A_{n-1}$;

(n.4) $\bar{I}_{ni} = [\alpha_{ni}, \alpha_{ni}']$ are α-intervals of $\varphi_m(x)$ and $\psi_m(x)$ $m \geqq n$;

(n.5) The intervals $[\varphi_n(\alpha_{ni}), \varphi_n(\alpha_{ni}')]$ and $[\psi_n(\alpha_{ni}), \psi_n(\alpha_{ni}')]$, $i = 1, 2, \cdots$, are α-intervals of $f(x)$ and $g(x)$, respectively;

(n.6) $|\varphi_n(\alpha_{ni}) - \varphi_n(\alpha_{ni}')| \leqq \frac{1}{n}$.

Put $\varphi_1(x) = \psi_1(x) = x$, $x \in I$ and $\{I_{1i}\} = \{(0,1)\}$, and suppose that there exist, for any $j \leqq n$, a set of $\varphi_j(x)$, $\psi_j(x)$ and the countable intervals $\{I_{ji}\}$ satisfying the relations (j.1), (j.2), ..., (j.6). We then construct a set of $\varphi_{n+1}(x)$ $\psi_{n+1}(x)$ and $\{I_{n+1\,i}\}$ as follows: From Lemma C, we can see for any i that there exists a set of points $\{a_0, a_1, \cdots, a_k\}$ in the closed interval $[\varphi_n(\alpha_{ni}), \varphi_n(\alpha_{ni}')] = I_{ni}'$ such that

$$a_0 = \varphi_n(\alpha_{ni}),\quad a_k = \varphi_n(\alpha_{ni}'),$$

$$|a_{m-1} - a_m| \leqq \frac{1}{n+1},\quad m = 1, 2, \cdots, k,$$

$[a_{m-1}, a_m]$ are an α-interval of $f(x)$, $m = 1, 2, \cdots, k$,

We then construct, for each i, a new polygonalized continuous function $f_i(x)$ on $I_{ni}' = [\varphi_n(\alpha_{ni}), \varphi_n(\alpha_{ni}')]$ as follows:

$$f(a_m) = f_i(a_m),\quad m = 0, 1, 2, \cdots, k.$$

$f_i(x)$ is linear on I_{ni}' except at the points $a_0, a_1, \cdots, a_k$. As we assume the validity of Lemma 2 we can find two continuous functions $\varphi_{ni}(x)$ and $\psi_{ni}(x)$ on $\bar{I}_{ni} = [\alpha_{ni}, \alpha_{ni}']$ such that

$$f_i(\varphi_{ni}(x)) = g(\psi_{ni}(x)),\quad x \in \bar{I}_{ni},$$

$$\varphi_{ni}(\alpha_{ni}) = \varphi_n(\alpha_{ni}),\quad \psi_{ni}(\alpha_{ni}) = \psi_n(\alpha_{ni}),$$

$$\varphi_{ni}(\alpha_{ni}') = \varphi_n(\alpha_{ni}'),\quad \psi_{ni}(\alpha_{ni}') = \psi_n(\alpha_{ni}').$$

Since a set C_i defined by $C_i = \{x ; \varphi_{n_i}(x) = a_m, m=1,\cdots,k\}$ is a closed subset of $\bar{I}_{n_i}$, the open set $\bar{I}_{n_i} - C_i$ consists of countable open intervals. Moreover, we can see that these open intervals consist of the following systems of countable open intervals $\{(\beta_m, \beta'_m)\}$ and $\{(\gamma_m, \gamma'_m)\}$ where

$$\varphi_{n_i}(\beta_m) = \varphi_{n_i}(\beta'_m) = a_s, \quad m = 1, 2, \cdots,$$

$$\varphi_{n_i}(\gamma_m) = a_t, \quad \varphi_{n_i}(\gamma'_m) = a_{t\pm 1}, \quad m = 1, 2, \cdots.$$

Thus we can find a point β''_m of the interval $(\beta_m \ \beta'_m)$ and a point a'_s of the interval $(a_s, \ a_{s\pm 1})$ such that

$f(a'_s) = g(\psi_{n_i}(\beta''_m))$,

$[a_s, a'_s]$ is an α -interval of $f(x)$,

$[\psi_{n_i}(\beta_m), \psi_{n_i}(\beta''_m)]$, $[\psi_{n_i}(\beta''_m), \psi_{n_i}(\beta'_m)]$ are α -intervals of $f(x)$.

Let us put $\{I_{n+1_i}\}$ $= \{(\gamma_m, \gamma'_m), (\beta_m, \frac{\beta_m+\beta'_m}{2}), (\frac{\beta_m+\beta'_m}{2}, \beta'_m)\}$, $m = 1, 2, \cdots$.

We shall construct $\varphi_{n+1}(x)$, $\psi_{n+1}(x)$ as follows:

$$\varphi_{n+1}(x) = \varphi_n(x), \quad \psi_{n+1}(x) = \psi_n(x), \quad x \in A_n;$$

$$\varphi_{n+1}(x) = \varphi_{n_i}(x), \quad \psi_{n+1}(x) = \psi_{n_i}(x), \quad x \in C_i, \quad i = 1, 2, \quad ;$$

$$\varphi_{n+1}\left(\frac{\beta_m+\beta'_m}{2}\right) = a'_s$$

$$\psi_{n+1}\left(\frac{\beta_m+\beta'_m}{2}\right) = \psi_{n_i}(\beta''_m)$$

$$n, i = 1, 2, \cdots ;$$

$\varphi_{n+1}(x)$ and $\psi_{n+1}(x)$ are linear on the intervals $[\beta_m, \frac{\beta_m+\beta'_m}{2}]$, $[\frac{\beta_m+\beta'_m}{2}, \beta'_m]$ and $[\gamma_m, \gamma'_m]$.

It can easily be seen that the functions $\varphi_{n+1}(x)$, $\psi_{n+1}(x)$ and the system of intervals $\{I_{n+1_i}\}$ satisfy the conditions $(n+1 . 1)$, ... $(n+1 . 6)$. The continuity of $\varphi_{n+1}(x)$ and $\psi_{n+1}(x)$ follows from that of $\varphi_n(x)$ and $\psi_n(x)$ and the condition (n.4). Next we shall show the uniform convergence of two sequences $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$. The uniform convergence of $\{\varphi_n(x)\}$ follows immediately from the conditions (n.2), (n.4), (n.6) and the continuity of each $\varphi_n(x)$. To prove the uniform convergence of $\{\psi_n(x)\}$, we shall show that for any $\varepsilon > 0$ there exists a positive integer N such that for $n > N$

$$(n.6') \quad |\psi_n(\alpha_{n_i}) - \psi_n(\alpha'_{n_i})| < \varepsilon.$$

By Lemma A there exists a positive number δ such that $|a - b| < \varepsilon$ provided $\sup\limits_{x\in[a,b]} f(x) - \inf\limits_{x\in[a,b]} f(x) < \delta$, and from the uniform continuity of $f(x)$ we can see that there exists a positive integer N such that $|f(x') - f(x)| < \delta$ for $|x - x'| < \frac{1}{N}$. From the condition (n.6) we have

$$|\varphi_n(\alpha_{n_i}) - \varphi_n(\alpha'_{n_i})| \leqq \frac{1}{n} < \frac{1}{N}$$

for any $n > N$, and hence

$$\delta > |f(\varphi_n(\alpha_{n_i})) - f(\varphi_n(\alpha'_{n_i}))|$$

$$= |g(\psi_n(\alpha_{n_i})) - g(\psi_n(\alpha'_{n_i}))|$$

$$= \sup_{x\in[\psi_n(\alpha_{n_i}), \psi_n(\alpha'_{n_i})]} g(x) - \inf_{x\in[\psi_n(\alpha_{n_i}), \psi_n(\alpha'_{n_i})]} g(x).$$

Thus we can say $|\psi_n(\alpha_{n_i}) - \psi_n(\alpha'_{n_i})| < \varepsilon$. Therefore $\{\psi_n(x)\}$ satisfies the condition (n.6'). The conditions (n.2), (n.4), (n.6') and the continuity of each $\psi_n(x)$ guarantee the uniform convergence of $\{\psi_n(x)\}$. Now we denote by $\varphi(x)$ and $\psi(x)$ the limit functions of $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$, then $\varphi(x)$ and $\psi(x)$ are the required functions. We have completed the proof of Theorem 1. We can extend Theorem 1 to the following one:

Theorem 2. Let $f_1(x), f_2(x), \cdots, f_k(x)$ be nowhere constant α -functions. Then there exist such α -functions $\varphi_1(x)$, $\varphi_2(x)$, $\cdot$, $\varphi_k(x)$ that

$$f_1(\varphi_1(x)) = f_2(\varphi_2(x)) = \cdots = f_k(\varphi_k(x))$$

for any $x \in I$.

Proof. By induction. Suppose Theorem 2 is true when the number

of given functions is $n-1$.
Then there exist $n-1$ α-functions $\varphi_1^*(x), \varphi_2^*(x), \dots, \varphi_{n-1}^*(x)$ such that $f_1(\varphi_1^*(x)) = f_2(\varphi_2^*(x)) = \cdots = f_{n-1}(\varphi_{n-1}^*(x))$ for any $x \in I$. Furthermore, we can assume that $\varphi_1^*(x), \varphi_2^*(x), \dots, \varphi_{n-1}^*(x)$ are nowhere constant functions, because, if otherwise, we can replace them by nowhere constant functions.
Let $g(x) = f_1(\varphi_1^*(x))$, so $g(x)$ is a nowhere constant α-function.

By Theorem 1 there exist two α-functions $\varphi(x), \varphi_n(x)$ such that $g(\varphi(x)) = f_n(\varphi_n(x))$ $x \in I$. Let $\varphi_1(x) = \varphi_1^*(\varphi(x)), \varphi_2(x) = \varphi_2^*(\varphi(x)), \dots, \varphi_{n-1}(x) = \varphi_{n-1}^*(\varphi(x))$, then $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are the required functions.

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