A REMARK ON MAUTNER'S DECOMPOSITION

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INTRODUCTION

F.I.Mautner has shown in his paper (1) that if an unitary representation $g \longrightarrow U(g)$ of a locally compact group G, which is the union of enumerable compact sets, decomposes into $U(g,t)$ by his direct integral, there exists a choice of representations $\tilde{U}(g,t)$ for the equivalence classes of operator-valued functions $U(g,t)$ with the following property: There exists for almost every t a strongly continuous unitary representation $V_t(g)$ with representation space $\mathfrak{H}_t$ and a subset N_t of G of right invariant Haar measure zero such that $g \bar{\in} N_t$ implies $\tilde{U}(g,t) = V_t(g)$. Recently he states in his [3] that if G is a connected Lie group, $\tilde{U}(g,t)$ are unitary representations exactly.

The aim of the present paper is to show that his hypothesis can be altered by the separably local compactness.

This result will be obtained by combining Mautner's work with a few modifications. The result will give us a little convenience. Although I have recently found that the result was obtained already by R.Godement in [4], his method is slightly different from ours.

PROPOSITION. Let G be a separable locally compact group. Let $U(g)$ be a strongly continuous unitary representation of G on $\mathfrak{H}$. Suppose $\mathfrak{H} = \int_\oplus \mathfrak{H}_t$ is a Mautner's direct integral such that every $U(g)$ decomposes, say into $U(g,t)$. Then there exists a choice of representations $U'(g,t)$ for the equivalence classes of operator valued functions $U(g,t)$ with the following property: The representations $U'(g,t)$ for almost all t are strongly continuous unitary representations of G.

To prove this proposition, we shall use the following Lemmas.

LEMMA 1. (Mautner [1]) Let G, $U(g)$, and $\mathfrak{H} = \int_\oplus \mathfrak{H}_t$ be as above, then there exists a choice of representations $\tilde{U}(g,t)$ for the equivalence classes of operator-valued functions $U(g,t)$ with the following property: There exists for almost every t a strongly continuous unitary representation $V_t(g)$ with representation space $\mathfrak{H}_t$ and a subset N_t of G of right invariant Haar measure zero such that $g \bar{\in} N_t$ implies $\tilde{U}(g,t) = V_t(g)$.

LEMMA 2. (Mackey [2]) Let Y be a measure space and let be a Borel measure in a separable locally compact metric space $\mathfrak{M}$. Let f be a complex valued function defined on $\mathfrak{M} \times Y$ which for each fixed y in Y is continuous on $\mathfrak{M}$ and for each fixed x in $\mathfrak{M}$ is measurable on Y. Then f is a measurable on the product space $\mathfrak{M} \times Y$.

PROOF OF PROPOSITION. Let us show that unitary representations $V_t(g)$ in Lemma 1 satisfy the property of Proposition.

Mautner proved the following in his paper [1] : Operators $F = \int U(g) f(g)\, dg$ decompose into

$$F(t) = \int f(g)\, \tilde{U}(g,t)\, dg = \int f(g)\, V_t(g)\, dg$$

for all $f \in L_1(G)$.

(We mean by dg right-invariant Haar measure.)

By changing the values on a set of $s(t)$ -measure zero, we can suppose $V_t(g)$ defines on the whole real line and the above equality is satisfied for all t. (for example, changing $V_t(g)$ into identity representation on a set of $s(t)$ -measure zero.)

Let x, y be arbitrary elements of $\mathfrak{H}$ and $x(t)$, $y(t)$ their components in $\mathfrak{H}_t$.

$$\langle F(t)x(t), y(t)\rangle = \int f(g) \langle V_t(g)x(t), y(t)\rangle\, dg$$

is $s(t)$ -measurable for all $f \in L_1(G)$.

Let f_n be sequence of continuous functions such that bounded measures $f_n(g)\,dg$ converge "étroitement" to point measure ε_{g_0} ($\varepsilon_{g_0}(g_0) = 1$), then for each fixed t,

$$\lim_n \langle F_n(t)x(t), y(t)\rangle = \lim_n \int f_n(g) \cdot$$
$$\cdot \langle V_t(g)x(t), y(t)\rangle\, dg = \langle V_t(g_0)x(t), y(t)\rangle$$

since $\langle V_t(g)x(t), y(t)\rangle$ is a continuous bounded function on G.

Hence for each fixed g, $\langle V_t(g)x(t), y(t)\rangle$ is $s(t)$-measurable. On the other hand, $\langle V_t(g)x(t), y(t)\rangle$ is clearly continuous on G for each fixed t.

By Lemma 2 we obtain the measurability of $\langle V_t(g)x(t), y(t)\rangle$ on $G\times R^1$. Since we can suppose that $V_t(g)$ defines unitary representations on the whole real line, we have $\|V_t(\varepsilon)\|=1$ for all t. Hence there exist unitary operators $V(g)$:

$$V(g) \sim V_t(g)$$

Let x, y again be arbitrary elements of $\mathfrak{H}$ and $x(t)$, $y(t)$ their components in $\mathfrak{H}_t$, then

$$\langle V(g)x, y\rangle = \int \langle V_t(g)x(t), y(t)\rangle\, ds(t)$$

Hence $\langle V(g)x, y\rangle$ is measurable on G, and $g\to V(g)$ is clearly algebraic homomorphism. Therefore $g\to V(g)$ is a measurable unitary representation. From the well-known fact $g\to V(g)$ is strongly continuous unitary representation.

We now show that $F'=\int f(g)V(g)\,dg$ decomposes into

$$F(t) = \int f(g)V_t(g)\,dg$$

for all $f\in L_1(G)$.

To prove this, it is sufficient to show that

$$\langle F'x, y\rangle = \int \langle F(t)x(t), y(t)\rangle\, ds(t)$$

where $x(t)$ and $y(t)$ are components of x and y respectively.

$$\int \langle F(t)x(t), y(t)\rangle\, ds(t) = \int \langle [\int f(g)V_t(g)\,dg]x(t),$$

$$y(t)\rangle\, ds(t) = \int \left(\int f(g)\langle V_t(g)x(t), y(t)\rangle\, dg\right) ds(t)$$

$$= \int f(g)\,dg \int \langle V_t(g)x(t), y(t)\rangle\, ds(t)$$

$$= \int f(g)\langle V(g)x, y\rangle\, dg$$

$$= \langle F'x, y\rangle$$

Hence $F'\sim F(t) = \int f(g)V_t(g)\,dg$

$$= \int f(g)\tilde{U}(g,t)\,dg$$

and $F = \int f(g)U(g)\,dg \sim F(t)$

$$F' = \int f(g)V(g)\,dg = F = \int f(g)U(g)\,dg$$

for all $f\in L_1(G)$,

and so

$$V(g) = U(g), \text{ for almost all } g.$$

By the continuity of $g\to U(g)$, and $g\to V(g)$, then we can conclude $U(g)=V(g)$ everywhere.

Put $V_t(g)$ for $U'(g,t)$, this completes the proof.

COROLLARY. (Bochner) Let $f(g)$ be a continuous positive definite function on G. Then there exists a system of elementary positive definite functions $f_t(g)$ such that

$$f(g) = \int f_t(g)\, ds(t)$$

where $s(t)$ is a suitable real valued non decreasing rightsemi continuous functions.

(*) Received Sept. 30, 1952.

[1] F.I.Mautner; Unitary representations of locally compact groups, II. Ann. of Math., 52(1950)528-556.

[2] G.W.Mackey; Induced representations of locally compact groups, I. Ann. of Math., 53(1952)101-139.

[3] F.I.Mautner; On the decomposition of unitary representation of Lie group, Proc. Amer. Math. Soc., 2(1951)490-496.

[4] R.Godement; Math. review, 13-1 p.11.

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