CONFORMAL MAPPING OF MULTIPLY CONNECTED DOMAINS, I.

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0. Introduction

In general theory of conformal mapping of multiply connected domains, various types of special domains have hitherto been used as canonical ones; in particular, for instance, whole plane slit along parallel segments, whole plane or circular disc or annulus slit along radial segments or circular arcs, etc. It is a basic problem in the theory to establish the existence of conformal mapping of a given domain onto such a canonical domain of respective type as well as to assert the uniqueness of mapping under suitable normalizing conditions. It is also an important problem to discuss various kinds of distortion concerning the families of univalent functions in a given canonical domain, some of which is not only interesting by itself but also useful as a clue of existence proof.

With respect to canonical domains of the above mentioned types, these problems have been investigated from various points of view; cf., for instance, Komatu [3] . The existence proofs have first been given by Hilbert [1], Koebe [1,2,6,7,8] and Courant [1,2] or by Koebe [3,4,5], especially based upon a potential-theoretic method or upon the so-called continuity method, respectively. On the other hand, the extremal properties belonging to such canonical domains have been clarified, with respect to distortion, by de Possel [1] , Grötzsch [1,2], Rengel [1] and others, and further been noticed to be available for establishing the existence proof. Indeed, the existence proof of mapping onto such a canonical domain has also been succeeded by means of purely function-theoretic methods alone; cf. de Possel [1], Rengel [2] , Grötzsch [3] . Such a proof may be regarded as a direct generalization of that of Riemann's mapping theorem concerning simply connected domains published by Radó [1] which is due to L. Fejér and F. Riesz.

Now, there are further types of canonical domains such as, for instance, whole plane slit along two sets of parallel segments being perpendicular each other, whole plane slit along radial segments as well as circular arcs, etc. The existence of conformal mapping onto such a canonical domain has also been shown by Koebe [3,4,5] by means of continuity method or potential-theoretic method.

In the present Note we shall clarify the extremal properties, with respect to distortion, belonging to such canonical domains by means of which we shall then notice that the existence proof of conformal mapping onto such a canonical domain can be reduced to the problem in case of extremely lower connectivity, in fact, the one concerning the essentially lowest connectivity. We shall further discuss the corresponding problems with regard to the related types of canonical domains, especially, parallel strip slit along horizontal and vertical segments in detail.

Although throughout the present Note we restrict ourselves to case of finite connectivity, some of the obtained results will immediately be extended to case of infinite connectivity.

1. Whole plane slit along horizontal and vertical segments.

Let us consider an n-ply connected domain D laid in the z-plane the boundary of which is supposed to be composed of n disjoint continua C_j $(j = 1, \cdots, n)$. Let $f(z)$ be a function univalent in D . In general, the image of D by mapping $w = f(z)$ be denoted by Δ and the boundary component of Δ corresponding to C_j be denoted by Γ_j . The assumption that every boundary component of D is a continuum does not restrict the generality. Otherwise, i.e., if some of them are isolated points, they are merely removable singularities of mapping function, and hence the problem will then reduce to a case of lower connectivity.

We now denote by A_p^α the family consisting of all n-ply connected domains Δ whose p boundary components Γ_j $(j = 1, \cdots, p)$ are segments with gradient α, and by $'A_p^\alpha$ the family consisting of all n-ply connected domains Δ whose $n-p$ boundary components Γ_j $(j = p+1, \cdots, n)$ are segments with gradient α; p being an integer such that $0 \leqq p \leqq n$. In particular, $A_0^\alpha = {}'A_n^\alpha$ is regarded as the family consisting of all univalent images of D.

Let z_∞ be an arbitrarily fixed point in D. Suppose that the functions $f(z)$ in consideration, being univalent in D, are normalized by the condition

$$\lim_{z \to z_\infty} \left(f(z) - \frac{1}{z - z_\infty} \right) = 0.$$

In case $z_\infty = \infty$, the condition must be replaced by a modified one, i.e.,

$$\lim_{z \to \infty} (f(z) - z) = 0$$

We then denote by $\mathcal{F}_p^\alpha(z_\infty)$ and $'\mathcal{F}_p^\alpha(z_\infty)$ the families consisting of normalized functions which map D onto domains belonging to A_p^α and $'A_p^\alpha$, respectively.

It is evident that neither of the families $\mathcal{F}_p^\alpha(z_\infty)$ and $'\mathcal{F}_p^\alpha(z_\infty)$ is empty for every possible values of α and p. In particular, $\mathcal{F}_0^\alpha(z_\infty) = {}'\mathcal{F}_n^\alpha(z_\infty)$ consists of all normalized functions univalent in D. On the other hand, as is well-known, the family $\mathcal{F}_n^\alpha(z_\infty) = {}'\mathcal{F}_0^\alpha(z_\infty)$ consists of the unique function mapping D, under the prescribed normalization at z_∞, onto whole plane slit along parallel segments with gradient α; cf. de Possel [1]. Moreover, the function belonging to the family $\mathcal{F}_n^\alpha(z_\infty)$ with any α is expressible by those with special α's; in fact, denoting by $f(z; z_\infty, \alpha)$, in general, the unique function belonging to $\mathcal{F}_n^\alpha(z_\infty)$, the identical relation

$$f(z; z_\infty, \alpha)$$
$$= e^{i\alpha}(f(z, z_\infty; 0)\cos\alpha - i f(z; z_\infty, \pi/2)\sin\alpha)$$

holds good; cf. Grunsky [1] or Schiffer [1]. This fact may be slightly generalized. Indeed, the same remains true also if we suppose, in general, $f(z, z_\infty; \alpha) \in \mathcal{F}_p^\alpha(z_\infty) \frown {}'\mathcal{F}_p^{\alpha+\pi/2}(z_\infty)$ for any p, while the general existence theorem for such functions is a main purpose of the present Note.

Let now $f \equiv f(z; z_\infty)$ be any function defined in D and satisfying the preassigned normalization at z_∞. All such functions being admitted, we then introduce a functional defined by

$$a[f] = \left[\frac{d}{dz} \left(f(z; z_\infty) - \frac{1}{z - z_\infty} \right) \right]^{z = z_\infty}.$$

Consequently, any admissible function is expanded around z_∞ in the form

$$f(z; z_\infty) = \frac{1}{z - z_\infty} + a[f](z - z_\infty) + \cdots,$$

the dotted part being composed of the terms of degrees higher than unity. In case $z_\infty = \infty$, an evident modification must, of course, take place; namely, $1/(z - z_\infty)$ must be replaced by z.

We first state a fundamental distortion theorem concerning $a[f]$, yielding a generalization of a theorem due to de Possel [1].

Theorem 1. If $f(z; z_\infty) \in \mathcal{F}_p^{\pi/2}(z_\infty)$ and $\phi(z; z_\infty) \in {}'\mathcal{F}_p^0(z_\infty)$, then

$$\Re\, a[f] \leqq \Re\, a[\phi];$$

the equality here is valid only if $f \equiv \phi$.

Proof. We shall follow a method due to Grunsky [1]. In view of the definition of $\mathcal{F}_p^{\pi/2}(z_\infty)$ and $'\mathcal{F}_p^0(z_\infty)$, we immediately deduce the functional relations, satisfied along boundary components, of the form

$$\bar{f} = -f + 2\gamma_j \quad (z \in C_j,\ j = 1, \cdots, p)$$

and

$$\bar{\phi} = \phi - 2i\delta_j \quad (z \in C_j;\ j = p+1, \cdots, n),$$

γ_j and δ_j denoting real constants. We may suppose that the basic domain is a bounded one enclosed by regular analytic closed curves; otherwise, it is only necessary to resort to a customary procedure of intermediate auxiliary mappings. The functions f and ϕ being then regular also on the whole boundary, we get, by means of Green's formula,

$$\iint_D |f' - \phi'|^2 d\omega_z$$
$$= \sum_{j=1}^{n} \frac{1}{2i} \int_{C_j} (\bar{f} - \bar{\phi})(f' - \phi')\,dz$$
$$= \sum_{j=1}^{n} \frac{1}{2i} \int_{C_j} (\bar{f}f' + \bar{\phi}\phi' - \bar{f}\phi' - \bar{\phi}f')\,dz,$$

where $d\omega_z$ denotes the areal element $dx\,dy$, $z = x + iy$.

We now estimate the curvilinear integrals in the right-hand side. It is evident that

$$\sum_{j=1}^{n} \frac{1}{2i} \int_{C_j} \bar{f} f' dz \leqq 0;$$

in fact, the left-hand side expresses exactly the negatively computed area of the complementary set of the image of D by the mapping $w = f(z)$. Because of the same reason, f being merely replaced by ϕ, we see that

$$\sum_{j=1}^{n} \frac{1}{2i} \int_{C_j} \bar{\phi} \phi' dz \leqq 0.$$

Since f and ϕ are, of course, one-valued, we get, for $j=1, \cdots, p$,

$$\frac{1}{2i} \int_{C_j} \bar{f} \phi' dz = \frac{1}{2i} \int_{C_j} (-f + 2\gamma_j) \phi' dz$$

$$= -\frac{1}{2i} \int_{C_j} f \phi' dz = \frac{1}{2i} \int_{C_j} \phi f' dz$$

and

$$\frac{1}{2i} \int_{C_j} \bar{\phi} f' dz = \frac{1}{2i} \int_{C_j} \bar{\phi}\, df$$

$$= -\frac{1}{2i} \int_{C_j} \bar{\phi}\, d\bar{f} = \overline{\frac{1}{2i} \int_{C_j} \phi\, df} = \overline{\frac{1}{2i} \int_{C_j} \phi f' dz};$$

we get similarly, for $j = p+1, \cdots, n$,

$$\frac{1}{2i} \int_{C_j} \bar{f} \phi' dz = \frac{1}{2i} \int_{C_j} \bar{f}\, d\phi$$

$$= \frac{1}{2i} \int_{C_j} \bar{f}\, d\bar{\phi} = -\overline{\frac{1}{2i} \int_{C_j} f\, d\phi} = \overline{\frac{1}{2i} \int_{C_j} \phi f' dz}$$

and

$$\frac{1}{2i} \int_{C_j} \bar{\phi} f' dz = \frac{1}{2i} \int_{C_j} (\phi - 2i\delta_j) f' dz$$

$$= \frac{1}{2i} \int_{C_j} \phi f' dz$$

We thus obtain

$$\sum_{j=1}^{n} \frac{1}{2i} \int_{C_j} (\bar{f} \phi' + \bar{\phi} f') dz$$

$$= 2 \mathcal{R} \left(\frac{1}{2i} \sum_{j=1}^{n} \int_{C_j} \phi f' dz \right)$$

By means of residue theorem, we further get supposing $z_\infty \neq \infty$

$$\sum_{j=1}^{n} \int_{C_j} \phi f' dz$$

$$= \sum_{j=1}^{n} \int_{C_j} \left(\frac{1}{z - z_\infty} + a[\phi](z - z_\infty) + \cdots \right) \left(\frac{-1}{(z - z_\infty)^2} + a[f] + \cdots \right) dz$$

$$= 2\pi i \left(a[f] - a[\phi] \right).$$

Hence, we deduce the relation

$$\iint_D |f' - \phi'|^2 d\omega_z \leqq 2\pi \mathcal{R} \left(a[\phi] - a[f] \right),$$

which implies immediately the inequality stated in the theorem. The equality sign there can evidently appear only if $f' \equiv \phi'$, from which the identity $f \equiv \phi$ must follow in view of the assigned case $z_\infty = \infty$ can be treated with an evident modification. The proof has thus been completed.

From the last inequality contained in the above proof yields a more precise result. Namely, we can state the following corollary.

Corollary 1. Under the same assumption as in the Theorem 1, we have

$$\mathcal{R} a[\phi] - \mathcal{R} a[f] \geqq \frac{1}{2\pi} \left(\Omega[f] + \Omega[\phi] \right),$$

where $\Omega[F]$ denotes, in general, the area of complementary set of the image of D by mapping $w = F$.

This corollary is further a generalization of a theorem due to Tsuji [1] stating that the unique function $\phi(z, \infty)$ of $\mathcal{F}^0(\infty) (\equiv \mathcal{F}_n^0(\infty))$ satisfies the inequality

$$\mathcal{R} a[\phi] \geqq \frac{1}{2\pi} \Omega,$$

where Ω denotes the area of complementary set of the basic domain D being supposed to contain the point at infinity; In fact, we may take $f(z, \infty) \equiv z$ in the corollary with $z_\infty = \infty$ and then get

$$a[f] = 0, \quad \Omega[f] = \Omega; \quad \Omega[\phi] = 0$$

Corollary 2. If $\phi_p(z, z_\infty) \in \mathcal{F}_p^{\pi/2}(z_\infty) \frown \mathcal{F}_p^0(z_\infty)$ $(p = 0, 1, \cdots, n)$, then

$$\mathcal{R} a[\phi_{p-1}] \geqq \mathcal{R} a[\phi_p] \quad (p = 1, \cdots, n)$$

Proof. In view of $\phi_{p-1} \in \mathcal{F}_{p-1}^0(z_\infty)$ and $\phi_p \in \mathcal{F}_p^{\pi/2}(z_\infty) \subset \mathcal{F}_{p-1}^{\pi/2}(z_\infty)$, the proposition follows immediately from the theorem.

By making use of the above proved theorem, we can now characterize the function which maps a given n-ply connected domain onto whole plane slit along horizontal and vertical segments, i.e., onto a domain of the type $\mathcal{F}_p^{\pi/2}(z_\infty) \frown \mathcal{F}_p^0(z_\infty)$, by its extremal property which is by itself available for existence proof of such a mapping.

In order to perform the existence proof entirely, it will remain only to give an existence proof in a direct manner concerning the doubly connected domains; namely, the proof of existence theorem in general case can thus be reduced to that in doubly connected case which will be supposed for a while as known.

We now precede the general existence theorem by a lemma stating a special case.

Lemma. Let any n-ply connected domain D in the z-plane be given, the boundary of which is composed of n continua C_j $(j=1,\cdots,n)$. Then, D can be mapped conformally and univalently in such a manner that $n-1$ components C_j $(j=1,\cdots,n-1)$ correspond to vertical slits and the remaining component C_n corresponds to a horizontal slit. Moreover, at an arbitrarily fixed point z_∞ interior to D, the mapping function $w=\phi(z,z_\infty)$ can be subject to a normalization such as

$$\phi(z;z_\infty)=\frac{1}{z-z_\infty}+o(1)\quad(z\to z_\infty)$$

— in case $z_\infty=\infty$, the condition being, of course, replaced by $\phi(z;\infty)=z+o(1)\ (z\to\infty)$ —. The mapping function is uniquely determined by this normalizing condition. In other words, the family $\mathfrak{F}_{n-1}^{\pi/2}(z_\infty)\frown{}'\mathfrak{F}_{n-1}^{0}(z_\infty)$ consists of a unique function.

Proof. We consider a variational problem to minimize the functional $\mathcal{R}a[f]$, any function f belonging to ${}'\mathfrak{F}_{n-1}^{0}(z_\infty)$ being admitted as an argument function. Since the family ${}'\mathfrak{F}_{n-1}^{0}(z_\infty)$ is normal in the Montel's sense and compact, a solution of the problem does surely exist. Let $\phi=\phi(z,z_\infty)$ be a minimizing function, i.e.,

$$\mathcal{R}a[\phi]=\operatorname*{Min}_{f\in{}'\mathfrak{F}_{n-1}^{0}(z_\infty)}\mathcal{R}a[f],\quad \phi\in{}'\mathfrak{F}_{n-1}^{0}(z_\infty).$$

We shall show that also $\phi\in\mathfrak{F}_{n-1}^{\pi/2}(z_\infty)$. For that purpose, we now suppose the contrary, i.e., that ϕ did not belong to $\mathfrak{F}_{n-1}^{\pi/2}(z_\infty)$. Then, the image of at least one among C_j $(j=1,\cdots,n-1)$, C_{n-1} say, by the mapping $w=\phi(z;z_\infty)$ would not be a vertical slit. Let the image of C_j be denoted by Γ_j. We denote by

$$\chi(w)=w+\frac{a[\chi]}{w}+\cdots,$$

expansion being valid around $w=\infty$, the function mapping the doubly connected domain enclosed by two continua Γ_{n-1} and Γ_n univalently in such a manner that these boundary continua correspond to a vertical and a horizontal slit respectively. Here, use is made of the existence in doubly connected case! Then, in view of Theorem 1 — $n, z_\infty, z; f, \phi$ in the theorem being replaced by $2,\infty,w,\chi,w$, respectively —, we get

$$\mathcal{R}a[\chi]<\mathcal{R}a[w]=0.$$

the equality sign in the last inequality being excluded because of $\chi(w)\not\equiv w$. It is evident that $\chi(\phi(z;z_\infty))\in{}'\mathfrak{F}_{n-1}^{0}(z_\infty)$ while we get

$$\mathcal{R}a[\chi(\phi)]=\mathcal{R}a[\chi]+\mathcal{R}a[\phi]<\mathcal{R}a[\phi]$$

which contradicts to the extremality of ϕ. Thus, we must really have $\phi\in\mathfrak{F}_{n-1}^{\pi/2}(z_\infty)$ and hence $\phi\in\mathfrak{F}_{n-1}^{\pi/2}(z_\infty)\frown{}'\mathfrak{F}_{n-1}^{0}(z_\infty)$.

Next, in order to show the uniqueness of the mapping function, we denote by $\phi^*(z;z_\infty)$ any function belonging to $\mathfrak{F}_{n-1}^{\pi/2}(z_\infty)\frown{}'\mathfrak{F}_{n-1}^{0}(z_\infty)$. The difference $\phi^*-\phi$ is then regular and bounded throughout D and possesses constant real parts along C_j $(j=1,\cdots,n-1)$ and a constant imaginary part along C_n. Hence, we must have

$$\phi^*-\phi\equiv[\phi^*-\phi]^{z=z_\infty}=0.$$

Cf. also the uniqueness proof for Theorem 2 stated below.

We are now in position to state a general theorem on existence as well as uniqueness of the function mapping a given domain onto whole plane slit along perpendicular segments.

Theorem 2. Any n-ply connected domain D bounded by n continua C_j $(j=1,\cdots,n)$ can be mapped conformally and univalently onto whole plane slit along horizontal and vertical segments in such a manner that its p boundary components C_j $(j\leq p)$ correspond to vertical slits and the remaining $n-p$ components C_j $(j>p)$ correspond to horizontal slits. Moreover, under the normalizing condition at a fixed point z_∞ interior to D, the mapping is uniquely determined. In other words, the family $\mathfrak{F}_p^{\pi/2}(z_\infty)\frown{}'\mathfrak{F}_p^{0}(z_\infty)$ for any p with $0\leq p\leq n$ consists of a uniquely determinate function.

Proof. The theorem is well-known in case $p=0$ or $p=n$ as de Possel's one and shown in the lemma also in case $p=n-1$. We may suppose $p>0$. The family $\mathfrak{f}_p^{\pi/2}(z_\infty)$ being normal and compact, the variational problem

$$\mathcal{R}a[\phi]=\max_{f\in \mathfrak{f}_p^{\pi/2}(z_\infty)} \mathcal{R}a[f], \quad \phi\in \mathfrak{f}_p^{\pi/2}(z_\infty)$$

possesses surely a solution $\phi = \phi(z;z_\infty)$. In order to show that also $\phi \in {}'\mathfrak{f}_p^{0}(z_\infty)$, we suppose the contrary. If, for instance, Γ_{p+1} were not a horizontal slit, then it is possible, based upon the preceding lemma, to map the $(p+1)$-ply connected domain enclosed merely by Γ_j $(j=1, \cdots, p, p+1)$ univalently in such a manner that the p continua Γ_j $(j \leqq p)$ corresponds to a horizontal slit, the mapping function $\chi(w)$ being supposed to be normalized at $w=\infty$. Since $\chi(w) \not\equiv w$, it follows, in view of Theorem 1, that

$$0=\mathcal{R}a[w]<\mathcal{R}a[\chi]$$

and consequently, for a function $\chi(\phi(z;z_\infty)) \in \mathfrak{f}_p^{\pi/2}(z_\infty)$,

$$\mathcal{R}a[\chi(\phi)]=\mathcal{R}a[\chi]+\mathcal{R}a[\phi]>\mathcal{R}a[\phi].$$

This contradicts to the maximizing character of ϕ. Thus, it is asserted that $\phi \in {}'\mathfrak{f}_p^{0}(z_\infty)$ and hence $\phi\in \mathfrak{f}_p^{\pi/2}(z_\infty) \frown {}'\mathfrak{f}_p^{0}(z_\infty)$.

We next prove the uniqueness of $\phi(z;z_\infty)$. Let $\phi^*(z;z_\infty)$ be also a function belonging to $\mathfrak{f}_p^{\pi/2}(z_\infty) \frown {}'\mathfrak{f}_p^{0}(z_\infty)$. Then, by means of Theorem 1, we get

$$\mathcal{R}a[\phi] \leqq \mathcal{R}a[\phi^*]$$

and

$$\mathcal{R}a[\phi^*] \leqq \mathcal{R}a[\phi]$$

and hence the equality $\mathcal{R}a[\phi^*] = \mathcal{R}a[\phi]$. Therefore, again in view of Theorem 1, we assert.

$$\phi^* \equiv \phi,$$

the desired result.

We have hitherto considered the families $\mathfrak{f}_p^{\alpha}(z_\infty)$ and ${}'\mathfrak{f}_p^{\alpha}(z_\infty)$ merely for special values of α, i.e., for $\alpha=\pi/2$ and $\alpha=0$, and entered upon the discussion of existence of a non-empty family $\mathfrak{f}_p^{\pi/2}(z_\infty) \frown {}'\mathfrak{f}_p^{0}(z_\infty)$. But, by means of a quite similar procedure, the result can be modified in a somewhat general form. For instance, corresponding to Theorem 1, the following proposition will be verified.

Theorem 3. Let α and β be any real constants, and let further $f(z;z_\infty)\in \mathfrak{f}_p^{\alpha}(z_\infty)$ and $\phi(z;z_\infty)\in {}'\mathfrak{f}_p^{\beta}(z_\infty)$. Then

$$-\mathcal{R}\left(e^{-2i\alpha}a[f]\right) \leqq \mathcal{R}\left(e^{-2i\beta}a[\phi]\right);$$

the equality sign is valid only if $f\equiv\phi$.

The Theorem 2 is generalized in a corresponding manner, stated as follows.

Theorem 4. The family $\mathfrak{f}_p^{\alpha}(z_\infty) \frown {}'\mathfrak{f}_p^{\beta}(z_\infty)$, for every set of possible values of p, α and β, consists of a unique function.

The results obtained in the present section will further be generalized in a following manner. Let α_k $(k=1,\cdots,k,\ k\leqq n)$ be any real number. Then, the problem establishing the existence of mapping of an n-ply connected domain D onto whole plane slit along segments with k gradients in such a manner that, among n boundary components C_j, the assigned p_k $(k=1,\cdots,k;\ \sum p_k = n)$ components correspond to segments with gradient α_k can be reduced to the problem in k-ply connected case, i.e., the problem establishing the existence of mapping of a k-ply connected domain onto whole plane slit along k segments with gradients α_k $(k=1,\cdots,k)$. The uniqueness proof is easy.

2. Whole plane slit along radial segments as well as circular arcs.

We consider again a domain D of the same character as in the preceding section and denote by Δ, in general, its conformal univalent image. Further, let the boundary components of D be denoted by C_j $(j=1,\cdots,n)$ and those of Δ by Γ_j $(j=1,\cdots,n)$, respectively.

We now denote by R_p the family consisting of all n-ply connected domains Δ whose p boundary components Γ_j $(j=1,\cdots,p)$

lie on radial half-lines $\arg w = c_j$ respectively, and by $'R_p$ the family consisting of all n-ply connected domains Δ whose $n-p$ boundary components Γ_j ($j = p+1, \cdots, n$) lie on radial half-lines $\arg w = c_j$ respectively.

We further introduce the families K_p and $'K_p$ similarly by taking the concentric circles $|w| = c_j$ instead of the radial half-lines $\arg w = c_j$ in case of R_p and $'R_p$, respectively.

Here also p is supposed to be any integer such that $0 \leq p \leq n$. In particular, $R_0 = 'R_n = K_0 = 'K_n$ is regarded as the family of all univalent images of D. It may also be noticed that we may suppose without loss of generality all the boundary components C_j to be continua but not isolated points.

Let z_0 and z_∞ be two different points interior to D, being arbitrarily fixed. Suppose that the functions $f(z)$ univalent in D are normalized by the conditions

$$f(z_0) = 0, \quad f(z) - \frac{1}{z - z_\infty} = O(1) \quad (z \to z_\infty)$$

In case $z_\infty = \infty$, the second condition must be replaced by a modified one, namely

$$f(z) - z = O(1) \quad (z \to \infty).$$

We then denote by $\mathcal{R}_p(z_0, z_\infty)$, $'\mathcal{R}_p(z_0, z_\infty)$, $\mathfrak{K}_p(z_0, z_\infty)$ and $'\mathfrak{K}_p(z_0, z_\infty)$ the families consisting of all normalized functions which map D univalently onto domains of R_p, $'R_p$, K_p and $'K_p$, respectively.

Evidently, neither of those families is empty. In particular, the family $\mathcal{R}_0(z_0, z_\infty) = '\mathcal{R}_n(z_0, z_\infty) = \mathfrak{K}_0(z_0, z_\infty) = '\mathfrak{K}_n(z_0, z_\infty)$ consists of all normalized functions univalent in D. It is also a well-known fact that each of the families $\mathcal{R}_n(z_0, z_\infty) = '\mathcal{R}_0(z_0, z_\infty)$ and $\mathfrak{K}_n(z_0, z_\infty) = '\mathfrak{K}_0(z_0, z_\infty)$ consists of a unique function mapping D, under the prescribed normalizing conditions at z_0 and z_∞ onto whole plane slit along radial segments or circular arcs alone, respectively; cf. Rengel [2].

Theorems concerning extremality on distortion of the derivatives of the last mentioned mapping functions at z_0, due to Grötzsch [1,2] and Rengel [1], are well-known. They now can be generalized to a fundamental distortion theorem stated in the following form.

Theorem 1. If $f(z; z_0, z_\infty) \in \mathcal{R}_p(z_0, z_\infty)$ and $\phi(z; z_0, z_\infty) \in '\mathfrak{K}_p(z_0, z_\infty)$, then

$$|f'(z_0; z_0, z_\infty)| \leq |\phi'(z_0; z_0, z_\infty)|;$$

the equality is valid only if $f \equiv \phi$.

Proof. We shall follow a method due to Rengel [1]. We consider an annulus $r < |w| < R$ containing the whole boundary of the image of D by the mapping $w = \phi(z; z_0, z_\infty)$. We then denote by

$$q(r)r < |\omega| < Q(R)R$$

the smallest annulus which contains the doubly-connected ring domain enclosed by the image curves of $|w| = r$ and $|w| = R$ by the composed mapping $\omega = f(\phi^{-1}(w; z_0, z_\infty); z_0, z_\infty)$. It is easily seen that

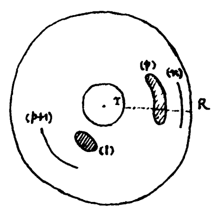
$$q(r) \to \left| \frac{f'(z_0; z_0, z_\infty)}{\phi'(z_0; z_0, z_\infty)} \right| \quad (r \to +0)$$

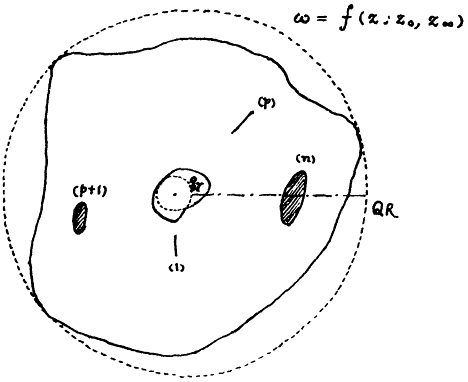
and

$$Q(R) \to 1 \qquad (R \to \infty).$$

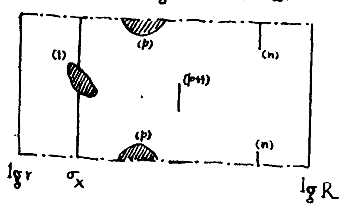
We now observe the parts of the images of D by the mappings $w = \phi$ and $\omega = f$ contained in the annuli $r < |w| < R$ and $qr < |\omega| < QR$, respectively. We cut these parts along positive real axis and then map the thus obtained domains — eventually pieces consisting of some domains — by the principal branch of logarithm:

$w = \phi(z, z_0, z_\infty)$

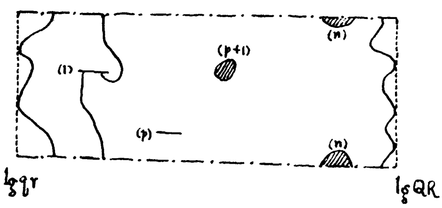




$Z = \lg \phi(z; z_0, z_\infty)$



$W = \lg f(z; z_0, z_\infty)$



$$Z = X + iY = \lg w, \qquad W = U + iV = \lg \omega,$$

respectively. The part G lying inside the rectangle $\lg r < X < \lg R$, $0 < Y < 2\pi$ which is originated from D is mapped by

$$W = \lg f(\phi^{-1}(\exp Z; z_0, z_\infty); z_0, z_\infty)$$

univalently onto a part contained in the rectangle $\lg qr < U < \lg QR$, $0 < V < 2\pi$, whence it follows immediately the inequality

$$\iint_G \left|\frac{dW}{dZ}\right|^2 d\omega_Z \leqq 2\pi \lg \frac{QR}{qr}.$$

We now consider in the Z-plane a segment or eventually some segments, σ_X say, lying on a vertical line with abscissa X ($\lg r < X < \lg R$) and inside the above rectangle. Then, the image of such a segment or segments has a total length not less than 2π, except a finite number of σ_X with abscissas which coincide with those of vertical lines bearing the slits originated from circular slits in the w-plane. Moreover, there exists an X-interval of a length a for any X of which the length of W-image of σ_X is always greater than $2\pi + c$, provided $f \not\equiv \phi$; a and c being certain fixed positive numbers. In fact, otherwise, it is easily seen that dW/dZ would remain real in a subdomain and hence, in view of the assigned normalizing conditions, $W = Z$ which would imply $f \equiv \phi$. By making use of Schwarz's inequality, we get

$$\begin{aligned}
&2\pi \iint_G \left|\frac{dW}{dZ}\right|^2 d\omega_Z \\
&= \int_{\lg r}^{\lg R} dX \cdot 2\pi \int_{\sigma_X} \left|\frac{dW}{dZ}\right|^2 dY \\
&\geqq \int_{\lg r}^{\lg R} dX \cdot \int_{\sigma_X} dY \int_{\sigma_X} \left|\frac{dW}{dZ}\right|^2 dY \\
&\geqq \int_{\lg r}^{\lg R} dX \left(\int_{\sigma_X} \left|\frac{dW}{dZ}\right| dY\right)^2 \\
&\geqq \left(\lg \frac{R}{r} - a\right)(2\pi)^2 + a(2\pi + c)^2 \\
&> (2\pi)^2 \lg \frac{R}{r} + 4\pi a c.
\end{aligned}$$

We therefore obtain the inequality

$$2\pi \cdot 2\pi \lg \frac{QR}{qr} > (2\pi)^2 \lg \frac{R}{r} + 4\pi a c,$$

namely

$$\lg \frac{Q(R)}{q(r)} > \frac{ac}{\pi}.$$

Let now r and R tend to $+0$ and ∞, respectively. Since the quantities a and c can be taken as fixed ones, this limit process implies

$$\lg \left|\frac{\phi'(z_0, z_0, z_\infty)}{f'(z_0, z_0, z_\infty)}\right| \geqq \frac{ac}{\pi} > 0.$$

We thus assert that the inequality stated in the theorem holds good and further in the strict sense unless $f \not\equiv \phi$.

The just proved theorem can also be stated in an equivalent form as follows.

Theorem 1a. If $g(z; z_0, z_\infty)$

$\in \mathfrak{S}_p(z_0, z_\infty)$ and $\psi(z; z_0, z_\infty) \in {}'\mathcal{R}_p(z_0, z_\infty)$, then

$$|g'(z_0; z_0, z_\infty)| \geqq |\psi'(z_0; z_0, z_\infty)|;$$

the equality is valid only if $g \equiv \psi$.

Corollary 1. Under the same assumption as in the theorem, we have

$$|f'(z_0; z_0, z_\infty)|$$

$$\leqq |\phi'(z_0; z_0, z_\infty)| \exp\left(-\frac{1}{2\pi}\Omega[\lg f]\right),$$

where $\Omega[\lg f]$ denotes the logarithmic area of the complement of the image of D by the mapping $w = f$.

Corollary 2. If $\phi_p(z; z_0, z_\infty) \in \mathcal{R}_p(z_0, z_\infty) \cap {}'\mathcal{R}_p(z_0, z_\infty)$ $(p = 0, 1, \cdots, n)$, then

$$|\phi'_{p-1}(z_0; z_0, z_\infty)| \geqq |\phi'_p(z_0; z_0, z_\infty)|$$

$$(p = 1, \cdots, n).$$

Corollary 2a. If $\psi_p(z; z_0, z_\infty) \in \mathfrak{S}_p(z_0, z_\infty) \cap {}'\mathcal{R}_p(z_0, z_\infty)$ $(p = 0, 1, \cdots, n)$, then

$$|\psi'_{p-1}(z_0, z_0, z_\infty)| \leqq |\psi'_p(z_0; z_0, z_\infty)|$$

$$(p = 1, \cdots, n).$$

The distortion theorem having thus been established, the arguments quite similar to those in the preceding section are here also valid in order to prove the existence of a mapping onto whole plane slit along radial segments and circular arcs together, i.e., onto a domain of the type $\mathcal{R}_p(z_0, z_\infty) \cap {}'\mathfrak{S}_p(z_0, z_\infty)$. Here the existence proof in general case can also be reduced to that in doubly connected case.

It will be almost unnecessary to describe the procedure of proof again in detail. We state here merely the corresponding results.

Lemma. The family $\mathcal{R}_{n-1}(z_0, z_\infty) \cap {}'\mathfrak{S}_{n-1}(z_0, z_\infty)$ consists of a unique function.

Theorem 2. The family $\mathcal{R}_p(z_0, z_\infty) \cap {}'\mathfrak{S}_p(z_0, z_\infty)$ for any p with $0 \leqq p \leqq n$ consists of a unique function.

The fact stated in the lemma expresses, of course, a special case of that in the theorem.

In the present section we have hitherto discussed merely the case of whole plane slit along radial segments and circular arcs. The discussion for cases of a circular disc or an annulus, instead of whole plane, cut along such slits can also take place in quite similar manners. Then, a slight modification will be necessary concerning normalization.

As normalizing conditions, we may take in case of a circular disc $|w| < R$:

$$z_0 \in D, \quad f(z_0) = 0;$$

$$|f(z)| < R \ (z \in D), \ |f(z)| = R \ (z \in C_1);$$

$$\arg f'(z_0) = 0 \ \left(\text{or} \ \ z_1 \in C_1, \ f(z_1) = R\right);$$

and in case of an annulus $r < |w| < R$.

$$r < |f(z)| < R \ (z \in D);$$

$$|f(z)| = R \ (z \in C_1), \ |f(z)| = r \ (z \in C_2);$$

$$z_1 \in C_1, \ f(z_1) = R \ \left(\text{or} \ z_2 \in C_2, \ f(z_2) = r\right).$$

In these cases, the general existence problems can completely be proved out provided that the problem concerning domain of connectivity 3 or 4 respectively has been done. But, if an argument due to Grötzsch [4] is taken into account, the problems in general cases can both be further reduced to that discussed in the present section, i.e., that concerning doubly connected case.

On the other hand, if the problem on a circular disc slit along radial segments and circular arcs has been worked out, those on whole plane and an annulus can then be obtained by usual procedure of constructing suitable quotients. With respect to a result on circular disc corresponding to corollary 2 of Theorem 1, cf. Bergman [1], p. D 35.

Similar results can also be obtained with regard to the problem where one of radial slits is replaced by a segment or a half-line starting from a finite point not coincident with the origin and reaching the origin or the point at infinity or by a half-line starting from the origin and reaching the point at infinity. We further get, in particular, the mapping onto a slit parallel strip if we combine the mapping by logarithmic function with the last mentioned one. Such a mapping will be discussed in detail in the next section; cf. also Ozawa [1,2].

On the other hand, Grunsky [1] as well as Koebe [8] considered, instead of radial or circular slits, also the slits lying on a system of logarithmic spirals of a given inclination which have the origin and the point at infinity as common asymptotic points. The latter may be regarded as a generalization of the former. In fact, such a system of logarithmic spirals is expressed by an equation of the form

$$\arg w - \alpha \lg |w| = c,$$

where α denotes the inclination, i.e., the tangent (gradient) of the constant angle between the spirals of the system and radius vectors centred at the origin and c denotes the constant specifying a spiral of the family. The spirals will reduce to half-lines or circles centred at the origin according to a specialization $\alpha = 0$ or $\alpha = \infty$, respectively.

Now, making use of a method due to Grunsky, the problem of mapping a given domain onto whole plane slit along arcs of logarithmic spirals of two systems with assigned inclinations orthogonal each other can easily be solved by combining the mappings considered in the present section. We can indeed state the following theorem, which has already been proved by Koebe [8] in a more general form but by a quite different way.

Theorem 3. Any n-ply connected domain D bounded by n continua $C_j\ (j = 1, \cdots, n)$ can be mapped conformally and univalently onto whole plane slit along arcs of logarithmic spirals of two systems in such a manner that its p boundary components $C_j\ (j \leqq p)$ correspond to slits of a system with an assigned inclination α and the remaining $n-p$ components $C_j\ (j > p)$ correspond to slits of another system orthogonal to the former, i.e., with the inclination $-1/\alpha$. Moreover, under the habitual normalizing conditions at fixed points z_0 and z_∞ interior to D, the mapping is uniquely determinate.

Proof. The method which has been used by Grunsky to prove an extreme case $p = 0$ or an equivalent case $p = n$ i.e., the case where spirals of one system alone are concerned, is valid with few modifications also for general case $0 \leqq p \leqq n$. Namely, for any given p, making use of the uniquely determinate functions

$$\phi_p(z, z_0, z_\infty) \in \mathcal{R}_p(z_0, z_\infty) \cap {}'\!\mathfrak{S}_p(z_0, z_\infty),$$

$$\psi_p(z, z_0, z_\infty) \in \mathfrak{S}_p(z_0, z_\infty) \cap {}'\!\mathcal{R}_p(z_0, z_\infty),$$

we now introduce two functions P_p and Q_p defined by

$$\begin{aligned} &P_p(z; z_0, z_\infty) \\ &= \phi_p(z, z_0, z_\infty)^{1/2}\, \psi_p(z; z_0, z_\infty)^{1/2}, \\ &Q_p(z; z_0, z_\infty) \\ &= \phi_p(z, z_0, z_\infty)^{1/2}\, \psi_p(z; z_0, z_\infty)^{-1/2}; \end{aligned}$$

the branches of square roots in the right-hand sides being determined in such a manner that P_p satisfies the same normalizing conditions at z_0 and z_∞ as ϕ_p or ψ_p and further that Q_p attains the value 1 at z_∞. It is evidently seen that P_p and Q_p are both one-valued in D, that P_p possesses a zero point and a pole only at z_0 and z_∞, respectively, both being of the first order, and that Q_p possesses neither zero point nor pole. By inverting the defining equations for P_p and Q_p, we immediately have

$$\phi_p = P_p Q_p, \qquad \psi_p = P_p Q_p^{-1}.$$

Now, since $\arg \phi_p$ and $\lg |\psi_p|$ remain constant along each of $C_j\ (j \leqq p)$, we get the relations of the form

$$\begin{aligned} c_j &= \arg P_p + \arg Q_p, \\ d_j &= \lg |P_p| - \lg |Q_p|; \end{aligned}$$

$$\lg Q_p = \overline{\lg P_p} - i\gamma_j \quad (\gamma_j \equiv c_j + i d_j)$$
$$(z \in C_j;\quad j \leqq p).$$

Similarly, since $\lg |\phi_p|$ and $\arg \psi_p$ remain constant along each of $C_j\ (j > p)$, we further get the relations of the form

$$\begin{aligned} c_j &= \lg |P_p| + \lg |Q_p|, \\ d_j &= \arg P_p - \arg Q_p; \end{aligned}$$

$$\lg Q_p = -\overline{\lg P_p} - \gamma_j \quad (\gamma_j \equiv c_j + i d_j)$$
$$(z \in C_j,\quad j > p)$$

We shall then show that the desired mapping function f is given by the relation

$$f(z, z_0, z_\infty) = P_p(z; z_0, z_\infty) Q_p(z; z_0, z_\infty)^k,$$
where the constant k is defined as

$$k = e^{2i\kappa}, \qquad \alpha = \tan\kappa$$

and Q_p^k denotes the branch taking the value 1 at z_∞.

It is evident that the so defined function f satisfies the assigned normalizing conditions at z_0 and z_∞. Its behavior on the boundary is as follows. For any $z \in C_j$ $(j \leq p)$,

$$\begin{aligned} \lg f &= \lg P_p + k \lg Q_p \\ &= \lg P_p + e^{2i\kappa}\overline{\lg P_p} - i\gamma_j e^{2i\kappa} \\ &= 2e^{i\kappa}\mathcal{R}(e^{-i\kappa}\lg P_p) - i\gamma_j e^{2i\kappa} \end{aligned}$$

and hence

$$\begin{aligned} &\arg f - \alpha \lg|f| \\ &= 2\sin\kappa \cdot \mathcal{R}(e^{-i\kappa}\lg P_p) - \mathcal{J}(i\gamma_j e^{2i\kappa}) \\ &-\tan\kappa\left(2\cos\kappa \cdot \mathcal{R}(e^{-i\kappa}\lg P_p) - \mathcal{R}(i\gamma_j e^{2i\kappa})\right) \\ &= -\sec\kappa \cdot \mathcal{R}(\gamma_j e^{i\kappa}); \end{aligned}$$

that is, the image of each C_j $(j \leq p)$ lies on a logarithmic spiral of inclination α. Similarly, for any $z \in C_j$ $(j > p)$,

$$\begin{aligned} \lg f &= \lg P_p + k \lg Q_p \\ &= \lg P_p - e^{2i\kappa}\overline{\lg P_p} - \gamma_j e^{2i\kappa} \\ &= 2ie^{i\kappa}\mathcal{J}(e^{-i\kappa}\lg P_p) - \gamma_j e^{2i\kappa} \end{aligned}$$

and hence

$$\begin{aligned} &\arg f + \frac{1}{\alpha}\lg|f| \\ &= 2\cos\kappa \cdot \mathcal{J}(e^{-i\kappa}\lg P_p) - \mathcal{J}(\gamma_j e^{2i\kappa}) \\ &+ \cot\kappa\left(-2\sin\kappa\ \mathcal{J}(e^{-i\kappa}\lg P_p) - \mathcal{R}(\gamma_j e^{2i\kappa})\right) \\ &= -\operatorname{cosec}\kappa \cdot \mathcal{R}(\gamma_j e^{i\kappa}); \end{aligned}$$

that is, the image of each C_j $(j > p)$ lies on a logarithmic spiral of inclination $-1/\alpha$ which is orthogonal to one of inclination α.

Since the function Q_p, as already mentioned, possesses neither zero point nor pole, it is obvious that the function $f = P_p Q_p^k$ is, like P_p, schlicht in respective neighborhoods of the points z_0 and z_∞. Hence, in view of the behavior of f on the boundary of D, we conclude that the image of D by the mapping $w = f(z; z_0, z_\infty)$ covers the whole plane just once except arcs of logarithmic spirals in question; that is, the function f maps D univalently onto whole plane slit along arcs of logarithmic spirals of two systems in the desired manner.

The uniqueness of the mapping may be shown as follows. In fact, let f^* be any function having the same properties as f with respect to the mapping character. Then, the quantities

$$\arg\frac{f^*}{f} - \alpha\lg\left|\frac{f^*}{f}\right| \equiv \arg f^* - \alpha\lg|f^*| - (\arg f - \alpha\lg|f|)$$

and

$$\arg\frac{f^*}{f} + \frac{1}{\alpha}\lg\left|\frac{f^*}{f}\right| \equiv \arg f^* + \frac{1}{\alpha}\lg|f^*| - \left(\arg f + \frac{1}{\alpha}\lg|f|\right)$$

remain constant along any C_j $(j \leq p)$ and any C_j $(j > p)$, respectively. Since the quotient f^*/f neither vanishes nor becomes infinite, we see from a quite similar reason as above that it reduces to a constant. Based upon the normalization at z_∞, f^* must coincide identically with f. Thus, the theorem has completely been proved.

3. Parallel strip slit along perpendicular segments.

We again consider an n-ply connected domain D possessing n continua C_j $(j = 1, \cdots, n)$ as boundary components. With regard to its univalent image Δ with corresponding boundary components Γ_j, we now introduce following notations.

We denote by S_p the family consisting of all such domains that Γ_1 is composed of two parallel lines $\mathcal{J}w = \pm\pi/2$, $-\infty < \mathcal{R}w < +\infty$ and the Γ_j $(j = 2, \cdots, p)$ are vertical segments contained in the strip $|\mathcal{J}w| < \pi/2$, and similarly by $'S_p$ the family consisting of all such domains Δ that Γ_1 is the same as above and Γ_j ($j = p+1, \cdots, n$) are vertical segments contained in the strip $|\mathfrak{J}w| < \pi/2$.

We further define the families T_p and $'T_p$ similarly by taking horizontal segments instead of vertical ones in cases of S_p and $'S_p$, respectively.

Here p is supposed to be an integer such that $1 \leq p \leq n$. In particular, $S_1 = 'S_n = T_1 = 'T_n$ is regarded as the family consisting of all univalent images of D contained in the strip $|\mathfrak{J}w| < \pi/2$ which is bounded by Γ_1 alone.

Let z_∞ and z_∞^- be any fixed different boundary elements lying on C_1. Suppose that the functions $f(z)$ univalent in D be normalized by the conditions

$$\lim_{z\to z_\infty} \mathfrak{R}f(z) = +\infty, \qquad \lim_{z\to z_\infty^-} \mathfrak{R}f(z) = -\infty.$$

We then denote by $\mathfrak{T}_p(z_\infty, z_\infty^-)$, $'\mathfrak{T}_p(z_\infty, z_\infty^-)$, $\mathfrak{I}_p(z_\infty, z_\infty^-)$ and $'\mathfrak{I}_p(z_\infty, z_\infty^-)$ the families of normalized functions which map D onto domains of S_p, $'S_p$, T_p and $'T_p$, respectively.

We now observe the simply connected domain bounded by C_1 alone and containing D in its interior. We then map it onto the parallel strip $|\mathfrak{J}w| < \pi/2$ in such a manner that z_∞ and z_∞^- correspond to $+\infty \equiv +\infty + i0$ and $-\infty \equiv -\infty + i0$, respectively, the mapping being determined uniquely except a translation parallel to the real axis. If this mapping function is restricted into the basic domain D, it belongs to $\mathfrak{T}_1(z_\infty, z_\infty^-) (= '\mathfrak{T}_n(z_\infty, z_\infty^-) = \mathfrak{I}_1(z_\infty, z_\infty^-) = '\mathfrak{I}_n(z_\infty, z_\infty^-))$

Because of the just noticed fact, we may suppose, for the sake of brevity, that the given domain D itself is of the type S_1, i.e., a sub-domain of the strip $|\mathfrak{J}z| < \pi/2$ among whose boundary components C_1 coincides with $\mathfrak{J}z = \pm\pi/2, -\infty < \mathfrak{R}z < +\infty$ and the remaining C_j ($j = 2, \cdots, n$) are contained in the strip. Accordingly, we take $z_\infty = +\infty$ and $z_\infty^- = -\infty$, and we shall write merely $\mathfrak{T}_p$ etc. instead of $\mathfrak{T}_p(+\infty, -\infty)$ etc.

We first prepare a lemma.

Lemma 1. In a domain D of the just mentioned type, any function $w = f(z)$ belonging to $\mathfrak{T}_1$ satisfies asymptotic relations expressed by

$$f(z) = z + b_\pm[f] + o(1)$$
$$(z \in D, \quad \mathfrak{R}z \to \pm\infty);$$

$b_\pm[f]$ being real constants.

Proof. We put $Z = e^z$ and $W = e^w$. Then, the function defined by

$$W = F(Z) \equiv \exp f(\lg Z),$$

the logarithm denoting its principal branch, is regular and univalent in the domain e^D obtained from D by $Z = e^z$. In view of inversion principle, $F(Z)$ remains analytic also in the domain containing 0 and ∞ as interior points which is bounded by the image curves e^{C_j}. Moreover, we have

$$F(0) = 0, \qquad F(\infty) = \infty,$$

and the orders of zero point $Z = 0$ and of pole $Z = \infty$ are both equal to 1. Since by the mapping $W = F(Z)$ the positive imaginary axes correspond each other, the derivatives $F'(0)$ and $F'(\infty)$ must both be real and positive. Hence, putting

$$F'(0) = e^{b_-}, \qquad F'(\infty) = e^{b_+},$$

both quantities $b_\pm \equiv b_\pm[f]$ are also real. On the other hand, we have

$$F'(0) = \left[\frac{de^w}{de^z}\right]^{e^z=0} = \lim_{z\to-\infty} e^{w-z}$$
$$(w = f(z)),$$

and hence, for $\mathfrak{R}z \to -\infty$

$$e^{w-z} = e^{b_-} + o(1),$$

yielding an asymptotic relation

$$f(z) - z = b_-[f] + o(1).$$

In a similar way, we get, for $\mathfrak{R}z \to +\infty$,

$$f(z) - z = b_+[f] + o(1).$$

As immediately seen from the above mentioned proof, more precise asymptotic relations

$$f(z) = z + b_\pm[f] + o(e^{\pm\mathfrak{R}z})$$
$$(\mathfrak{R}z \to \pm\infty)$$

may be derived. Remembering further the analytic continuability across the boundary component C_1, we see that the last limit relations remain to hold, for each f, uniformly in $|\mathcal{J}z| \leqq \pi/2$ as $\mathcal{R}z \to \pm\infty$.

We now introduce a quantity defined by

$$\beta[f] = \ell_+[f] - \ell_-[f]$$

$$\equiv \lim_{z\to+\infty} (f(z) - f(-z) - 2z),$$

f being any function of $\mathcal{T}_1$. The fundamental distortion theorem can then be stated as follows.

Theorem 1. If $f(z) \in \mathcal{T}_p$ and $\phi(z) \in {}'\mathcal{T}_p$, then

$$\beta[f] \leqq \beta[\phi],$$

the equality is valid only if $f \equiv \phi + c$, c being a real constant.

Proof. By means of the transformations $Z = \exp\phi(z)$ and $W = \exp f(z)$ followed by the inversions with respect to the imaginary axes of Z- and W-planes, based upon the inversion principle, the function defined by

$$W = F(Z) \equiv \exp f(\phi^{-1}(\lg Z))$$

can be regarded as the one mapping the $2(n-1)$-ply connected domain which is bounded by $n-1$ continua in the Z-plane originated from C_j $(j = 2, \cdots, n)$ and their inverses with respect to the imaginary axis onto the domain in the W-plane which is obtained in a similar manner. In view of $f \in \mathcal{T}_p$, the boundary continua in the Z-plane originated from C_j $(j = 2, \cdots, p)$ as well as their inverses with respect to the imaginary axis are all radial slits centred at the origin. On the other hand, in view of $\phi \in {}'\mathcal{T}_p$ the boundary continua in the W-plane originated from C_j $(j = p+1, \cdots, n)$ as well as their inverses with respect to the imaginary axis are all circular slits around the origin. Hence, by Theorem 1 of §2 —— taking $2(n-1)$; Z, 0, ∞; Z, $F(Z)/F'(\infty)$ instead of n; z, z_0, z_∞; f, ϕ there, respectively ——, we get

$$1 \leqq \left| \frac{F'(0)}{F'(\infty)} \right|;$$

$F'(0)$ and $F'(\infty)$ are really both real quantities. The equality in the last inequality is valid only if

$$F(Z)/F'(\infty) \equiv Z.$$

Now, the expansion of $F(Z)$ around $Z = 0$ becomes

$$F(Z) = \exp(\phi^{-1}(\lg Z) + \ell_-[f] + o(1))$$

$$= \exp(\lg Z - \ell_-[\phi] + o(1) + \ell_-[f] + o(1))$$

$$= Z\exp(\ell_-[f] - \ell_-[\phi] + o(1)) \quad (Z \to 0)$$

and that around $Z = \infty$ becomes

$$F(Z) = \exp(\phi^{-1}(\lg Z) + \ell_+[f] + o(1))$$

$$= \exp(\lg Z - \ell_+[\phi] + o(1) + \ell_+[f] + o(1))$$

$$= Z\exp(\ell_+[f] - \ell_+[\phi] + o(1)) \quad (Z \to \infty).$$

We thus have

$$F'(0) = \exp(\ell_-[f] - \ell_-[\phi])$$

and

$$F'(\infty) = \exp(\ell_+[f] - \ell_+[\phi]).$$

Consequently, the above inequality $1 \leqq |F'(0)/F'(\infty)|$ implies

$$\ell_+[f] - \ell_+[\phi] \leqq \ell_-[f] - \ell_-[\phi],$$

whence the desired result $\beta[f] \leqq \beta[\phi]$.

The equality sign can appear, as noticed above, only if $F(Z) \equiv F'(\infty)Z$. We then get in turn

$$\exp f(\phi^{-1}(\lg Z)) \equiv F'(\infty)Z,$$

$$f(\phi^{-1}(\lg Z)) \equiv \lg Z + \lg F'(\infty);$$

$$f(z) \equiv \phi(z) + \lg F'(\infty);$$

here $\lg F'(\infty) = \ell_+[f] - \ell_+[\phi]$ being a real constant.

The result just proved can also be stated in an equivalent form as follows.

Theorem 1a. If $g(z) \in \mathcal{T}_p$ and $\psi(z) \in {}'\mathcal{T}_p$, then

$$\beta[g] \geqq \beta[\psi];$$

the equality is valid only if $g \equiv \psi + d$, d being a real constant.

Corollary 1. Under the same assumption as in Theorem 1, we have more precisely

$$\beta[\phi] - \beta[f] \geqq \frac{1}{\pi}\Omega[f],$$

$\Omega[f]$ denoting the area of the part, contained in the strip $|\Im w| < \pi/2$, of complement of the image of D by $w = f(z)$.

Proof. We consider the image of D by $Z = \exp\phi(z)$ and its inverse with respect to imaginary axis. By means of corollary 1 of Theorem 1 of § 2, the union of these domains are mapped by $W = F(Z)/F'(\infty)$ onto a set whose complement has a logarithmic area $\Omega[\lg(F/F'(\infty))]$ satisfying

$$\exp\left(-\frac{1}{2\pi}\Omega[\lg(F/F'(\infty))]\right)$$

$$\geqq 1\Big/\left|\frac{F'(0)}{F'(\infty)}\right| = \exp(\beta[f] - \beta[\phi]),$$

whence it follows

$$\beta[\phi] - \beta[f] \geqq \frac{1}{2\pi}\Omega[\lg(F/F'(\infty))].$$

But, since the inversions with respect to imaginary axes in Z- and W-planes have taken place, we have

$$\Omega[\lg(F/F'(\infty))] = 2\Omega[f],$$

yielding the required inequality.

Corollary 2. If $\phi_p(z) \in \mathfrak{T}_p \cap {}'\mathfrak{I}_p$ $(p = 1, \cdots, n)$, then

$$\beta[\phi_{p-1}] \geqq \beta[\phi_p] \quad (p = 2, \cdots, n).$$

Corollary 2a. If $\psi_p(z) \in \mathfrak{I}_p \cap {}'\mathfrak{T}_p$ $(p = 1, \cdots, n)$, then

$$\beta[\psi_{p-1}] \leqq \beta[\psi_p] \quad (p = 2, \cdots, n).$$

Thus, the distortion theorem having been established, the existence proof of a mapping onto a strip slit along perpendicular segments, i.e., a domain of the type $\mathfrak{T}_p \cap {}'\mathfrak{I}_p$ can be performed quite similarly as in case of the preceding sections. And, the existence proof in general case can now be reduced to that in triply connected case. It will suffice merely to state the corresponding results.

Lemma 11. The family $\mathfrak{T}_{n-1} \cap {}'\mathfrak{I}_{n-1}$ consists of a function uniquely determined except any translation parallel to the real axis (any real additive constant).

Theorem 2. The family $\mathfrak{T}_p \cap {}'\mathfrak{I}_p$ for any p with $1 \leqq p \leqq n$ consists of a function uniquely determined except any translation parallel to the real axis.

Of course, the fact stated in the lemma corresponds to a special case of that in the theorem itself.

While, as already stated above, a general existence problem can be reduced to triply connected one, the particular case where a mapping onto a parallel strip slit along horizontal or vertical segments alone, i.e., a domain of the type $\mathfrak{T}_n = {}'\mathfrak{T}_1$ or $\mathfrak{I}_n = {}'\mathfrak{I}_1$, respectively, is in question, can further be reduced to doubly connected one. By means of auxiliary mapping by exponential function as in the above proof of Theorem 1 and inversion with respect to the imaginary axis, the last particular case can be reduced to a well-known theorem concerning the mapping of a $2(n-1)$-ply connected domain onto whole plane slit along radial segments or circular arcs alone; the domains in question being especially symmetric with respect to imaginary axis. But in such a particular case the function $\phi(z)$ of $\mathfrak{T}_n$ or $\mathfrak{I}_n$ which is uniquely determined except any real additive constant can also be characterized in a direct manner by the variational problem

$$\beta[\phi] = \operatorname*{Min}_{f \in \mathfrak{T}_1} \beta[f]$$

or

$$\beta[\phi] = \operatorname*{Max}_{f \in \mathfrak{I}_1} \beta[f],$$

respectively, the range of admissible argument functions $f(z)$ being the same family $\mathfrak{T}_1 = \mathfrak{I}_1$. Consequently, the general existence proof can thus be reduce to doubly connected one.

On the other hand, any ring domain, that is, a doubly connected domain possessing two disjoint continua as boundary, can be mapped conformally and univalently onto an annulus, i.e., a concentric circular ring; the fact having been proved in various ways; cf. Carathéodory [1], Teichmüller [1], Komatu [4], etc. Further, the function which maps an annulus onto whole plane slit along radial segments or circular arcs alone can explicitly, by means of elliptic functions; cf. Komatu [1]. Consequently, by combining an elementary transformation, the mapping onto a parallel strip slit along a horizontal or vertical segment can also be written down in an explicit form; cf. also, for instance, Kubo [1].

Moreover, in a proof of general existence theorem concerning $\mathfrak{T}_n$ or $\mathfrak{I}_n$, based upon a variational method, only the doubly connected case $(n=2)$ of theorem I will be used, as noticed above. The existence theorem is, in general, essentially equivalent to Grötzsch-Rengel's distortion theorem. But, in a particular case of connectivity two, there exists a further equivalent distortion theorem; cf. Komatu [2]. Hence, in order to prove the general existence theorem in such a case, the last mentioned distortion theorem will also suffice.

It may be noticed that a potential-theoretic proof for existence of mapping onto a parallel strip slit along a horizontal segment has recently been given by Kubo [1].

(*) Received July 28, 1951.

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