A NEW PROOF OF LIBER'S THEOREM

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The purpose of this paper is to give a very simple proof of Liber's theorems, published in the Doklady Akad. Nauk. S.S.S.R. (N.S), 1949. Another proof were obtained by S.Sasaki and published in Proc. Acad. Japan 27 (1951).

Theorem. Suppose that the holonomy group H of a given affinely connected space without torsion A_n be a one parametric group. If we denote the symbol of the infinitesimal transformation of H by $Xf = a^i_j x^j \frac{\partial f}{\partial x^i}$ (a^i_j : const.), then the rank of the matrix $\| a^i_j \|$ is at most 2.

The writer's proof is as follows: We shall first state E. Cartan's fundamental theorem: (1). If the holonomy group of a non-holonomic space E with a fundamental group G is g, then the frames at every point of the space E can be chosen so that the connexion of the space in consideration is analytically the same as that of a space with the fundamental group g .

According to the Cartan's theorem we can choose frames at every point of the space A_n , so that the connexion of A_n is analytically the same as that of a space with the one parameter group H Hence, if we denote the infinitesimal transformation of H by

$$Xf = a^i_j x^j \frac{\partial f}{\partial x^i}$$

$$(a^i_j \ : \text{const.})$$

then the connexion of the space is given by the equations,

$$(1) \qquad dp = \omega^i e_i \quad de_j = \omega^i_j e_i$$

$$(i , j = 1,2,3,\dots,n)$$

provided that

$$(2) \qquad \omega^i_j = a^i_j \rho$$

where ω^i, ω^i_j and ρ are Pfaffians. The equations of structure of the space A_n are:

$$(3) \qquad \Omega^i_j = -(\omega^i_j)' + [\omega^k_j \omega^i_k]$$

where Ω^i_j is the curvature tensor and dash means exterior derivative.

From (2) and (3) we get

$$(4) \qquad \Omega^i_j = +a^i_j \rho'$$

Now, the absence of torsion can be expressed by

$$(\omega^i)' + [\omega^i_j \omega^j] = 0$$

If we take the exterior derivative of the last equation, we get

$$[a^i_j \rho' \omega^j] = 0.$$

Hence, we obtain

$$[\rho' a^i_j \omega^j] = 0$$

$$(i , j = 1,2,\dots,n)$$

According to E.Cartan (2), if at least three of the n pfaffians ($a^i_j \omega^j$ $(i=1,\dots,n)$) are linearly independent ρ' vanishes. Therefore, by virtue of (4), Ω^i_j vanishes. Consequently, the affinely connected space in consideration is flat, which contradicts to our assumption. Hence $\rho' \neq 0$. If $\rho' \neq 0$, at most two of the Pfaffians $a^i_j \omega^j$ ($i = 1,\dots,n$) are linearly independent. Consequently the rank of Matrix $\| a^i_j \|$ is at most 2. Thus the theorem is proved.

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lonomie des espaces generalises. Acta Math., vol.48(1926).
(2) E.Cartan: Lecons sur la theorie des espace a connexion projective. 1937.
(3) Liber: On spaces of linear affine connection with one-parameter holonomy groups. Doklady Acad. Nauk SSSR (N.S.) 66 (1949). Russian.
(4) S.Sasaki: An alternative proof of Liber's theorem. Proc. Acad. Japan. Vol. 27 (1951), pp.73-80.