ON A THEOREM OF HARDY-LITTLEWOOD

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The object of this note is to give a proof of the following theorem and certain of its applications.

Theorem. If $S_n^p = o(n^{p+\beta})$, $\beta > 0$, $p > -1$, where S_n^p denotes the n-th Cesaro sum of order p for the series $\sum a_n$, then the series $\sum n^{-\beta} a_n$ is either summable (C, p) or not summable by any Cesaro means. Conversely if the series $\sum n^{-\beta} a_n$ is summable (C, p); $\beta > 0$, $p > -1$, then $S_n^p = o(n^{p+\beta})$.

Hardy and Littlewood [2] have proved this theorem for a non-negative integer p. Prof. Bosanquet kindly remarked me that the first part of the theorem is contained in a paper by A. Zygmund [4]. But it seems to me that this theorem is not popular (see Hyslop [3]). On the other hand Bosanquet [1] has succeeded in completing the convergence and summability factor theorem. Following his method we can prove the theorem. The method of proof is different from that of Zygmund. The converse part is also well known, but we give a new proof in the same idea.

Before going to the proof we need some lemmas.

Lemma 1. If $\alpha + p > -1$, $\delta > 0$, and $S_n^\alpha = o(n^{\alpha+p})$, then $S_n^{\alpha+\delta} = o(n^{\alpha+p+\delta})$.

Proof. We have

$$S_n^{\alpha+\delta} = \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} S_\nu^\alpha = \sum_{\nu=0}^{n} O\{(n-\nu+1)^{\delta-1}\}\, o(\nu^{\alpha+p}) = o(n^{\alpha+p+\delta})$$

Lemma 2. (Bosanquet [1]). If

$$J = \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1}, \quad 0 < \beta \leq \alpha < 1,$$

then, for $0 \leq \mu \leq \nu \leq n$, we have

$$|J| \leqq K A_{n-\mu}^{\beta-1} A_{\nu-\mu}^{-\alpha}, \qquad (1)$$

Where K is independent of μ, ν, and n.

Proof. We first observe that $A_0^{-\alpha-1} = 1 > 0$ and

$$A_\mu^{-\alpha-1} = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+\mu-1)}{1\cdot 2\cdots \mu} < 0 \ (\mu \geq 1) \qquad (2)$$

for $0 < \alpha < 1$. Moreover $A_0^{\beta-1} = 1 > 0$ and

$$0 < A_\mu^{\beta-1} = \frac{\beta+\mu-1}{\mu} A_{\mu-1}^{\beta-1} < A_{\mu-1}^{\beta-1} \ (\mu \geq 1) \qquad (3)$$

for $0 < \beta < 1$. It follows by (2) and (3) that

$$\begin{aligned} J &= \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1} \\ &= A_{n-\mu}^{\beta-1} |A_0^{-\alpha-1}| - A_{n-\mu-1}^{\beta-1} |A_1^{-\alpha-1}| - \cdots - A_{n-\nu}^{\beta-1} |A_{\nu-\mu}^{-\alpha-1}| \\ &\leq A_{n-\mu}^{\beta-1} (|A_0^{-\alpha-1}| - |A_1^{-\alpha-1}| - \cdots - |A_{\nu-\mu}^{-\alpha-1}|) \\ &= A_{n-\mu}^{\beta-1} \sum_{\sigma=0}^{\nu-\mu} A_\sigma^{-\alpha-1} \\ &= A_{n-\mu}^{\beta-1} A_{\nu-\mu}^{-\alpha} \end{aligned}$$

On the other hand by (2) and (3), if $0 \leq \mu \leq \nu \leq n$,

$$\begin{aligned} J &= \sum_{\rho=\mu}^{n} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1} \geqq \sum_{\rho=\mu}^{n} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1} \\ &= A_{n-\mu}^{\beta-\alpha-1} \geqq -K_1 (n-\mu+1)^{\beta-\alpha-1} \\ &\geq -K_1 (n-\mu+1)^{\beta-1} (\nu-\mu+1)^{-\alpha} \\ &\geqq -K_2 A_{n-\mu}^{\beta-1} A_{\nu-\mu}^{-\alpha}. \end{aligned}$$

Thus (1) holds, with $K = \max(1, K_2)$.

Proof of the first part of Theorem. The thesis is equivalent to the problem that if $S_n^p = o(n^{p+\beta})$, $\beta > 0$, $p > -1$, then the series $\sum n^{-\beta} a_n$ is summable (C, p), provided that it is summable $(C, p+1)$. We divide the proof into the case $n < p < n+1$ $(n = -1, 0, 1, 2, \cdots)$. We give the proof for the case $n = 0$ and 1, since that of the remaining cases is quitely similar.

Case 1. Suppose that $\beta > 0$, $-1 < p < 0$.

For the convenience we replace p by $-p$, then $0 < p < 1$ and

$$S_n^{-p} = o(n^{-p+\beta}) \qquad (4)$$

Since $\sum n^{-\beta} a_n$ is $(C, p+1)$ summable, the necessary and sufficient condition that it is summable (C, p) is

$$T_n^{-p} \equiv \sum_{\nu=0}^{n} A_{n-\nu}^{-p} \nu^{1-\beta} a_\nu = o(n^{1-p}). \qquad (5)$$

Now $\tau_n^{-p} = \sum_{\nu=0}^{n} A_{n-\nu}^{-p} \nu^{1-\beta} a_\nu$

$$= \sum_{\nu=0}^{n} A_{n-\nu}^{-p} \nu^{1-\beta} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{p-2} S_{\mu}^{-p}$$

$$= \sum_{\mu=0}^{n} S_{\mu}^{-p} \sum_{\nu=\mu}^{n} A_{n-\nu}^{-p} A_{\nu-\mu}^{p-2} \nu^{1-\beta}$$

$$= \sum_{\mu=0}^{n} S_{\mu}^{-p} I. \qquad (6)$$

By Abel's transformation we have

$$I = \sum_{\nu=\mu}^{n} A_{n-\nu}^{-p} A_{\nu-\mu}^{p-2} \nu^{1-\beta}$$

$$= (n+1)^{1-\beta} \sum_{\nu=\mu}^{n} A_{n-\nu}^{-p} A_{\nu-\mu}^{p-2}$$

$$+ \sum_{\nu=\mu}^{n} \Delta \nu^{1-\beta} \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{-p} A_{\rho-\mu}^{p-2}$$

$$\doteq I_1 + I_2 \qquad (7)$$

where

$$I_1 = (n+1)^{1-\beta} \sum_{\rho=0}^{n-\mu} A_{n-\mu-\rho}^{-p} A_{\rho}^{p-2}$$

$$= \begin{cases} (n+1)^{1-\beta}, & (\mu = n) \\ 0, & (0 \le \mu \le n-1) \end{cases}$$

Hence by (4)

$$\sum_{\mu=0}^{n} S_{\mu}^{-p} I_1 = o(n^{-p+\beta})\, O(n^{1-\beta}) = o(n^{1-p}) \quad (8)$$

Secondly

$$I_2 = \sum_{\nu=\mu}^{n} \Delta \nu^{1-\beta} \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{-p} A_{\rho-\mu}^{p-2}$$

$$= \sum_{\nu=\mu}^{n} O(\nu^{-\beta}) J.$$

where $J = \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{-p} A_{\rho-\mu}^{p-2}$.

In Lemma 2, we put $0 < \alpha = \beta = 1-p < 1$. Then, by (1), we have

$$|J| \le K A_{n-\mu}^{-p} A_{\nu-\mu}^{p-1}.$$

Hence

$$|I_2| = O\left(\sum_{\nu=\mu}^{n} A_{n-\mu}^{-p} A_{\nu-\mu}^{p-1} \nu^{-\beta}\right)$$

$$= O\left(A_{n-\mu}^{-p} \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{p-1} \nu^{-\beta}\right)$$

$$\leqq O\left(A_{n-\mu}^{-p} \mu^{-\beta} \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{p-1}\right)$$

$$= O\left(A_{n-\mu}^{-p} \mu^{-\beta} A_{n-\mu}^{p}\right)$$

$$= O(\mu^{-\beta})$$

If $\mu = 0$, we put $\mu^{-\beta} = 1$. It follows that

$$\sum_{\mu=0}^{n} S_{\mu}^{-p} I_2 = \sum_{\mu=0}^{n} o(\mu^{-p+\beta})\, O(\mu^{-\beta})$$

$$= \sum_{\mu=0}^{n} o(\mu^{-p}) = o(n^{1-p}) \qquad (9)$$

Thus by (6), (7), (8) and (9), we have

$$\tau_n^{-p} = o(n^{1-p}).$$

Case 2. Suppose that $\beta > 0$, $1 > p > 0$

It is sufficient to prove

$$\tau_n^{p} = \sum_{\nu=0}^{n} A_{n-\nu}^{p} \nu^{1-\beta} a_\nu = o(n^{1+p}) \qquad (10)$$

If we denotes

$$S_n^{\alpha}(u_\nu) = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} u_\nu.$$

then

$$\tau_n^{p} = S_n^{p+1}(\nu^{1-\beta} a_\nu) \qquad (11)$$

We have by partial summation

$$\sum_{\nu=0}^{n} \nu^{1-\beta} a_\nu = S_n (n+1)^{1-\beta} + \sum_{\nu=0}^{n} S_\nu \Delta \nu^{1-\beta}$$

i.e.

$$S_n^{1}(\nu^{1-\beta} a_\nu) = S_n (n+1)^{1-\beta} + S_n^{1}(S_\nu \Delta \nu^{1-\beta}),$$

and hence

$$S_n^{p+1}(\nu^{1-\beta} a_\nu) = S_n^{p}\{S_\nu^{1}(\mu^{1-\beta} a_\nu)\}$$

$$= S_n^{p}\{S_\nu (\nu+1)^{1-\beta}\} + S_n^{p+1}(S_\nu \Delta \nu^{1-\beta})$$

$$= J_1 + J_2 \qquad . \qquad (12)$$

Then

$$J_1 = S_n^{p}\{S_\nu (\nu+1)^{1-\beta}\}$$

$$= \sum_{\nu=0}^{n} A_{n-\nu}^{p-1} S_\nu (\nu+1)^{1-\beta}$$

$$= \sum_{\nu=0}^{n} A_{n-\nu}^{p-1} (\nu+1)^{1-\beta} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{-p-1} S_{\mu}^{p}$$

$$= \sum_{\mu=0}^{n} S_{\mu}^{p} \sum_{\nu=\mu}^{n} A_{n-\nu}^{p-1} A_{\nu-\mu}^{-p-1} (\nu+1)^{1-\beta}$$

$$= \sum_{\mu=0}^{n} S_{\mu}^{p} K,$$

Since $-1 < p-1 < 0$ and $-2 < -p-1 < -1$, K is analogous to I in Case 1, and we get

$$J_1 = o(n^{1+p}). \qquad (13)$$

Next

$$J_2 = S_n^{p+1}(S_\nu \Delta \nu^{1-\beta})$$

$$= S_n^{1}\{S_\nu^{p}(S_\mu \Delta \mu^{1-\beta})\}. \qquad (14)$$

Estimating similarly as I in Case 1, we have

$$S_\nu^{p}(S_\mu \Delta \mu^{1-\beta})$$

$$= \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{p-1} S_\mu \mu^{-\beta}$$

$$= \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{p-1} \mu^{-\beta} \sum_{\rho=0}^{\nu} A_{\mu-\rho}^{-p-1} S_{\rho}^{p}$$

$$= \sum_{\rho=0}^{\nu} S_{\rho}^{p} \sum_{\mu=\rho}^{\nu} A_{\nu-\mu}^{p-1} A_{\mu-\rho}^{-p-1} \mu^{-\beta}$$

$$= o(\nu^{p}).$$

Hence

$$J_2 = S_n^{1}(o(\nu^{p})) = o(n^{p+1}).$$

Collecting the estimations (11), (12), (13) and (14), we get (10).

Proof of the second part of Theorem. If

$$\sum_{n=0}^{\infty} n^{-\beta} a_n = s\ (C, p),$$

then $a_0 - s + \sum_{n=1}^{\infty} n^{-\beta} a_n = o\ (C. p).$

But

$$s_n^{p+1}(s) = A_n^p s = o\ (n^{p+\beta}),\quad \beta > 0,$$

and

$$s_n^{p+1}(\nu^{-\beta} a_\nu) = s_n^{p+1}(s) + s_n^{p+1}(a_0 - s, \cdots \nu^{-\beta} a_\nu \cdots).$$

Hence we can suppose without loss of generality

$$\sum_{n=0}^{\infty} n^{-\beta} a_n = o\ (C. p)$$

Now

$$\begin{aligned} s_n^p &= \sum_{\nu=0}^{n} A_{n-\nu}^p a_\nu \\ &= \sum_{\nu=0}^{n} A_{n-\nu}^p \nu^\beta \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{-p-2} t_\mu^p \\ &= \sum_{\mu=0}^{n} t_\mu^p \sum_{\nu=\mu}^{n} A_{n-\nu}^p A_{\nu-\mu}^{-p-2} \nu^\beta \end{aligned}$$

where

$$t_\mu^p = \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{p-1} a_\nu / \nu^\beta = o(\mu^p)$$

Thus we can now follow the line of proof of the first part to reach the result.

We will now give some applications to the theory of Fourier series.

Let $f(x)$ be a function absolutely integrable in $(0, 2\pi)$ and of period 2π and let its Fourier series be

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

Then we have

Theorem. If

$$\int_0^t |\varphi_x(u)|\, du = o(t),$$

$$\varphi_x(t) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2f(x) \} \qquad (15)$$

then the series $\sum_{n=1}^{\infty} A_n(x)/n^\delta$ $(0 < \delta < 1)$ is summable $(C, -\delta)$.

Proof. By a theorem of Jacob [4], (15) implies

$$\sigma_n^{-\delta} = s_n^{-\delta}(x) / {}_n^{-\delta} = O(n^\delta)$$

that is,

$$s_n^{-\delta}(x) = O(1)$$

By a well known theorem, since

$$\sum_{n=1}^{\infty} A_n(x)/n^\delta$$

is convergent at that point, we get the required result by the theorem already proved.

(*) Received June 1, 1951.

1 Bosanquet, L.S., Note on convergence and summability factors (III), Proc. London Math. Soc., 50 (1949) 482-496.

2 Hardy, G.H. and Littlewood, J.E., Contributions to the arithmetic theory of series, Proc. London Math. Soc., 11 (1913) 411-478.

3 Hyslop, J.M., On the approach of a series to its Cesaro limit, Proc. Edinburgh Math. Soc., 5 (1938) 182-201.

4 Jacob, M., Uber die Verallgemeinerung einigen Theoreme von Hardy in der Theorie der Fourierschen Reihen, Proc. London Math. Soc., 26 (1927) 470-492.

5 Zygmund, A., Uber einige Satze aus der Theorie der divergenten Reihen, Bull. 1 Academie Polonaise, 1927, 309-331.

6 Jacob (4) proved the following theorem: If $\int_0^t |\varphi(u)|\, du = o(t)$ then $\sum_{n=1}^{\infty} A_n(x)/n^\delta \varphi_n$ $(0 < \delta < 1)$ is $(C, -\delta)$ summable where $\{1/\varphi_n\}$ is a convex sequence tending to zero. He says that "Der Faktor $1/\varphi_n$ kann nicht weggelassen werden", but this is incorrect as the theorem shows.

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