NOTE ON (LM)-GROUPS OF FINITE ORDERS[*]

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(Communicated by H. Toyama)

In the present note, we study some properties of a finite group whose lattice of subgroups is lower semi-modular. We, however, use no result of the general theory of lattices.

I give my hearty thanks to Mr. M.SUZUKI for his kind remarks and advices.

NOTATIONS: $S_i(X)=S_{p_i}(X)$, $H_i(X)=H_{p_i}(X)$, $C(X)$, $C_\infty(X)$, $\theta(X)$ and $\Phi(X)$ denote a p_i-Sylow subgroup, a p_i-Sylow complement, the centre, the hypercentre, the commutator subgroup and Φ-subgroup of a group X respectively; (X) may be often omitted. $\mathfrak{N}_Y(X)$ denotes the normalizer of a subgroup X in a group Y.

1. On the p-nilpotency.

DEFINITION 1. A finite group is called p-nilpotent when it has a normal p-Sylow complement.

PROPOSITION 1. Let G be a group whose order has at least three distinct prime factors and let p be one of them. Then G is p-nilpotent if every proper subgroup of G is so.

PROOF. Let G be a group which satisfies our condition. If G is not p-normal in GRUN's sense[1], there exist a p-subgroup P and a p-regular element A in G such that A induces a non-identical automorphism into P, by virtue of a theorem of W.BURNSIDE[2]. Since $P\cdot\{A\}$ is non-p-nilpotent, we have $G=P\cdot\{A\}$. Let $A=A_1A_2\cdots A_r$ be the Sylow decomposition of A. Then $r\geqq 2$ by our condition. Clearly $G\neq P\cdot\{A_i\}$, whence $P\cdot\{A_i\}=P\times\{A_i\}$. Therefore $G=P\times\{A\}$ which is a contradiction. Hence G is p-normal. Now by a theorem of O.GRÜN[3],

$$S_p(G/\theta(G))\cong S_p(\mathfrak{N}(C(S_p))/\theta(\mathfrak{N}(C(S_p))))$$

If $G\neq\mathfrak{N}(C(S_p))$, since the latter is p-nilpotent by our condition, $S_p(\mathfrak{N}(C(S_p))/\theta(\mathfrak{N}(C(S_p))))\neq e$ whence $S_p(G/\theta(G))\neq e$. Therefore, $G\neq\theta(G)$ whence it is easily verified that G is p-nilpotent. If $G=\mathfrak{N}(C(S_p))$ and $S_p\neq C(S_p)$, then induction argument can be applied to $G/C(S_p)$ and we can see that $G/C(S_p)$ is p-nilpotent whence it is easily verified that G is p-nilpotent. Finally if $G=\mathfrak{N}(C(S_p))$ and $S_p=C(S_p)$, then there exists, by a theorem of I.SCHUR[4], one H_p in G. Since $G\neq S_p\cdot S_q(H_p)$, by our condition, $S_p\cdot S_q(H_p)=S_p\times S_q(H_p)$ whence $G=S_p\times H_p$. Therefore, of course, G is p-nilpotent.

PROPOSITION 2. Let G be a non-p-nilpotent group whose every proper subgroup is p-nilpotent. Then $G=S_p\cdot S_q$ where S_p is normal, $S_q=\{Q\}$ is cyclic, non-normal. And every proper subgroup of G is nilpotent. In particular it is soluble. The converse is also valid.

PROOF. Let G be a group which satisfies our condition. Follow the proof of PROPOSITION 1. First it is evident that the order of is $p^m q^n$ by PROPOSITION 1. Therefore if G is not p-normal, then $A=A_1$, using the same notations as in the proof of PROPOSITION 1, and this proves PROPOSITION 2. Now assume that G is p-normal. Then $\mathfrak{N}(C(S_p))=G$, since if $\mathfrak{N}(C(S_p))\neq G$, G is p-nilpotent, as is easily seen by virtue of the proof of PROPOSITION 1. If $C(S_p)\neq S_p$, induction can be applied to $G/C(S_p)$ and we can easily prove PROPOSITION 2. Finally if $C(S_p)=S_p$, then $G=S_p\cdot S_q$. And if $S_q=T\cup U$ where T and $U\subsetneqq S_q$, since $G\neq S_pT$ and S_pU, $S_p\cdot T=S_p\times T$ and $S_p\cdot U=S_p\times U$ whence $G=S_p\times S_q$ which is a contradiction. Therefore S_q is cyclic and this proves PROPOSITION 2. The converse is obvious.

REMARK 1. Similar results as PROPOSITION 1 and 2 have been obtained by many authors, for instance, O.SCHMIDT[5], D.KOLIANKOWSKY[6], S.TCHOUNIKHIN[7] and K.IWASAWA[8]. And our result is a slight modification of theirs. But it seems to me that our formulation is a little more general and applicable than antecedents. (Cf. M.SUZUKI[9]).

PROPOSITION 3[10]. A simple non-abelian group G has a proper subgroup which satisfies the condition in PROPOSITION 2 for every prime factor p of its order.

PROOF. Clearly G is not p-nilpotent. Therefore G has at least one non-p-nilpotent subgroup, for instance, G itself. Choose a minimal one of such subgroups. Then it is a group of PROPOSITION 2 and soluble. Therefore it does not coincide with G .

PROPOSITION 4[11]. Let the order g of a group G have just n distinct prime factors. If G has at most $n-1$ non-isomorphic proper non-nilpotent subgroups, G is soluble.

PROOF. Clearly we may assume that G is p-nilpotent for some p which is a prime factor of g , as is easily seen by virtue of the proof of PROPOSITION 3. The p-Sylow complement has clearly at most $n-2$ non-isomorphic proper non-nilpotent subgroups. Now for $n=1$ G is nilpotent. Therefore we can easily prove PROPOSITION 4 by induction for n .

2. On (C)-groups.

DEFINITION 2. A finite group is called a (C)-group if every maximal subgroup of any subgroup has a prime index.

PROPOSITION 5. A (C)-group is p-nilpotent for the least prime factor p of its order. In particular, it is soluble.

PROOF. Let G be a group satisfying the condition in PROPOSITION 2. If G is a (C)-group, then G has a subgroup H of index p as a maximal subgroup containing S_q and H is normal since p is the least. Since H is nilpotent, $S_q(H) = S_q(G)$ is normal in H and therefore in G . This is a contradiction.

REMARK 2. Groups of this type investigated first by O.ORE[12] and complemented by G.ZAPPA[13] and K. IWASAWA[14].

We shall refer only to

PROPOSITION 6. A minimal normal subgroup of a (C)-group has a prime order. Therefore, it has a chief series each of whose factors is of a prime order. The converse is also true.

PROOF. We shall show that G has a normal subgroup of order p_1 where p_1 is the maximum prime factor of the order of G . If $C(S_1)\cdot H_1 \neq G$ then $C(S_1)\cdot H_1$ has a normal subgroup of order p_1 and clearly this is also normal in G . Assume that $C(S_1)\cdot H_1 = G$. Then G has a subgroup M of index p_1 as a maximal subgroup containing H_1 . If $S_1(M) \neq e$ then M has a normal subgroup of order p_1 and this clearly is also normal in G . Finally if $S_1(M) = e$ then $S_1(G)$ is normal in G and of order p_1 . The remainder and the converse are obvious.

3. On (LM)-groups.

DEFINITION 3. A finite group G is called an (LM)-group if every intersection of two distinct maximal subgroups of any subgroup is maximal respectively in such two maximal subgroups.

PROPOSITION 7. An (LM)-group is p-nilpotent for the least prime factor p of its order. In particular, it is soluble.

PROOF. Let G be a group as in PROPOSITION 2. Assume that G is an (LM)-group and we kick out a contradiction. To do this we use induction argument. If S_p is not minimal normal we take such P contained in S_p and consider G/P . Then a contradiction easily tumbles out. Hence we may assume that S_p is minimal normal. Further if S_q is not of order q , a maximal subgroup T of S_q is normal in G . If we observe G/T , a contradiction easily tumbles out. Hence we may assume that S_q is of order q . Then S_p and S_q are maximal in G and obviously $S_p \cap S_q = e$. Since G is assumed to be an (LM)-group, S_p must be of order p and S_q is normal in G since $p < q$. This is a contradiction.

PROPOSITION 8. An (LM)-group is a (C)-group. The converse is not true.

PROOF. Assume that every proper subgroup of G is a (C)-group. We show first that the number of prime factors of $G:M$ is invariant by the choice of maximal subgroup M . In fact, let N be another maximal subgroup if any, then $G : M\cap N = (G:M)(M:M\cap N) = (G:N)(N:M\cap N)$ and $M\cap N$ is maximal in M and N since G is (LM), therefore, $M:M\cap N$ and $N:M\cap N$ are prime whence the assertion is obvious. If there exists no such N, then G is cyclic and the assertion is trivial. Now since G is soluble, $G:M$ is prime and G is (C). Thus induction completes our proof.

PROPOSITION 9. Let the order of a group G have the following prime factor decomposition: $p_1p_2p_3$ or $p_1p_2^2$ where $p_1 > p_2 > p_3$. Then G is a (C)-group, except the case that $G \cong \mathfrak{A}_4$. Further G is a (C)-group and not an (LM)-group if and only if H_1 induces an automorphism of order p_2p_3 or p_2^2 into S_1.

PROOF. If S_1 is not normal then $\pi(S_1) = S_1$, therefore, $p_2^2 \equiv 1 \pmod{p_1}$[15] whence $p_1 = 3$, $p_2 = 2$ and $G \cong \mathfrak{A}_4$. Hence S_1 is normal if not $G \cong \mathfrak{A}_4$. This proves our first assertion. Assume that G is not isomorphic to $\mathfrak{A}_4$. Now Θ is nilpotent by a theorem of O.ORE[16] and if $\Theta \neq S_1$ or more generally G is not fully irreducible, then G is clearly an (LM)-group. Further assume that G is not an (LM)-group. Therefore $S_1 = \Theta$. Then H_1 is cyclic and is considered as a group of automorphisms of S_1, as is easily seen. Conversely, assume that this is the case. Putting $S_1 = \{A\}$ we have $H_1^A \cap H_1 = e$. Therefore G is not an (LM)-group.

REMARK 3. PROPOSITION 9 was suggested to the author by Mr. S. SATO and I give him my hearty thanks. (Cf. S.SATO[17])

PROPOSITION 10. Assume that the order of a group G have the following prime factor decomposition: $p_1p_2^{e_2}p_3^{e_3}$ $(p_1 > p_2 > p_3)$. If G is an (LM)-group, then $S_1S_2 = S_1 \times S_2$ *or* $S_1S_3 = S_1 \times S_3$.

PROOF. Assume that the assertion is true for all groups of smaller order. Now Θ is nilpotent by a theorem of O.ORE[18] and if $\Theta \nleq S_1$, then $S_2(\Theta)$ or $S_3(\Theta)$ is distinct from e. Further if $S_2(\Theta) = S_2$ or $S_3(\Theta) = S_3$ then $S_1S_2 = S_1 \times S_2$ or $S_1S_3 = S_1 \times S_3$; and if $S_2(\Theta) \neq S_2$ then induction can be applied to $G/S_2(\Theta)$ and $[S_1, S_2]$ or $[S_1, S_3] \subseteq S_2(\Theta)$. Since $[S_1, S_2]$ and $[S_1, S_3] \leq S_1$, $[S_1, S_2]$ or $[S_1, S_3] \leq S_2(\Theta) \cap S_1 = e$ whence $S_1S_2 = S_1 \times S_2$ or $S_1S_3 = S_1 \times S_3$. Hence we may assume that $\Theta \leq S_1$.

Whence H_1 is abelian. Putting $S_1 = \{A\}$, we consider $H_1 \cap H_1^A$ then this contains S_2 or S_3, whence we can easily see that S_2 or S_3 is normal in G. Therefore $S_1S_2 = S_1 \times S_2$ or $S_1S_3 = S_1 \times S_3$. Thus induction completes the proof.

PROPOSITION 11. Let G be a (C)-group whose order g has at least four distinct prime factors. If every proper subgroup is an (LM)-group, then G is so, too.

PROOF. Let M and N be any two distinct maximal subgroups of G. We have to show that $M\cap N$ is maximal in M and N. Now if M and N are not conjugate $MN = NM = G$, by a theorem of O.ORE[19], whence we can easily see that $G:M = N:M\cap N =$ prime and $G:N = M:M\cap N =$ prime. Therefore $M\cap N$ is clearly maximal in M and N. Hence we may assume that $N = M^x$ for some element x of G. Now let g have the following prime factor decompositions: $p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(p_1 > p_2 > \cdots > p_r$ and $r \geq 4)$. If $G:M = p_i$, $i > 1$, then $G/S_1 \gneq M/S_1$ and N/S_1 and $G/S_1 \cong H_1$ is an (LM)-group, whence the assertion trivially holds. Hence we may assume that $G:M = p_1$. Now assume that the assertion is true for all groups of smaller order. If $e_1 > 1$ then $S_1(M)$ is normal in G and we can apply induction argument to $G/S_1(M) \gneq M/S_1(M)$ and $N/S_1(M)$. Then we can see that $G/S_1(M)$ is an (LM)-group whence the assertion clearly holds. Hence we may assume that $e_1 = 1$ and put $S_1 = \{A\}$ and $x = A$. Now PROPOSITION 10 can be applied to this case: We have $S_i^A = S_i$ except at most one k $(1 < k \leq r)$, where $M = S_2S_3 \cdots S_r$. Finally consider S_1S_k and $S_k^A \cap S_k$. Since S_1S_k is an (LM)-group, $S_k^A \cap S_k$ is maximal in S_k and S_k^A. Hence $M\cap N$ is clearly maximal in M and N. Therefore PROPOSITION 11 has been completely proved by induction.

PROPOSITION 12. Let G be a soluble group. Then G is a (C)- or an (LM)-group according to that G/C_∞ is a (C)- or an (LM)-group. The converse is also true.

PROOF. First assume that G/C_∞ is a (C)-group. We use induction for the length of the upper central series. Then we may assume that $C_\infty = C > e$. Let M be any maximal subgroup of G. If $M \supset C$, then since $G/C \supset M/C$ is a (C)-group, $G:M =$ prime. If $M \not\supset C$

then M is normal in G and obviously we have $G:M =$ prime. Next we assume that G/C_∞ is an (LM)-group. As above we may assume that $C_\infty = C \supset e$. Let M and N be any two distinct maximal subgroups of G. If M and N are not conjugate, we have $MN = NM = G$ by a theorem of O.ORE[20] and easily see $G:M = N:M\cap N =$ prime and $G:N = M:M\cap N =$ prime. Therefore $M\cap N$ is clearly maximal in M and N. Hence we may assume that M and N are conjugate. Then M and therefore $N \supset C$, since if not we can easily see that M is normal and $M = N$ which is a contradiction. Then since $G/C \geq M/C$ and N/C is an (LM)-group, $M\cap N$ is clearly maximal in M and N. Therefore induction proves PROPOSITION 12. The converse is trivial.

PROPOSITION 13. Let G be a (C)- but non-(LM)-group whose every proper subgroup is an (LM)-group. Then G has a homomorphic image which is a group as in PROPOSITION 9.

PROOF. Follow the proof of PROPOSITION 11. Let G be a group which satisfies our condition. We may replace G/C_∞ for G by virtue of PROPOSITION 12; therefore we may assume that $C = e$. Now the order of has the following prime factor decomposition:
$p_1^{e_1} p_2^{e_2} p_3^{e_3}$ or $p_1^{e_1} p_2^{e_2}$ $(p_1 > p_2 > p_3)$
and G has a homomorphic image of order $p_1 p_2^{e_2} p_3^{e_3}$ or $p_1 p_2^{e_2}$ as is easily seen in virtue of the proof of PROPOSITION 11. Therefore we may assume that the order of G is $p_1 p_2^{e_2} p_3^{e_3}$ or $p_1 p_2^{e_2}$. Now $G/H_1(\theta)$ satisfies the same condition that G does. Therefore we may assume that $\theta \subsetneqq S_1$. Hence H_1 is abelian. Now put $S_1 = \{A\}$ and let K be a maximal subgroup of H_1 and we consider $K \cap K^A$. Since $S_1 K$ is an (LM)-group, $K\cap K^A$ is maximal in K. On the other hand, it is contained in C. Therefore $K \cap K^A = e$. Therefore H_1 is of order $p_2 p_3$ or p_2^2 and this completes the proof of PROPOSITION 13.

Now we shall apply the method, by which P.HALL[21] studied complemented (C)-groups, to general (LM)-groups. And this is proposed by Mr. M.SUZUKI.

PROPOSITION 14. Let $G = G_1 \times G_2$ be a soluble group. Every maximal subgroup M, such that $M \not\supseteq G_1$ and G_2, is normal.

PROOF. Assume that the assertion is true for all groups of smaller order. And we shall prove PROPOSITION 14 by induction. Now if $M \cap G_1 = e$ and $M \cap G_2 = e$, then $G = MG_1 = MG_2$ and $G:M = p^e$ where p is a prime factor of the order of G. Therefore $G_1 : e = G_2 : e = p^e = G:M$ and, in particular, G is a p-group. Hence M is obviously normal in G. Then we may assume that, for instance, $M\cap G_1 = N_1 \supset e$. Since $\mathfrak{N}(N_1) \supset M$ and G_2, N_1 is normal in G. Therefore we can apply induction to $G/N_1 \cong G_1/N_1 \times G_2 \supset M/N_1$. Hence M is normal in G. And induction completes the proof of PROPOSITION 14.

PROPOSITION 15. If G_1 and G_2 are (LM)-groups, then $G = G_1 \times G_2$ is so, too.

PROOF. Let the assertion be secured for groups of smaller order. And we shall prove PROPOSITION 15 by induction argument. First it is trivial that G is a (C)-group by PROPOSITION 6. Let M and N be any two distinct maximal subgroups of G. If M and N are not conjugate, then, $MN = NM = G$ by a theorem of O.ORE[22]. Hence $G:M = N:M\cap N =$ prime and $G:N = M:M\cap N =$ prime. If M and N are conjugate, then M and N contain G_1 or G_2 by PROPOSITION 14 and therefore $M\cap N$ contains G_1 or G_2, whence it is clear that $M\cap N$ is maximal in M and N. Now it is sufficient to show that every proper subgroup of G is an (LM)-group. So we shall assume that there exists at least one non-(LM) subgroup in G; let H be a minimal one. Then every proper subgroup of H is an (LM)-group. Now we may assume that $G_1 H = G_2 H = G$. For if not, say $G_1 H \neq G$ then induction can be applied to $G_1 H = G_1 \times (G_2 \cap G_1 H)$ and we see that $G_1 H$ is an (LM)-group. Then, of course, H is (LM) which is a contradiction. Further we may assume that any minimal normal subgroup L of G which is contained in G_1 or G_2 is contained in H. For if not, it is evident that $\mathfrak{N}(H\cap L) \geqq H$ and G_1 or G_2. Therefore $\mathfrak{N}(H\cap L) = G$, that is, $H\cap L$ is normal in G. Since $H\cap L$ is distinct from L and since L is minimal, $H \cap L = e$. Then induction can be applied to $G/L \cong G_1/L \times G_2 \geqq HL/L \cong H$ and we see that G/L is (LM). Then, of course, $HL/L \cong H$ is (LM) which is a contradiction. In particular H contains a minimal normal subgroup $\mathfrak{P}_1$ which is contained in $\mathfrak{G}_1$ or $\mathfrak{G}_2$, say $\mathfrak{G}_1$ of order p_1 where p_1 is the maximum prime factor of the order of $\mathfrak{G}$: The existence of $\mathfrak{P}_1$ is secured since $\mathfrak{G}$ is a (C)-group. Now as is easily seen in virtue of the proof of PROPOSITION 11 there exists a maximal subgroup M of index p_1 in H such that the intersection $M \cap M^h$ is not maximal in M or M^h, say M, for a suitable element h of H. Now we may assume that M contains no minimal normal subgroup which is contained in $\mathfrak{G}_1$ or $\mathfrak{G}_2$. For if not, induction can be applied and we see that $M \cap M^h$ is maximal in M and M^h which is a contradiction. In particular, M does not contain $\mathfrak{P}_1$. Then $\mathfrak{G}_1 M = \mathfrak{G}_1 \mathfrak{P}_1 M = \mathfrak{G}_1 H = \mathfrak{G}$. Similarly $\mathfrak{G}_2 M = \mathfrak{G}$. Now consider $\mathfrak{G}_1 \cap M$, then $\mathfrak{N}(\mathfrak{G}_1 \cap M) \geqq M$ and $\mathfrak{G}_2$, that is, $\mathfrak{N}(\mathfrak{G}_1 \cap M) = \mathfrak{G}$ and $\mathfrak{G}_1 \cap M$ is normal in $\mathfrak{G}$. Therefore $\mathfrak{G}_1 \cap M = e$. Similarly $\mathfrak{G}_2 \cap M = e$. Then it is easily seen that $H \cap \mathfrak{G}_1 = \mathfrak{P}_1$ and $H \cap \mathfrak{G}_2 = \mathfrak{P}_2$ where $\mathfrak{P}_2$ is a minimal normal subgroup which is contained in $\mathfrak{G}_2$ of order p_1. Therefore $H = \mathfrak{P}_1 \cdot M = \mathfrak{P}_2 \cdot M$. Since $H \neq \mathfrak{P}_1 \cdot H_1(M)$, $\mathfrak{P}_1 \cdot H_1(M)$ is an (LM)-group. Then $S_i \cdot (H_1(M)) \cdot \mathfrak{P}_1 = S_i(H_1(M)) \times \mathfrak{P}_1$ except at most only one i by PROPOSITION 10. Then as in the proof of PROPOSITION 11 it is easily seen that $M \cap M^h$ is maximal in M which is a contradiction. Therefore induction completes the proof of PROPOSITION 15.

Lastly we shall analyse a structure of fully irreducible (LM)-groups. Since it is evident that a p-group belongs to this class if and only if its centrum is cyclic, we shall treat in the following only non-p-groups. We however, contrary to Hall's case, have not succeeded in writing out a structure of such groups.

Let $\mathfrak{G}$ be a fully-irreducible (LM)-group. Then since $\mathfrak{G}$ is a (C)-group, θ is nilpotent by a theorem of O.ORE[23]. Therefore $\theta \subsetneqq S_1$ by our assumption where p_1 is the largest prime factor of the order of $\mathfrak{G}$ and hence H_1 is abelian. Let $\Omega(C(S_1))$ be a subgroup of $C(S_1)$ which is consisted by all elements of order p_1 of $C(S_1)$ and consider $\Omega(C(S_1)) \cdot H_1$. Then it is easily verified that $\Omega(C(S_1))$ is of order p_1 since if not $\mathfrak{G}$ has at least two minimal normal subgroups. Therefore $C(S_1)$ is cyclic and S_1 is fully irreducible. Further it is evident that H_1 is considered as a group of some automorphisms of S_1 and that every prime factor q of the order of H_1 satisfies the condition! $p_1 \equiv 1 \pmod q$. We have used no fact that $\mathfrak{G}$ is an (LM)-group in above observation.

PROPOSITION 15. Let $\mathfrak{G}$ be an (LM)-group and H be its proper subgroup. If M is a maximal subgroup of $\mathfrak{G}$, then $M \gneqq H$ or $H \cap M$ is maximal in H. In particular $\Phi(H) \subseteqq \Phi(\mathfrak{G})$.

PROOF. Let N be a maximal subgroup of $\mathfrak{G}$ which contains H. If $N = M$ then $M \geqq H$. If $N \neq M$, then $N \cap M$ is maximal in M and N since $\mathfrak{G}$ is an (LM)-group. Now induction can be applied to N, H and $N \cap M$ and we can see that $N \cap M \cap H = M \cap H$ is maximal in H. Thus induction proves PROPOSITION 15.

Again let $\mathfrak{G}$ be a fully irreducible (LM)-group. Since $\Phi(\mathfrak{G})$ is nilpotent, $\Phi(\mathfrak{G}) \subseteqq S_1$ from our assumption. Therefore $\Phi(H_1) = e$ and H_1 is a direct product of elementary abelian q-groups where q runs all the prime factors of the order of H_1. Finally let

$$S = T_0 \supset T_1 \supset \cdots \supset T_{e_1} = e$$

be a part of principal series of $\mathfrak{G}$. Then it is easily verified that H_1 induces a group of automorphism of at most prime order into each T_i / T_{i+1} $(i = 0, \cdots, e_1 - 1)$. Conversely such a group is evidently a fully irreducible (LM)-group. Such a characterization, however, is not constructive at all, we think.

EXAMPLE. Let p be a prime such that $p - 1 = q_1 q_2 \cdots q_n \cdot r$ where $q_1, q_2, \cdots$, and q_n are primes and r is a positive integer. Let S be a p-group of order p^{2n+1} defined by following relations: $[A_{2i-1}, A_{2i}] = A_0$ for $i = 1, 2, \cdots, n$, $[A_k, A_l] = e$ for $(k, l) \neq (2i-1, 2i)$ or $(2i, 2i-1)$ and $A_j^p = e$. Then, as is easily verified, S is fully irreducible and of class 2. Denote by T_i the subgroup which is generated by A_{2i-1} and A_{2i} for $i = 1, 2, \cdots, n$. Then, as is easily verified, T_i has a cyclic group $Q_i = \{B_i\}$ of prime order q_i as a group of automorphisms such that $A_{2i-1}^{B_i} = A_{2i-1}^{x_i}$, $A_{2i}^{B_i} = A_{2i}^{x_i}$, and $x_i y_i \equiv x_i^q \equiv y_i^q \equiv 1$ (mod p). Let $H = Q_1 \times Q_2 \times \cdots \times Q_n$ be a group of automorphisms of S where Q_k induces an automorphism of order q_k into T_k in the same manner as above and does an identical automorphism into T_l with $l \neq k$. Let $G_T = S \cdot H$ be a holomorph of S by H. Then, as is easily verified, G_T of order $p^{2n+1} q_1 \cdots q_n$ is a fully irreducible (LM)-group.

(*) Received December 29, 1950.

(*) I have obtained the following results during 1947-1948. After I had accomplished this work, it was reported that Mr. A.JOHNES had studied groups of similar type. Yet his proof is not communicated to me.

1) Cf. H.ZASSENHAUS, Lehrbuch der Gruppentheorie I (1937).
2) l.c. (1).
3) l.c. (1).
4) l.c. (1).
5) Cf. O.SCHMIDT, Rec. Math. (31), (1924).
6) Cf. D.KOLIANKOWSKY, C.R.URSS (19), (1938).
7) Cf. S.TCHOUNIKHIN, Rec. Math. (46), (1938).
8) Cf. K.IWASAWA, Proc. Phys.-Math. Soc. Jap. (23), (1941).
9) Cf. M.SUZUKI, On the -nomomorphisms of finite groups, Forthcoming.
10) Cf. D.KOLIANKOWSKY, Rec. Math. (61), (1946).
11) l.c. (10).
12) Cf. O.ORE, Duke Math. J. (6), (1939).
13) Cf. G.ZAPPA, Duke Math. J. (7), (1940) etc.
14) Cf. K.IWASAWA, J. Fac. Sci. Univ. Tokyo (1941) I.
15) l.c. (1).
16) l.c. (12).
17) Cf. S.SATO, Osaka Math. J. (1), (1949).
18) l.c. (12).
19) l.c. (12).
20) l.c. (12).
21) Cf. P.HALL, J. London Math. Soc., (12), (1937).
22) l.c. (12).
23) l.c. (12).