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CURVATURE AND REAL ANALYSIS

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1. Introduction. In a recent paper with S.-S. Chern [3], the author studie the volume decreasing property of a class of harmonic mappings thereby obtain ing a real analogue of the classical Schwarz-Ahlfors lemma. The domain M was taken to be the unit open ball with the hyperbolic metric of constan negative curvature, and the image space was a negatively curved Riemannia manifold with sectional curvature bounded away from zero. In this paper, i is shown that M may by taken to be any complete Riemannian manifold o nonpositive curvature provided its sectional curvatures are bounded below b a negative constant (see [5]). The technique employed also yields a distanc decreasing theorem when the map is volume preserving.

2. Harmonic mappings. Let M and N be C^∞ oriented Riemannian manifold of the same dimension n with metrics $ds_M{}^2$ and $ds_N{}^2$, respectively, and volum elements dv_M and dv_N. Let $f: M\to N$ be a C^∞ mapping and $A=f^*dv_N/dv_M$ be th ratio of volume elements. We calculate the Laplacian Δ of $u=A^2$ as in [3] and so recall the necessary Riemannian geometry. Locally, then, $ds_M{}^2=\sum \omega_i$ and $ds_N{}^2=\sum \omega_a{}^{*2}$, where the ω_i and $\omega_a{}^*$ are linear differential forms in M an N, respectively. The structure equations in M are

$$d\omega_i=\sum_j \omega_j\wedge\omega_{ji},\ d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}-\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l\,.$$

The Ricci tensor is defined by $R_{ij}=\sum_k R_{ikjk}$, and the scalar curvature by $R=\sum_i R_{ii}$. (The corresponding quantities in N will be denoted with an asterisk.)

Let $f^*: \Lambda(N)\to\Lambda(M)$ be the pull-back map, and set $f^*\omega_a{}^*=\sum_i A_i{}^a\omega_i$. (I the sequel, we will drop f^* from such formulas when its presence is clear fro the context.) The covariant differential of the tangent mapping f_* is define by

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$$dA_i{}^a+\sum_j A_j{}^a\omega_{ji}+\sum_b A_i{}^b\omega_{ba}{}^*\equiv\sum_j A_{ij}{}^a\omega_j \qquad \text{(say)}$$

where $A_{ij}{}^a=A_{ji}{}^a$. The mapping f is called *harmonic* if $\sum_i A_{ii}{}^a=0$. The following formula for the Laplacian Δ of u was obtained in [3]:

$$\frac{1}{2}\Delta u=2\sum_j (A_j)^2-C+A\sum_{a,i,j} B_a{}^i A^a{}_{jji}+u(R-\sum_{b,c,j} R_{bc}{}^* A_j{}^b A_j{}^c)\,,$$

where $(B_a{}^i)$ is the adjoint matrix of $(A_j{}^a)$, $C=\sum B_a{}^i B_b{}^k A_{kj}{}^a A_{ij}{}^b$ is a scalar invariant of the mapping, $dA=\sum A_j\omega_j$, and the $A^a{}_{ijk}$ are defined by

$$dA_{ij}{}^a+\sum_b A_{ij}{}^b\omega_{ba}{}^*+\sum_k A_{kj}{}^a\omega_{ki}+\sum_k A_{ik}{}^a\omega_{kj}\equiv\sum_k A^a{}_{ijk}\omega_k\,.$$

The mapping f is said to be *totally degenerate* if u vanishes everywhere.

3. Distortion theorem. We sketch the proof of the following.

THEOREM. *Let M be a complete Riemannian manifold whose sectional curvatures are nonpositive and bounded below by a negative constant $-A$. Let $f: M\to N$ be a harmonic mapping of equidimensional spaces of dimension n satisfying the condition $C\leqq 0$. If N is an Einstein space with scalar curvature $R^*\leqq -n(n-1)A$, or if its sectional curvatures are $\leqq -A$, then f is volume decreasing. If f is volume preserving and either N is Einsteinian with $R^*\leqq -n^2(n-1)A$, or if its sectional curvatures are $\leqq -nA$, then it is distance decreasing.*

The technique employed is to distort the metric of the domain M conformally in such a way that the ratio of volume elements attains its maximum on M. Let $d\tilde{s}^2$ be a Riemannian metric of M conformally related to ds^2. Then, there is a function $p>0$ on M such that $d\tilde{s}^2=p^2ds^2$. In the sequel, we distinguish the elements of M referred to $d\tilde{s}^2$ with a tilda. Put $d\log p=\sum p_i\omega_i$. Then, if f is harmonic

$$\frac{1}{2}\tilde{\Delta}\tilde{u}=2\sum_j (\tilde{A}_j)^2-\tilde{C}+(n-2)q^{2n+2}[A\sum_{a,i,j} B_a{}^i A_{ij}{}^a p_j+u\Delta\log p$$
$$-2u\sum_j (p_j)^2]+\tilde{u}(\tilde{R}-\sum_{b,c,j}\tilde{A}_j{}^b A_j{}^c R_{bc}{}^*),\qquad q=1/p\,.$$

LEMMA 1. *If f is a harmonic mapping, then*

$$\tilde{C}=q^{2n+2}[C-(n-2)u\sum_j (p_j)^2]\,.$$

Thus, if C is nonpossitive, so is $\tilde{C}$.
If $\tilde{u}$ attains its maximum at $x\in M$, then at x,

$$A\sum_{a,i,j} B_a{}^i A_{ji}{}^a p_j+u\Delta\log p-2u\sum_j (p_j)^2=u[(n-2)\sum_j (p_j)^2+\Delta\log p]\,.$$

LEMMA 2. *Let f be harmonic with respect to $(ds_M{}^2, ds_N{}^2)$ with the property $C\leqq 0$, and let $\tilde{u}$ attain its maximum at $x\in M$. If $n=2$, or if the function*

$P=(n-2)\sum_j (p_j)^2+\Delta \log p$ *is nonnegative everywhere on* M, *then either* f *is totally degenerate, or else* $-\sum_{b,c,j} R_{bc}{}^*\tilde{A}_j{}^b\tilde{A}_j{}^c \leqq -\tilde{R}$ *at* x.

The remainder of the proof is due to Har'El [5] except for the method used to establish the boundedness of $\Delta\tau$. Let y be a point of M and denote by $d(x, y)$ the distance-from-y function. Then, $t(x)=(d(x, y))^2$, $x\in M$, is C^∞ and convex on M (see [2]). (If M is not simply connected, consider its simply connected covering.) The function $\tau(x)=d(x, y)$ is also convex, but it is only continuous on M. The convex open submanifolds $M_\rho=\{x\in M \mid t(x)<\rho\}$ of M exhaust M, that is, $M=\bigcup_{\rho<\infty} M_\rho$.

Consider the metric $d\tilde{s}^2=(\rho/\rho-t)^2ds^2$ on M_ρ. Then $\tilde{u}=(\rho-t/\rho)^{2n}u$ is nonnegative and continuous on the closure $\bar{M}_\rho$ of M_ρ and vanishes on ∂M_ρ. Since $\bar{M}_\rho$ is compact, $\tilde{u}$ has a maximum in M_ρ. Since $t(x)$ is convex, the function P is positive, so we obtain the conclusion of Lemma 2.

Relating the scalar curvatures $\tilde{R}$ of M_ρ and R of M, we obtain

$$\tilde{R}=\frac{(\rho-t)^2}{\rho^2}R-2(n-1)\frac{\rho-t}{\rho}\frac{\Delta t}{\rho}-4n(n-1)\frac{t}{\rho^2}, \qquad t<\rho.$$

LEMMA 3. *For each* ρ, *there exists a positive constant* $\varepsilon(\rho)$ *such that the inequality*

$$\tilde{R}\geqq -n(n-1)A-\varepsilon(\rho)$$

holds om $M(\rho)$. *Moreover* $\varepsilon(\rho)\to 0$ *as* $\rho\to\infty$.

To see that $\Delta\tau$ is bounded as $\tau\to\infty$, observe that the level hypersurfaces of τ are spheres S with y as center. The hessian of τ can be identified with the second fundamental form h of those spheres, extended to be 0 in the normal direction. It follows that $\Delta\tau=$ trace $h=(n-1)\cdot$ mean relative curvature of S. If the curvature $K\geqq a^2$, then from [1; pp. 247-255], $\Delta\tau\leqq(n-1)a\cot a\tau$. If we put $a^2=-\alpha^2$, then $\Delta\tau\leqq(n-1)\alpha\coth\alpha\tau$.

The theorem is now a consequence of Lemmas 1—3.

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