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ON THE LOWER ORDER OF AN ENTIRE FUNCTION

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1. Introduction. Let $f(z)$ be an entire function and let $T(r, f)$ be its characteristic function in the sense of Nevanlinna. Then the lower order $\rho(f)$ and the order $\lambda(f)$ of $f(z)$ are defined by the relations

$$\liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} = \rho(f),$$

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \lambda(f).$$

If $f(z)$ is of finite order, the concept of genus $q(f)$ can be defined. For convenience we say that $q(f)$ is infinite if $f(z)$ is of infinite order.

It is well known that the growth of $f(z)$ is closely related to the distribution of its zeros.

Indeed Edrei and Fuchs proved the following

THEOREM. *Let $f(z)$ be an entire function of finite order having only negative zeros. If the order is greater than one, then*

$$q(f) \leqq \rho(f) \leqq \lambda(f) \leqq q(f)+1.$$

In this paper we shall be concerned with the relation between the growth of an entire function and the distribution of its zeros.

THEOREM 1. *Let $f(z)$ be an entire function whose zeros lie in the sector*

$$\left\{z \,\middle|\, |\arg z| \leqq \beta < \frac{\pi}{2}\right\}$$

for some β. Then $q(f) \geqq 1$ implies $\rho(f) \geqq 1$.

From Theorem 1, we have

THEOREM 2. *Let $f(z)$ be an entire function of finite genus $q(f) \geqq 1$. If its zeros $\{a_n\}$ lie in the sector*

$$\left\{z \,\middle|\, |\arg z| \leqq \gamma < \frac{\pi}{2q(f)}\right\}$$

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for some γ, *then*

$$q(f)\leqq\rho(f)\leqq\lambda(f)\leqq q(f)+1\,.$$

THEOREM 3. *Let* $f(z)$ *be an entire function of infinite order having only real zeros. Then*

$$\rho(f)=\lambda(f)=+\infty\,.$$

In Theorem 2, the value $\pi/2q$ is best in the following sense.

THEOREM 4. *For each integer* $q\geqq 1$, *there exists an entire function* $g(z)$ *of genus* q *whose zeros lie in the sector*

$$\left\{z\,\middle|\,|\arg z|\leqq\frac{\pi}{2q}\right\}$$

and

$$\rho(g)<q\leqq\lambda(g)\,.$$

The above results can be generalized to meromorphic functions with some modifications. For instance, a rather more detailed discussion leads to the following theorem which we state without proof.

THEOREM 5. *Let* $f(z)$ *be a meromorphic function of infinite order having only negative zeros and positive poles. Then* $f(z)$ *is of regular growth.*

Finally, it should be remarked that the results of this note remain true even if infinitely many zeros and poles have unknown arguments, but are sufficiently rare.

2. Proof of Theorem 1. In order to prove Theorem 1 we need some preliminary facts.

LEMMA 1 [4; p. 235]. *Let* $f(z)$ *be an entire function of finite genus* q. *Then*

$$T(r,f)=o(r^{q+1})\,.$$

LEMMA 2 [2; p. 50]. *If* $g(z)$, $h(z)$ *are transcendental entire functions and* $f(z)=g(h(z))$, *then*

$$3T(r,f)\geqq T(r^{N+1},g)$$

for arbitrarily fixed positive integer N *and sufficiently large* r.

We may suppose, as we may do without loss in generality, that $f(0)\neq 0$.
If its zeros $\{a_n\}$ satisfy

$$\sum_n\frac{1}{|a_n|}<+\infty\,,$$

then

$$f(z)=e^{g(z)}\prod_n\left(1-\frac{z}{a_n}\right),$$

where $g(z)$ is a non-constant entire function. Hence by Lemma 1, we have

$$T(r, f(z))=T(r, e^{g(z)})+o(r)\,.$$

Therefore Lemma 2 gives

$$\lim_{r\to\infty}\frac{T(r, f)}{r}\neq 0\,,$$

which means that the lower order of $f(z)$ is at least one.

Next we consider the case that

$$\sum_n\frac{1}{|a_n|}=+\infty\,.$$

By a well known formula [2; p. 22],

$$r\operatorname{Re}\left(\frac{f'(0)}{f(0)}\right)=\frac{1}{\pi}\int_0^{2\pi}\log|f(re^{it})|\cos t\,dt$$
$$+\sum_{|a_n|<r}\operatorname{Re}\left(\frac{\bar{a}_n}{r}-\frac{r}{a_n}\right)$$

for each positive r. Since

$$|\arg a_n|\leqq\beta<\frac{\pi}{2},$$

we have

$$4\,T(r, f)\geqq Cr+\sum_{|a_n|<r}\left(\frac{r}{|a_n|}-\frac{|a_n|}{r}\right)\cos\beta\,,$$

with $C=\operatorname{Re}\left(\frac{f'(0)}{f(0)}\right)$. Further from

$$\sum_{|a_n|<r}\left(\frac{r}{|a_n|}-\frac{|a_n|}{r}\right)$$
$$=2N(r, 0, f)+\int_0^r N(t, 0, f)\left(\frac{r}{t^2}-\frac{1}{r}\right)dt\,,$$

we obtain

$$4T(r, f)\geqq Cr+\cos\beta\int_0^r N(t, 0, f)\left(\frac{r}{t^2}-\frac{1}{r}\right)dt\,.$$

Hence

$$\frac{4T(r, f)}{r}\geqq C+\frac{3}{4}\cos\beta\int_0^{r/2}N(t, 0, f)\frac{1}{t^2}\,dt\,.$$

Therefore we conclude that

$$\lim_{r\to\infty}\frac{T(r, f)}{r}=+\infty\,.$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2. Firstly, we suppose that

$$\sum_n \frac{1}{|a_n|^q} < +\infty \qquad (q=q(f)).$$

Then

$$f(z)=e^{P(z)}E(z, \{a_n\}),$$

where $P(z)$ is a polynomial of degree just q and $E(z, \{a_n\})$ is the canonical product formed with the $\{a_n\}$ as zeros. Since the genus of $E(z, \{a_n\})$ is at most $q-1$, Lemma 1 yields

$$T(r, f(z))=T(r, e^{P(z)})+o(r^q).$$

Hence we have

$$\lim_{r\to\infty} \frac{T(r, f)}{r^q}=C\neq 0,$$

which gives $\rho(f)\geqq q$.

In the case that

$$\sum_n \frac{1}{|a_n|^q}=+\infty,$$

we consider the auxiliary entire function $F(z)$ defined by

$$F(z)=\prod_{k=1}^{q} f(\omega^k \sqrt[q]{z}),$$

where

$$\omega=\exp\left(i\frac{2\pi}{q}\right).$$

Evidently the zeros of $F(z)$ are $\{a_n{}^q\}$ and

$$T(r^q, F)\leqq qT(r, f)$$

for each positive r. Hence the genus of $F(z)$ must be at least one and

$$q\rho(F)\leqq \rho(f).$$

Therefore by Theorem 1, we obtain

$$\rho(f)\geqq q.$$

On the other hand the inequalities

$$\rho(f)\leqq \lambda(f)\leqq q(f)+1$$

are well known. Thus we have the desired fact.

4. Proof of Theorem 3. In the first place we assume that the zeros $\{a_n\}$ satisfy

$$\sum_n \frac{1}{|a_n|^s} < +\infty$$

for some finite positive value of s. Then

$$f(z)=e^{h(z)}E(z, \{a_n\}),$$

where $E(z, \{a_n\})$ is the canonical product formed with the $\{a_n\}$ as zeros and $h(z)$ is a transcendental entire function. Thus by Lemma 1, we have

$$T(r, f(z))=T(r, e^{h(z)})+o(r^s).$$

Applying Lemma 2 to $e^{h(z)}$,

$$3T(r, f(z))\geqq T(r^{N+1}, e^z)+o(r^s)$$

for arbitrarily fixed positive integer N and sufficiently large r. Hence we deduce

$$\rho(f)\geqq N$$

for all $N>s$. Therefore the lower order $\rho(f)$ must be infinite.

It remains to prove the case that

$$\sum_n \frac{1}{|a_n|^s}=+\infty$$

for every finite positive value of s. In this case, for each positive integer N, let us consider the auxiliary entire function $F_N(z)$ defined by

$$F_N(z)=\prod_{k=1}^{2N} f(\omega^k \sqrt[2N]{z}),$$

where

$$\omega=\exp\left(i\frac{\pi}{N}\right).$$

Since the zeros of $F_N(z)$ are $\{a_n^{2N}\}$, the genus of $F_N(z)$ is not less than one. Hence from

$$T(r^{2N}, F_N(z))\leqq 2N\, T(r, f(z)),$$

we have

$$1\leqq \rho(F_N)\leqq \frac{1}{2N}\rho(f).$$

Thus we deduce

$$2N\leqq \rho(f)$$

for every positive integer N. Therefore $\rho(f)$ must be infinite. This completes the proof of Theorem 3.

5. Lemmas. Before proceeding with the proof of Theorem 4, we need the following lemmas.

LEMMA 3. *Let q be a positive integer and let $b_1, b_2, \cdots, b_N$ be*

$$1 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_N .$$

Then

$$\sum_{n=1}^{N} \log\left|E\left(\frac{r}{w_q b_n}, q\right)E\left(\frac{r}{\bar{w}_q b_n}, q\right)\right|$$

$$\begin{cases} \leqq N\log(1+r^2) & q=1, \\ \leqq \dfrac{2N}{q-1}\operatorname{cosec}\dfrac{\pi}{2q}r^{q-1} & q\geqq 2 \end{cases}$$

for each positive r. Here $E(z, q)$ is the Weierstrass primary factor of genus q and

$$w_q = \exp\left(i\frac{\pi}{2q}\right).$$

Proof. It is sufficient to prove the result for $q \geqq 2$. Let us denote the counting function of the finite sequence $\{b_n\}_{n=1}^{N}$ by $n(t)$. Then by a simple calculation we have

$$\sum_{n=1}^{N} \log E\left(\frac{z}{b_n}, q\right) = -z^{q+1}\int_0^\infty \frac{n(t)}{t^{q+1}(t-z)}dt$$

excepting real positive z. Hence

$$\sum_{n=1}^{N} \log E\left(\frac{r}{w_q b_n}, q\right)E\left(\frac{r}{\bar{w}_q b_n}, q\right)$$

$$= 2r^{q+1}\sin\frac{\pi}{2q}\int_0^\infty \frac{n(t)}{t^q|w_q t - r|^2}dt .$$

Since

$$|w_q t - r| = |t - \bar{w}_q r| \geqq r\sin\frac{\pi}{2q},$$

we have

$$\sum_{n=1}^{N} \log\left|E\left(\frac{r}{w_q b_n}, q\right)E\left(\frac{r}{\bar{w}_q b_n}, q\right)\right|$$

$$\leqq 2r^{q-1}\operatorname{cosec}\frac{\pi}{2q}\int_1^\infty \frac{N}{t^q}dt ,$$

which gives the desired result.

LEMMA 4. *Under the same notations as in Lemma 3, set*

$$F_q(r, x)=\log |E(w_q re^{ix}, q)E(\overline{w}_q re^{ix}, q)|$$

$$=\frac{1}{2}\log L_q(r, x)+\sum_{n=1}^{q}\frac{2}{n}r^n\cos nx\cos n\frac{\pi}{2q},$$

where

$$L_q(r, x)=1-4r\cos\frac{\pi}{2q}\cos x-2r^2+4r^2\cos^2 x$$

$$+4r^2\cos^2\frac{\pi}{2q}-4r^3\cos\frac{\pi}{2q}\cos x+r^4.$$

Then there exists $R_q>1$ *such that*

$$F_q(r, x)\leqq F_q(r, 0)$$

for each $r\geqq R_q$ *and* $0\leqq x\leqq\pi$.

Proof. From [3; Lemma 1], we have

$$\frac{L_q(r, x)}{2r^{q+1}}\frac{d}{dx}F_q(r, x)$$

$$=-\left(\cos(q-1)\frac{\pi}{2q}\sin(q-1)x\right)r^2+\left(\sin\frac{\pi}{q}\sin qx\right)r$$

$$+\cos(q+1)\frac{\pi}{2q}\sin(q+1)x.$$

Hence

$$\frac{L_q(r, x)}{2r^{q+1}}\frac{d}{dx}F_q(r, x)$$

$$=-\sin\frac{\pi}{2q}\left[r^2\sin(q-1)x-2r\cos\frac{\pi}{2q}\sin qx+\sin(q+1)x\right].$$

If $q=1$, it is clear that

$$F_1(r, x)\leqq F_1(r, 0)$$

for each x and each positive r. If $q=2$,

$$\frac{L_2(r, x)}{2r^3}\frac{d}{dx}F_2(r, x)$$

$$=-\frac{\sqrt{2}}{2}\sin x(r^2-2\sqrt{2}r\cos x+3-4\sin^2 x).$$

Since

$$r^2-2\sqrt{2}r\cos x+3-4\sin^2 x=\left(2\cos x-\frac{r}{\sqrt{2}}\right)^2+\frac{r^2}{2}-1,$$

$(d/dx)F_2(r, x)$ is non-positive for $r\geqq\sqrt{2}$ and $0\leqq x\leqq\pi$. Therefore $F_2(r, x)\leqq F_2(r, 0)$ there.

Next we consider the case that $q\geqq 3$. Evidently there exist $0<\gamma<\pi/2(q-1)$

and $A>0$ such that

$$1-\cos(q-2)x \geqq A>0$$

for

$$\left|x-\frac{k}{q-1}\pi\right| \leqq \gamma \qquad (k=1, 2, \cdots, q-2).$$

Hence

$$\begin{aligned} F_q(r, 0)-F_q(r, x) &= \frac{1}{2}\log L_q(r, 0)-\frac{1}{2}\log L_q(r, x) \\ &\quad +2\sum_{n=1}^{q-1}\frac{r^n}{n}(1-\cos nx)\cos n\frac{\pi}{2q} \\ &\geqq o(r)+\frac{2}{q-2}r^{q-2}(1-\cos(q-2)x)\cos\frac{q-2}{2q}\pi \\ &\geqq o(r)+\frac{2}{q-2}A\,r^{q-2}\cos\frac{q-2}{2q}\pi \end{aligned}$$

for

$$\left|x-\frac{k}{q-1}\pi\right| \leqq \gamma \qquad (k=1, 2, \cdots, q-2).$$

Then we can fix $R_1(q)>1$ such that

$$F_q(r, 0) \geqq F_q(r, x)$$

for $r \geqq R_1(q)$ and $|x-(k/q-1)\pi| \leqq \gamma$ $(k=1, 2, \cdots, q-2)$. On the other hand $\sin(q-1)x$ never take zeros on the following intervals

$$I_0=\left(0, \frac{\pi}{q-1}-\gamma\right],$$

$$I_k=\left[\frac{k}{q-1}\pi+\gamma, \frac{k+1}{q-1}\pi-\gamma\right] \qquad (k=1, 2, \cdots, q-3),$$

$$I_{q-2}=\left[\frac{q-2}{q-1}\pi+\gamma, \pi\right).$$

Therefore we can write $(d/dx)F_q(r, x)$ on these intervals such that

$$\frac{L_q(r, x)}{2r^{q+1}}\frac{d}{dx}F_q(r, x)=-\sin\frac{\pi}{2q}\sin(q-1)x\,H_q(r, x),$$

where

$$H_q(r, x)=r^2-2r\cos\frac{\pi}{2q}\frac{\sin qx}{\sin(q-1)x}+\frac{\sin(q+1)x}{\sin(q-1)x}.$$

Since

$$\frac{\sin qx}{\sin(q-1)x}, \qquad \frac{\sin(q+1)x}{\sin(q-1)x}$$

are both bounded on the intervals I_k $(k=0, 1, \cdots, q-2)$, there exists $R_2(q)>1$ such that, for each $r\geqq R_2(q)$,

$$H_q(r, x)>0$$

on these intervals. Hence $(d/dx)F_q(r, x)$ has the same sign as $-\sin(q-1)x$ there. Thus for each $r\geqq R_1(q)+R_2(q)$,

$$\max(F_q(r, 0),\ F_q(r, \pi))\geqq F_q(r, x)$$

on these intervals, so that for every x. Further from

$$F_q(r, 0)\geqq F_q(r, \pi)$$

for sufficiently large r, we can fix $R_q>1$ such that

$$F_q(r, 0)\geqq F_q(r, x)$$

for $r\geqq R_q$ and $0\leqq x\leqq \pi$. Thus the proof of Lemma 4 is complete.

6. Proof of Theorem 4. Let ρ and λ be

$$q-1<\rho<q<\lambda<q+1\,.$$

Set

$$s=\rho-q+1>0$$

and set an integer N satisfying

$$N>2+\frac{2\lambda}{s(q+1-\lambda)}\,.$$

Next, with this N, define the sequence $\{Z_n\}$ by the relations

$$Z_1=2\,,\qquad Z_n=Z_{n-1}^N+1\qquad (n=2, 3, \cdots)\,.$$

Further from this sequence $\{Z_n\}$, we construct the sequence $\{a_n\}$ such that

$$a_1=a_2=a_{Z_1}=2^{1/\lambda}\,,$$

$$a_{Z_k+1}=\cdots=a_{Z_{k+1}}=Z_{k+1}^{1/\lambda}\qquad (k=1, 2, \cdots)\,.$$

Then the counting function $n(t)$ of the sequence $\{a_n\}$ is

$$n(t)=Z_k$$

for

$$Z_k^{1/\lambda}\leqq t<Z_{k+1}^{1/\lambda}\qquad (k=1, 2, \cdots)\,,$$

and hence the order of $\{a_n\}$ must be λ.

Now we consider the entire function $g(z)$ defined by

$$g(z)=\prod_{n=1}^{\infty}E\Big(\frac{z}{w_q a_n},\, q\Big)E\Big(\frac{z}{\overline{w}_q a_n},\, q\Big)\,,$$

where $E(z, q)$ is the Weierstrass primary factor of genus q and

$$w_q=\exp\left(\imath\frac{\pi}{2q}\right).$$

Since the order of $\{a_n\}$ is λ, this function $g(z)$ will be well defined and the order and the genus of $g(z)$ are λ and q, respectively. Put

$$X_k=Z_k^{1/s} \qquad (k=1, 2, \cdots).$$

Then

$$\lim_{k\to\infty}\frac{Z_k^{1/\lambda}}{X_k}=0,$$

since $1/s-1/\lambda$ is positive. Therefore for sufficiently large k, we have

$$R_q Z_k^{1/\lambda}\leqq X_k,$$

where R_q is a positive constant of Lemma 4. Hence from

$$a_n\leqq Z_k^{1/\lambda}\leqq\frac{X_k}{R_q} \qquad (n\leqq Z_k),$$

Lemma 4 gives

$$\begin{aligned}\sum_{n\leqq Z_k}&\log\left|E\left(\frac{X_k e^{\imath u}}{w_q a_n}, q\right)E\left(\frac{X_k e^{\imath u}}{\bar{w}_q a_n}, q\right)\right|\\ &=\sum_{n\leqq Z_k} F_q\left(\frac{X_k}{a_n}, u\right)\\ &\leqq\sum_{n\leqq Z_k} F_q\left(\frac{X_k}{a_n}, 0\right)\\ &=\sum_{n\leqq Z_k}\log\left|E\left(\frac{X_k}{w_q a_n}, q\right)E\left(\frac{X_k}{\bar{w}_q a_n}, q\right)\right|\end{aligned}$$

for sufficiently large k and every u. Further using Lemma 3, we obtain

$$\begin{aligned}\sum_{n\leqq Z_k}&\log\left|E\left(\frac{X_k e^{\imath u}}{w_q a_n}, q\right)E\left(\frac{X_k e^{\imath u}}{\bar{w}_q a_n}, q\right)\right|\\ &\leqq Z_k\log(1+X_k^2)\end{aligned}$$

for $q=1$, and if $q\geqq 2$,

$$\sum_{n\leqq Z_k}\log\left|E\left(\frac{X_k e^{\imath u}}{w_q a_n}, q\right)E\left(\frac{X_k e^{\imath u}}{\bar{w}_q a_n}, q\right)\right|\leqq A_q Z_k X_k^{q-1},$$

where A_q is an absolute constant. On the other hand from

$$\log|E(z, q)|\leqq B_q|z|^{q+1},$$

we have

$$\sum_{n>Z_k} \log\left|E\left(\frac{X_k e^{iu}}{w_q a_n}, q\right)E\left(\frac{X_k e^{iu}}{\overline{w}_q a_n}, q\right)\right|$$

$$\leqq 2B_q \sum_{n>Z_k}\left(\frac{X_k}{a_n}\right)^{q+1},$$

with a positive constant B_q. Evidently by the definitions,

$$\sum_{n>Z_k}\left(\frac{X_k}{a_n}\right)^{q+1} \leqq X_k^{q+1} \sum_{j\geqq 1} Z_{k+j}^{1-(q+1)/\lambda}.$$

Since

$$\log Z_{k+j} \geqq N^j \log Z_k \geqq j(N-1)\log Z_k,$$

we obtain

$$Z_{k+j}^{1-(q+1)/\lambda} \leqq (Z_k^{\gamma})^j \qquad (j=1, 2, \cdots),$$

where

$$\gamma=(N-1)\left(1-\frac{q+1}{\lambda}\right)<0.$$

Then

$$\sum_{j\geqq 1} Z_{k+j}^{1-(q+1)/\lambda} \leqq \frac{Z_k^{\gamma}}{1-Z_k^{\gamma}} \leqq 2Z_k^{\gamma}$$

for sufficiently large k. Further from

$$q+1+s\gamma<q-1,$$

we deduce

$$\sum_{n>Z_k} \log\left|E\left(\frac{X_k e^{iu}}{w_q a_n}, q\right)E\left(\frac{X_k e^{iu}}{\overline{w}_q a_n}, q\right)\right|$$

$$\leqq 4B_q X_k^{q+1} Z_k^{\gamma}$$

$$\leqq 4B_q X_k^{q-1}.$$

Therefore for every u and sufficiently large k, we obtain

$$\log|g(X_k e^{iu})| \leqq X_k^{\rho} \log(1+X_k^{2})+4B_q$$

in the case that $q=1$, and if $q\geqq 2$,

$$\log|g(X_k e^{iu})| \leqq A_q Z_k X_k^{q-1}+4B_q X_k^{q-1}$$

$$\leqq (A_q+4B_q)X_k^{\rho}.$$

Thus the lower order $\rho(g)$ satisfies

$$\rho(g) \leqq \rho < q,$$

which yields the desired result.

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