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# ON CONFORMAL DIFFEOMORPHISMS OF 4-DIMENSIONAL RIEMANNIAN MANIFOLDS

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**Introduction.** Let $M$ and $M^*$ be $n$-dimensional connected Riemannian manifolds with metric tensor fields $g$ and $g^*$ respectively, and consider a conformal diffeomorphism $f$ of $M$ into $M^*$. Then the metric tensor fields are related by

$$g^* = \frac{1}{\rho^2} g,$$

where $\rho$ is a positive-valued scalar field on $M$ and said to be associated with $f$.

In his previous paper [3], the present author proved the following theorems, the first of which is of local character and the second of global character:

THEOREM A. *Assume that $M$ and $M^*$ are Riemannian manifolds of dimension $n \geqq 4$, $M$ is the Pythagorean product of two Riemannian manifolds $M_1$ and $M_2$ of dimension $n_1$ and $n_2$ respectively, and the Ricci tensor of $M^*$ is parallel. If there is a non-homothetic conformal diffeomorphism of $M$ into $M^*$ such that the associated scalar field $\rho$ depends on both $M_1$ and $M_2$ in an open subset in $M$, then both the parts $M_1$ and $M_2$ of $M$ are Einstein manifolds, except the case $n_1 = n_2 = 2$, and the scalar curvatures $\kappa_1$ and $\kappa_2$ of the parts possess one of the following properties:*

1) *$\kappa_1 = -\kappa_2 = k$, $k$ being a non-zero constant,*
2) *$\kappa_1 = k$ and $M_2$ is one-dimensional, $\kappa_2 = 0$,*
3) *$\kappa_1 = \kappa_2 = 0$ so that $M$ is an Einstein manifold of zero scalar curvature.*

THEOREM B. *In addition to the assumptions of the theorem above, we assume that $M$ and $M^*$ are complete and $M$ is reducible in place of being the Pythagorean product. Then there exists no non-homothetic conformal diffeomorphism of $M$ onto $M^*$ such that the associated scalar field $\rho$ depends on both $M_1$ and $M_2$ in an open subset in $M$.*

The purpose of this paper is to discuss conformal diffeomorphisms in the exceptional case $n_1 = n_2 = 2$ of Theorem A and to prove the following theorem of local character:

THEOREM. *Assume that $M$ is the Pythagorean product $M_1 \times M_2$ of two-dimensional manifolds $M_1$ and $M_2$ with metric tensor $g_1$ and $g_2$ respectively and*

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*the scalar curvature $\kappa$ of $M$ is not constant and that $M^*$ is a 4-dimensional manifold with parallel Ricci tensor. If there is a conformal diffeomorphism $f$ of $M$ into $M^*$ and the associated scalar field $\rho$ depends on both the parts $M_1$ and $M_2$ in an open subset of $M$, then so does the field $\rho$ in the whole manifold $M$, and*

(1) *the scalar curvature $\kappa$ of $M$ is proportional to $\rho$ with constant coefficient, say $\kappa=C\rho$ $(C\neq 0)$,*

(2) *the associated scalar field $\rho$ is written as $\rho=\rho_1+\rho_2$, where $\rho_1$ and $\rho_2$ are scalar fields on $M_1$ and $M_2$ respectively and satisfy the equations*

$$\nabla\nabla\rho_1=(-3C\rho_1^2+B)g_1\,,\qquad \nabla\nabla\rho_2=(-3C\rho_2^2+B)g_2$$

*and hence the lengths of their gradient vectors are given by*

$$|\nabla\rho_1|^2=-2C\rho_1^3+2B\rho_1-A_1\,,\qquad |\nabla\rho_2|^2=-2C\rho_2^3+2B\rho_2-A_2\,,$$

*$\nabla$ denoting covariant differentiation and $A_1$, $A_2$ and $B$ being constants, that is, $\rho_1$ and $\rho_2$ are concircular scalar fields of elliptic type on $M_1$ and $M_2$ respectively, and*

(3) *$M^*$ is a 4-dimensional Einstein manifold with scalar curvature $\kappa^*$ equal to*

$$\kappa^*=A_1+A_2\,.$$

The arguments up to the equation (2.9) of the previous paper [3] are valid in the envisaged case of $n=4$ and $n_1=n_2=2$. Hence in § 1 we shall briefly repeat the arguments in the general case of dimension $n$ and state formulas needed later. Confining ourselves to the case $n_1=n_2=2$ in § 2, we shall prove the theorem stated above.

**§ 1. Formulas in the general case of dimension $n\geqq 4$.** With respect to a local coordinate system, we shall denote the metric tensor $g$ of $M$ by components $g_{\mu\lambda}$, the Christoffel symbols by $\left\{{\kappa \atop \mu\lambda}\right\}$, the curvature tensor by $K_{\nu\mu\lambda}{}^{\kappa}$, the Ricci tensor by $K_{\mu\lambda}$, and the scalar curvature by $\kappa$, where $\kappa$ is defined by

$$\kappa=\frac{1}{n(n-1)}K_{\mu\lambda}g^{\mu\lambda} \tag{1.1}$$

for $n\geqq 2$ and $\kappa=0$ for $n=1$, and put

$$L_{\mu\lambda}=K_{\mu\lambda}-\frac{n}{2}\kappa g_{\mu\lambda}\,. \tag{1.2}$$

Denoting quantities of $M^*$ corresponding to those of $M$ under a conformal diffeomorphism $f$ by asterisking and putting $\rho_\lambda=\nabla_\lambda\rho$, we have the transformation formulas

$$\left\{{\kappa \atop \mu\lambda}\right\}^*=\left\{{\kappa \atop \mu\lambda}\right\}-\frac{1}{\rho}(\delta_\mu^\kappa\rho_\lambda+\delta_\lambda^\kappa\rho_\mu-g_{\mu\lambda}\rho^\kappa)\,, \tag{1.3}$$

$$K^*_{\nu\mu\lambda}{}^{\kappa}=K_{\nu\mu\lambda}{}^{\kappa}+\frac{1}{\rho}(\delta_\nu{}^\kappa\nabla_\mu\rho_\lambda-\delta_\mu{}^\kappa\nabla_\nu\rho_\lambda+g_{\mu\lambda}\nabla_\nu\rho^\kappa-g_{\nu\lambda}\nabla_\mu\rho^\kappa)$$
$$-\frac{1}{\rho^2}\rho_\kappa\rho^\kappa(\delta_\nu{}^\kappa g_{\mu\lambda}-\delta_\mu{}^\kappa g_{\nu\lambda}), \tag{1.4}$$

$$K^*_{\mu\lambda}=K_{\mu\lambda}+\frac{1}{\rho}(n-2)\nabla_\mu\rho_\lambda+\frac{1}{\rho}g_{\mu\lambda}\nabla_\kappa\rho^\kappa-\frac{1}{\rho^2}(n-1)\rho_\kappa\rho^\kappa g_{\mu\lambda}, \tag{1.5}$$

$$\kappa^*=\rho^2\kappa+\frac{2}{n}\rho\nabla_\kappa\rho^\kappa-\rho_\kappa\rho^\kappa, \tag{1.6}$$

$$L^*_{\mu\lambda}=L_{\mu\lambda}+\frac{1}{\rho}(n-2)\nabla_\mu\rho_\lambda-\frac{1}{2\rho^2}(n-2)g_{\mu\lambda}\rho_\kappa\rho^\kappa, \tag{1.7}$$

and

$$\nabla^*_\nu L^*_{\mu\lambda}=\nabla_\nu L_{\mu\lambda}+\frac{1}{2\rho^2}(n-2)[\nabla_\nu\nabla_\mu\nabla_\lambda\rho^2-g_{\mu\lambda}\nabla_\nu(\rho_\kappa\rho^\kappa)$$
$$-g_{\nu\lambda}\nabla_\mu(\rho_\kappa\rho^\kappa)-g_{\nu\mu}\nabla_\lambda(\rho_\kappa\rho^\kappa)]+\frac{1}{2\rho^2}[2L_{\mu\lambda}\nabla_\kappa\rho^2$$
$$+L_{\nu\mu}\nabla_\lambda\rho^2+L_{\nu\lambda}\nabla_\mu\rho^2-(g_{\nu\lambda}L_{\mu\kappa}+g_{\nu\mu}L_{\lambda\kappa})\nabla^\kappa\rho^2], \tag{1.8}$$

where we have denoted covariant differentiation in $M$ and $M^*$ by $\nabla$ and $\nabla^*$ respectively.

If the Ricci tensor of $M^*$ is parallel, that is, $\nabla^*_\nu K^*_{\mu\lambda}=0$, then the scalar curvature $\kappa^*$ is constant, the tensor $L^*_{\mu\lambda}$ is also parallel, and the tensor $L_{\mu\lambda}$ of $M$ satisfies the equation

$$2\rho^2\nabla_\nu L_{\mu\lambda}+(n-2)\nabla_\nu\nabla_\mu\nabla_\lambda\rho^3$$
$$=(n-2)[g_{\mu\lambda}\nabla_\nu(\rho_\kappa\rho^\kappa)+g_{\nu\lambda}\nabla_\mu(\rho_\kappa\rho^\kappa)+g_{\nu\mu}\nabla_\lambda(\rho_\kappa\rho^\kappa)]$$
$$-[2L_{\mu\lambda}\nabla_\nu\rho^2+L_{\nu\lambda}\nabla_\mu\rho^2+L_{\nu\mu}\nabla_\lambda\rho^2-(g_{\nu\lambda}L_{\mu\kappa}+g_{\nu\mu}L_{\lambda\kappa})\nabla^\kappa\rho^2]. \tag{1.9}$$

Applying Ricci's formula to $\nabla_\nu\nabla_\mu\nabla_\lambda\rho^2$ in this equation, we obtain the equation

$$\rho(\nabla_\nu L_{\mu\lambda}-\nabla_\mu L_{\nu\lambda})-(n-2)K_{\nu\mu\lambda}{}^\kappa\rho_\kappa$$
$$=L_{\nu\lambda}\rho_\mu-L_{\mu\lambda}\rho_\nu+(g_{\nu\lambda}L_{\mu\kappa}-g_{\mu\lambda}L_{\nu\kappa})\rho^\kappa. \tag{1.10}$$

Now we suppose that the manifold $M$ is the Pythagorean product $M_1\times M_2$ of two Riemannian manifolds $M_1$ and $M_2$, and the dimensions are $n_1$ and $n_2$ respectively, $n=n_1+n_2$. The manifolds $M_1$ and $M_2$ are called *parts* of $M$. Let $(x^h, x^p)$ be a separate coordinate system of $M$, such that $(x^h)$ and $(x^p)$ are local coordinate systems of the parts $M_1$ and $M_2$ respectively ($h, i, j, k=1, 2, \cdots, n_1$; $p, q=n_1+1, \cdots, n$). In such a system, the metric tensor $g=(g_{\mu\lambda})$ is represented as the direct sum of the metrices $g_1=(g_{ji})$ of $M_1$ and $g_2=(g_{qp})$ of $M_2$. The Christoffel symbols, the curvature tensor and the Ricci tensor have pure components only. The parts $\nabla_j$ and $\nabla_q$ of the covariant differentiation $\nabla$ in $M$ coincide with the covariant differentiations in the parts $M_1$ and $M_2$, and commute with each other.

The scalar curvatures $\kappa_1$ of $M_1$ and $\kappa_2$ of $M_2$ satisfy the relation

$$n_1(n_1-1)\kappa_1+n_2(n_2-1)\kappa_2=n(n-1)\kappa\,, \tag{1.11}$$

$\kappa$ being the scalar curvature of $M$, even if $n_1=1$ or $n_2=1$. Since the scalar curvature $\kappa$ depends in general on $M_1$ and $M_2$, the covariant derivative $\nabla_\nu L_{\mu\lambda}$ has hybrid components

$$\nabla_q L_{ji}=-\frac{n}{2}g_{ji}\nabla_q\kappa\,,\qquad \nabla_j L_{qp}=-\frac{n}{2}g_{qp}\nabla_j\kappa \tag{1.12}$$

besides pure components.

Putting the indices $\lambda=i$, $\mu=j$, $\nu=q$ and $\lambda=p$, $\mu=q$, $\nu=j$ in the equation (1.10) referred to a separate coordinate system, we obtain

$$\begin{aligned} L_{ji}\rho_q&=\Big(\frac{n}{2}\rho\nabla_q\kappa-L_{qp}\rho^p\Big)g_{ji}\,,\\ L_{pq}\rho_j&=\Big(\frac{n}{2}\rho\nabla_j\kappa-L_{ji}\rho^i\Big)g_{qp}\,. \end{aligned} \tag{1.13}$$

If the associated scalar field $\rho$ depends on both $M_1$ and $M_2$, $\rho_j\neq0$ and $\rho_q\neq0$, in an open subset $U$ of $M$, then we may put

$$L_{ji}=\lambda_1 g_{ji}\,,\qquad L_{qp}=\lambda_2 g_{qp} \tag{1.14}$$

in $U$, where $\lambda_1$ and $\lambda_2$ are proportional factors. The definition (1.2) of $L_{\mu\lambda}$ implies

$$K_{ji}=\Big(\frac{n}{2}\kappa+\lambda_1\Big)g_{ji}\,,\qquad K_{qp}=\Big(\frac{n}{2}\kappa+\lambda_2\Big)g_{qp}\,, \tag{1.15}$$

and, by contraction of these equations, we see that the scalar curvatures $\kappa_1$ and $\kappa_2$ satisfy the relations

$$(n_1-1)\kappa_1=\frac{n}{2}\kappa+\lambda_1\,,\qquad (n_2-1)\kappa_2=\frac{n}{2}\kappa+\lambda_2\,. \tag{1.16}$$

Substituting (1.14) into (1.13), we have

$$(\lambda_1+\lambda_2)\rho_j=\frac{n}{2}\rho\nabla_j\kappa\,,\qquad (\lambda_1+\lambda_2)\rho_q=\frac{n}{2}\rho\nabla_q\kappa \tag{1.17}$$

or the tensor equation

$$(\lambda_1+\lambda_2)\rho_\mu=\frac{n}{2}\rho\nabla_\mu\kappa\,. \tag{1.18}$$

§ 2. **Proof of the theorem.** Now we suppose that $n=4$ and $n_1=n_2=2$. Let $U$ be an open subset of $M$ in which $\rho_j\neq0$ and $\rho_q\neq0$. We shall first show that the theorem is valid in the subset $U$.

The relations (1.11) and (1.16) reduce to

(2.1) $$\kappa_1+\kappa_2=6\kappa$$

and

(2.2) $$\kappa_1=2\kappa+\lambda_1\,, \qquad \kappa_2=2\kappa+\lambda_2\,.$$

respectively. Hence we have

(2.3) $$\lambda_1+\lambda_2=2\kappa\,,$$

and the equation (1.18) turns out to be

(2.4) $$\kappa\rho_\mu=\rho\nabla_\mu\kappa\,,$$

from which we may put

(2.5) $$\kappa=C\rho\,,$$

$C$ being a constant. This is the part (1) of the theorem.

If the scalar curvature $\kappa$ of $M$ is constant, then so are the curvatures $\kappa_1$ and $\kappa_2$, and $\kappa=0$ by virture of (2.4). Therefore the parts $M_1$ and $M_2$ are two-dimensional manifolds of constant curvature with reversed sign. This is the case 1) or 3) treated in Theorem A, and will be excluded from our present consideration, and we shall suppose $C\neq 0$ from now on.

Now, by virtue of the equation (2.5), we may put

(2.6) $$\rho=\rho_1+\rho_2\,,$$

where $\rho_1$ and $\rho_2$ are scalar fields depending only on $M_1$ and $M_2$ respectively. It follows then from (2.1) and (2.5) that

(2.7) $$\kappa_1=6C\rho_1\,, \qquad \kappa_2=6C\rho_2$$

and from (2.2) that

(2.8) $$\lambda_1=C(4\rho_1-2\rho_2)\,, \qquad \lambda_2=C(4\rho_2-2\rho_1)\,.$$

The equation (1.9) for $n=4$ is rewritten in the form

(2.9) $$\begin{aligned}2\nabla_\nu\nabla_\mu\nabla_\lambda\rho^2+2\nabla_\nu(\rho^2L_{\mu\lambda})=&2[g_{\mu\lambda}\nabla_\nu(\rho_\kappa\rho^\kappa)+g_{\nu\lambda}\nabla_\mu(\rho_\kappa\rho^\kappa)+g_{\nu\mu}\nabla_\lambda(\rho_\kappa\rho^\kappa)]\\&-[L_{\nu\lambda}\nabla_\mu\rho^2+L_{\nu\mu}\nabla_\lambda\rho^2-(g_{\nu\lambda}L_{\mu\kappa}+g_{\nu\mu}L_{\lambda\kappa})\nabla^\kappa\rho^2]\,.\end{aligned}$$

Referring this equation to a separate coordinate system, putting $\lambda=i$, $\mu=j$, $\nu=p$ and $\lambda=p$, $\mu=q$, $\nu=\imath$ and substituting (1.14), we have the equations

(2.10) $$\begin{aligned}\nabla_p\nabla_j\nabla_i\rho^2&=g_{ji}\nabla_p(\rho_\kappa\rho^\kappa-\rho^2\lambda_1)\,,\\ \nabla_i\nabla_q\nabla_p\rho^2&=g_{qp}\nabla_i(\rho_\kappa\rho^\kappa-\rho^2\lambda_2)\,.\end{aligned}$$

Applying $\nabla_q$ to the first equation and $\nabla_j$ to the second and comparing the results, we have

$$g_{ji}\nabla_q\nabla_p(\rho_\kappa\rho^\kappa-\rho^2\lambda_1)=g_{qp}\nabla_j\nabla_i(\rho_\kappa\rho^\kappa-\rho^2\lambda_2)\,.$$

Therefore we may put

$$\nabla_j \nabla_i(\rho_\kappa\rho^\kappa-\rho^2\lambda_2)=\psi g_{ji}\,, \qquad \nabla_q \nabla_p(\rho_\kappa\rho^\kappa-\rho^2\lambda_1)=\psi g_{qp}\,. \tag{2.11}$$

Substituting these into the covariant derivatives of the equations (2.10), we have

$$\nabla_q \nabla_p \nabla_j \nabla_i \rho^2=\psi g_{qp} g_{ji}\,. \tag{2.12}$$

On the other hand, the successive derivatives of $\rho^2$ are given by

$$\begin{aligned} \nabla_i\rho^2&=2\rho\nabla_i\rho_1\,,\\ \nabla_j\nabla_i\rho^2&=2(\rho\nabla_j\nabla_i\rho_1+\nabla_j\rho_1\nabla_i\rho_1)\,,\\ \nabla_p\nabla_j\nabla_i\rho^2&=2(\nabla_p\rho_2)\nabla_j\nabla_i\rho_1\,,\\ \nabla_q\nabla_p\nabla_j\nabla_i\rho^2&=2(\nabla_q\nabla_p\rho_2)\nabla_j\nabla_i\rho_1\,. \end{aligned} \tag{2.13}$$

Comparing the fourth of (2.13) with (2.12), we may put

$$\nabla_j\nabla_i\rho_1=\psi_1 g_{ji}\,, \qquad \nabla_q\nabla_p\rho_2=\psi_2 g_{qp}\,, \tag{2.14}$$

where $\psi_1$ and $\psi_2$ are functions on $M_1$ and $M_2$ respectively and satisfy the relation $2\psi_1\psi_2=\psi$. Substituting (2.14) into the second and the third equations of (2.13), we have

$$\nabla_j\nabla_i\rho^2=2(\rho\psi_1 g_{ji}+\nabla_j\rho_1\nabla_i\rho_1)\,, \tag{2.15}$$

$$\nabla_p\nabla_j\nabla_i\rho^2=2\psi_1 g_{ji}\nabla_p\rho^2\,. \tag{2.16}$$

Since

$$\rho_\kappa\rho^\kappa=(\nabla_i\rho_1)(\nabla^i\rho_1)+(\nabla_p\rho_2)(\nabla^p\rho_2)\,, \tag{2.17}$$

we have

$$\nabla_i(\rho_\kappa\rho^\kappa)=2\psi_1\nabla_i\rho_1\,, \qquad \nabla_p(\rho_\kappa\rho^\kappa)=2\psi_2\nabla_p\rho_2\,. \tag{2.18}$$

Referring the equation (2.9) to a separate coordinate system, putting $\lambda=i$, $\mu=p$, $\nu=j$ and $\lambda=p$, $\mu=i$, $\nu=q$ and substituting (1.14), we have

$$\nabla_j\nabla_p\nabla_i\rho^2=g_{ji}\nabla_p(\rho_\kappa\rho^\kappa)-(\lambda_1-\lambda_2)g_{ji}\rho\nabla_p\rho_2\,,$$

$$\nabla_q\nabla_i\nabla_p\rho^2=g_{qp}\nabla_i(\rho_\kappa\rho^\kappa)-(\lambda_2-\lambda_1)g_{qp}\rho\nabla_i\rho_1\,.$$

Substituting (2.16) and the similar equation of $\nabla_i\nabla_q\nabla_p\rho^2$ and (2.18) into one of these equations, we can obtain

$$\psi_1-\psi_2=-\frac{1}{2}(\lambda_1-\lambda_2)\rho$$

and, substituting (2.6) and (2.8) into this relation,

$$\psi_1-\psi_2=-3C(\rho_1-\rho_2)(\rho_1+\rho_2)=-3C(\rho_1^2-\rho_2^2)\,.$$

Since $\phi_1$ and $\phi_2$ are functions on $M_1$ and $M_2$ respectively, these functions may be expressed as

$$\phi_1=-3C\rho_1^2+B\,, \qquad \phi_2=-C\rho_2^2+B\,, \tag{2.19}$$

$B$ being a constant.

Applying $\nabla_k$ to the equation (2.15), we have

$$\nabla_k\nabla_j\nabla_i\rho^2=2(\rho g_{ji}\nabla_k\phi_1+\phi_1 g_{ji}\nabla_k\rho_1+\phi_1 g_{kj}\nabla_i\rho_1+\phi_1 g_{ki}\nabla_j\rho_1)\,. \tag{2.20}$$

On the other hand, putting $\lambda=\iota$, $\mu=j$, $\nu=k$ in (2.9), we have

$$\nabla_k\nabla_j\nabla_i\rho^2+\nabla_k(\rho^2\lambda_1)g_{ji}=g_{ji}\nabla_k(\rho_\kappa\rho^\kappa)+g_{ki}\nabla_j(\rho_\kappa\rho^\kappa)+g_{kj}\nabla_i(\rho_\kappa\rho^\kappa)\,.$$

Substituting (2.18) and (2.20) into this equation, we have

$$\nabla_k(\rho^2\lambda_1)+2\rho\nabla_k\phi_1=\rho[2\lambda_1\nabla_k\rho_1+\rho\nabla_k\lambda_1+2\nabla_k\phi_1]=0\,.$$

However it is verified that this equation is satisfied by means of (2.8) and (2.19), that is, the equation (2.9) referred to $M_1$ implies no further condition for $\rho$.

Thus we have seen that the scalar fields $\rho_1$ and $\rho_2$ satisfy the equations

$$\begin{aligned}\nabla_j\nabla_i\rho_1&=(-3C\rho_1^2+B)g_{ji}\,,\\ \nabla_q\nabla_p\rho_2&=(-3C\rho_2^2+B)g_{qp}\end{aligned} \tag{2.21}$$

respectively, that is, they are concircular on $M_1$ and $M_2$. Transvecting the equations (2.21) with $\nabla^i\rho_1$ and $\nabla^p\rho_2$ respectively, we have

$$\nabla_j\{(\nabla_i\rho_1)(\nabla^i\rho_1)\}=2(-3C\rho_1^2+B)\nabla_j\rho_1\,,$$

$$\nabla_q\{(\nabla_p\rho_2)(\nabla^p\rho_2)\}=2(-3C\rho_2^2+B)\nabla_q\rho_2\,,$$

and, integrating these equations, we find

$$\begin{aligned}(\nabla_i\rho_1)(\nabla^i\rho_1)&=-2C\rho_1^3+2B\rho_1-A_1\,,\\ (\nabla_p\rho_2)(\nabla^p\rho_2)&=-2C\rho_2^3+2B\rho_2-A_2\,,\end{aligned} \tag{2.22}$$

where $A_1$ and $A_2$ are constants. This is the part (2) of the theorem.

Substituting the expressions (2.22) into (2.17), we have

$$\begin{aligned}\rho_\kappa\rho^\kappa&=-2C(\rho_1^3+\rho_2^3)+2B(\rho_1+\rho_2)-A_1-A_2\\ &=-2C\rho(\rho_1^2-\rho_1\rho_2+\rho_2^2)+2B\rho-(A_1+A_2)\end{aligned} \tag{2.23}$$

and, from (2.21),

$$\nabla_\kappa\rho^\kappa=-6C(\rho_1^2+\rho_2^2)+4B\,. \tag{2.24}$$

Substituting (2.5), (2.23) and (2.24) into the equation (1.6) for $n=4$, we can see by direct computation that the scalar curvature $\kappa^*$ of $M^*$ is equal to

$$\kappa^*=A_1+A_2\,. \tag{2.25}$$

Moreover, since $M_1$ and $M_2$ are two-dimensional and consequently Einstein manifolds and the scalar curvatures $\kappa_1$ and $\kappa_2$ are given by (2.7), we have

$$K_{ji}=6C\rho_1 g_{ji}, \qquad K_{qp}=6C\rho_2 g_{qp}. \tag{2.26}$$

Then, referring the formula (1.5) to a separate coordinate system and using (2.21), (2.23) and (2.24), we find that the components $K^*_{ji}$ of the Ricci tensor $K^*_{\mu\lambda}$ of $M^*$ are equal to

$$\begin{aligned} K^*_{ji} &= 6C\rho_1 g_{ji}+\frac{2}{\rho}\nabla_j\rho_i+\frac{1}{\rho}g_{ji}\nabla_\kappa\rho^\kappa-\frac{3}{\rho^2}g_{ji}\rho_\kappa\rho^\kappa \\ &= [6C\rho^2\rho_1+2\rho(-3C\rho_1^2+B)-6C\rho(\rho_1^2+\rho_2^2)+4B\rho\,, \\ &\quad +6C\rho(\rho_1^2-\rho_1\rho_2+\rho_2^2)-6B\rho+3(A_1+A_2)]\frac{1}{\rho^2}g_{ji} \end{aligned}$$

or

$$K^*_{ji}=3(A_1+A_2)g^*_{ji},$$

and the components $K^*_{pq}$ are equal to similar expressions, and we have

$$K^*_{ji}=3\kappa^* g^*_{ji}, \qquad K^*_q=3\kappa^* g^*_p.$$

Noting $\nabla_p\rho_i=0$, we see $K^*_{pi}=0$ from (1.5) and hence obtain the tensor equation

$$K^*_{\mu\lambda}=3\kappa^* g^*_{\mu\lambda},$$

which means that $M^*$ is a 4-dimensional Einstein manifold of scalar curvature $\kappa^*=A_1+A_2$. This is the part (3) of the theorem and the theorem is valid in the subset $U$.

Denote the parts through a point $P$ by $M_1(P)$ and $M_2(P)$. Let $P$ be a point of the subset $U$, $Y(P)$ an arbitrary vector at $P$ tangent to $M_2(P)$ and $Y$ the natural extension of $Y(P)$ on $M_1(P)$. The intersection $M_1(P)\cap U$ is relatively open in $M_1(P)$ and the equation (2.16) means that the derivative $Y\rho^2=Y^p\nabla_p\rho^2$ along the direction $Y$ is a concircular scalar field in $M_1(P)\cap U$. If the complement $M_1(P)-U$ contained inner points, we would have $Y\rho^2=2\rho Y\rho=0$ in $M_1(P)-U$ and $Y\rho^2$ itself would be a concircular scalar field on $M_1(P)$ by continuity. Since the stationary point of a concircular scalar field is isolated [1], [2, p. 15], the point where $\nabla_i(Y\rho^2)=0$ is isolated in $M_1(P)$ unless it vanishes identically. However we would have $\nabla_i(Y\rho^2)=0$ in $M_1(P)-U$; this is a contradiction. Therefore the closure of the subset $M_1(P)\cap U$ coincides with the parts $M_1(P)$ and all the equations in the above proof are valid in $M_1(P)$ and similarly in $M_2(P)$ for points $P\in U$.

It follows from the equations (2.7) that the scalar curvatures $\kappa_1$ and $\kappa_2$ are also concircular in $M_1(P)\cap U$ and $M_2(P)\cap U$ respectively. On the other hand, it follows from (1.13) that $\kappa$ and $\kappa_2$ are independent of points of $M_1$ in $M-U$ or $\kappa$ and $\kappa_2$ are independent of points of $M_2$ in $M-U$. The above arguments on $\rho$ in the parts $M_1(P)$ and $M_2(P)$ through a point $P\in U$ are also applicable to the scalar curvatures $\kappa_1$ and $\kappa_2$ on the parts $M_1$ and $M_2$. Therefore the closure of

$U$ coincides with the manifold $M$. Thus the proof of the theorem has been completed.

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