S. KOBAYASHI
KŌDAI MATH. SEM. REP.
27 (1976), 291—299

SCHWARZ'S LEMMA IN H_p SPACES

BY SHŌJI KOBAYASHI

§ 1. Introduction.

Let R be a Riemann surface and let $t \in R$ be any fixed point. For $0<p<\infty$, let $H_p(R)$ denote the class of all functions f analytic on R for which the subharmonic function $|f|^p$ has a harmonic majorant. We put for any $f \in H_p(R)$

$$\|f\|_p=(u(t))^{1/p}, \tag{1}$$

where u is the least harmonic majorant of $|f|^p$ on R. Then, for $1 \leqq p<\infty$, $H_p(R)$ is a Banach space with the norm $\| \ \|_p$, and for $0<p<1$, $H_p(R)$ is not a Banach space but a Fréchet space with the metric $d(\ , \ )$ defined by $d(f, g)=\|f-g\|_p^p$ $(f, g \in H_p(R))$. Although the "norm" $\| \ \|_p$ defined by (1) depends on the choice of t, the induced topology does not ([11]). Let $H_\infty(R)$ be the Banach algebra of all functions which are analytic and bounded on R, with the uniform norm $\| \ \|_\infty$. These H_p spaces, which generalize the classical Hardy classes in the unit disc, were introduced by Parreau [10] and independently by Rudin [11].

In this paper we are concerned with the problem of maximizing $|f'(t)|$ under the restrictions $f \in H_p(R)$, $f(t)=0$ and $\|f\|_p \leqq 1$. Let H_p^0 denote the class which consists of all $f \in H_p(R)$ such that $f(t)=0$ and $\|f\|_p \leqq 1$. We put for $0<p \leqq \infty$

$$\alpha_p=\sup_{f \in H_p^0} |f'(t)| .$$

We shall investigate some properties of α_p as a function of p on $(0, \infty]$. It is easily shown by the normal family argument that there exists a function $f \in H_p^0$ for which $f'(t)=\alpha_p$. Such a function is called an extremal function for H_p^0 and denoted by f_p. If $1<p<\infty$, then the uniform convexity of $H_p(R)$ implies that f_p is unique for any Riemann surface. It is well known that for any plane region there is a unique extremal function f_∞ for H_∞^0 ([5]). In this paper we shall also investigate the convergence of f_p as p approaches to some p_0 with $1<p_0 \leqq \infty$. In Section 5, we shall consider another extremal problem similar to the above one.

I would like to express my deep gratitude to Professor N. Suita for his constant encouragement and helpful suggestions.

Received May 4, 1974.

§ 2. Some lemmas on L_p.

We begin with three lemmas on L_p spaces. Lemma 1 is easily proved by applying Holder's inequality, and Lemma 2 by Fatou's lemma. Lemma 3, which is a generalization of Clarkson's result, is proved by the same method as his ([2], p. 403).

LEMMA 1. *Let $(X, d\mu)$ be a measure space with total mass 1. If $0<r<s\leqq\infty$ and $f\in L_s(d\mu)$, then $f\in L_r(d\mu)$ and*

$$\|f\|_r\leqq\|f\|_s. \tag{2}$$

Equality holds in (2) *if and only if* $|f|=$const. a. e. *on* X.

LEMMA 2. *Let $(X, d\mu)$ be a measure space with tatal mass 1, and let $0<p\leqq\infty$. If $f\in L_q(d\mu)$ for all $q<p$, then*

$$\|f\|_p=\lim_{q\uparrow p}\|f\|_q. \tag{3}$$

In the case that $f\notin L_p(d\mu)$, the left side of (3) *should be interpreted as $+\infty$.*

LEMMA 3. *Let $(X, d\mu)$ be a measure space and let $1<a<b<\infty$. Then, for any positive number ε, there exists a positive number δ such that if $a\leqq p\leqq b$, $f, g\in L_p(d\mu)$, $\|f\|_p$, $\|g\|_p\leqq 1$ and $\|1/2(f+g)\|_p\geqq 1-\delta$, then $\|f-g\|_p<\varepsilon$.*

Remark. Since we can regard $H_p(R)$ as a subspace of $L_p(C, (1/2\pi)d\theta)$, where C is the unit circle and $(1/2\pi)d\theta$ is the normalized Lebesgue measure on C ([11], p. 51), the above three lemmas are also valid for $H_p(R)$.

§ 3. Continuity of α_p and convergence of f_p.

THEOREM 1. *α_p is nonincreasing and left-continuous on $(0, \infty]$.*

We need a lemma.

LEMMA 4. *Let $0<p\leqq\infty$ and $g_k\in H_p(R)$ for $k=1, 2, \cdots$. If g_k converges to some g uniformly on every compact subset of R, then*

$$\|g\|_p\leqq\varliminf_{k\to\infty}\|g_k\|_p. \tag{4}$$

In the case that $g\notin H_p(R)$, the left side of (4) *should be interpreted as $+\infty$.*

Proof. Let $\{R_m\}$ be a regular exhaustion of R such that $t\in R_1$. Let μ_m denote the harmonic measure for t on the boundary ∂R_m of R_m. It is known that for $0<p<\infty$ and for any function f analytic on R

$$\|f\|_p=\lim_{m\to\infty}\left(\int_{\partial R_m}|f|^p d\mu_m\right)^{1/p}, \tag{5}$$

where the sequence of the right side is nondecreasing in m and the limit does not depend on the choice of $\{R_m\}$. In the case that $f\notin H_p(R)$, the left side of (5) should be interpreted as $+\infty$ ([10], p. 137).

If $p<\infty$, then we see by (5)

$$\|g\|_p=\lim_{m\to\infty}\Big(\int_{\partial R_m}|g|^p d\mu_m\Big)^{1/p}$$

$$=\lim_{m\to\infty}\lim_{k\to\infty}\Big(\int_{\partial R_m}|g_k|^p d\mu_m\Big)^{1/p}$$

$$\leqq\varliminf_{k\to\infty}\|g_k\|_p\,.$$

If $p=\infty$, then the assertion of the lemma is almost trivial.

Proof of Theorem 1. It is evident from Lemma 1 that α_p is nonincreasing on $(0,\infty]$. Let $p_0\in(0,\infty]$. Since $\{f_p\,;\,c<p<p_0\}$ forms a normal family, where $0<c<p_0$, we can choose a sequence $\{p_k\}$ which converges increasingly to p_0 so that f_{p_k} converges to some g uniformly on every compact subset of R as $k\to\infty$. It is easily shown that $g(t)=0$ and $g'(t)\geqq\alpha_{p_0}$. By Lemma 4, we see

$$\|g\|_p\leqq\varliminf_{k\to\infty}\|f_{p_k}\|_p\leqq\varliminf_{k\to\infty}\|f_{p_k}\|_{p_k}=1\,,$$

for any $p<p_0$. Then, by Lemma 2, we see that $g\in H^0_{p_0}$, and hence g is an extremal function for $H^0_{p_0}$ and $\lim_{p\uparrow p_0}\alpha_p=\alpha_{p_0}$.

COROLLARY 1. *If* $1<p_0<\infty$, *then* f_p *converges to* f_{p_0} *uniformly on every compact subset of* R *as* $p\uparrow p_0$.

THEOREM 2. *If* $1<p_0<\infty$, *then* $\lim_{p\uparrow p_0}\|f_p-f_{p_0}\|_p=0$.

Proof. Let $a=1/2(p_0+1)$ and $b=p_0$. Let ε be any positive number. Applying Lemma 3, we can find a positive number δ such that if $a\leqq p\leqq b$, $f, g\in H_p(R)$, $\|f\|_p, \|g\|_p\leqq 1$ and $\|1/2(f+g)\|_p\geqq 1-\delta$, then $\|f-g\|_p<\varepsilon$. Since f_p converges to f_{p_0} uniformly on every compact subset of R as $p\uparrow p_0$ by Corollary 1, we see by (5) and Fatou's lemma

$$\varliminf_{p\uparrow p_0}\Big\|\frac{1}{2}(f_p+f_{p_0})\Big\|_p\geqq\varliminf_{p\uparrow p_0}\Big(\int_{\partial R_m}\Big|\frac{1}{2}(f_p+f_{p_0})\Big|^p d\mu_m\Big)^{1/p}$$

$$\geqq\Big(\int_{\partial R_m}|f_{p_0}|^{p_0}d\mu_m\Big)^{1/p_0},$$

for any m. Letting $m\to\infty$, we have

$$\varliminf_{p\uparrow p_0}\Big\|\frac{1}{2}(f_p+f_{p_0})\Big\|_p\geqq\|f_{p_0}\|_{p_0}=1\,.$$

Therefore we can find p_1 such that $\|1/2(f_p+f_{p_0})\|_p\geqq 1-\delta$ for $p_1\leqq p<p_0$. Thus we have $\|f_p-f_{p_0}\|_p<\varepsilon$.

THEOREM 3. *If* $1<p_0<\infty$, *then the following conditions are equivalent:*
(a) α_p *is continuous at* p_0.
(b) $\lim_{p\downarrow p_0}\|f_p-f_{p_0}\|_{p_0}=0$.

Proof. Suppose that (a) holds. Since $\{f_p\,;\,p>p_0\}$ forms a normal family, we can find a sequence $\{p_k\}$ which converges decreasingly to p_0 so that f_{p_k} converges to some g uniformly on every compact subset of R. Applying Lemma 1 and 4, we see $\|g\|_{p_0}\leqq 1$. Since α_p is continuous at p_0, we see $g'(t)=\alpha_{p_0}$. Then the uniqueness of f_{p_0} implies that $g=f_{p_0}$ and that f_p converges to f_{p_0} uniformly on every compact subset of R as $p\downarrow p_0$. Applying Lemma 4, we see

$$\varliminf_{p\downarrow p_0}\left\|\frac{1}{2}(f_p+f_{p_0})\right\|_{p_0}\geqq\|f_{p_0}\|_{p_0}=1\,,$$

and hence the uniform convexity of $H_{p_0}(R)$ implies (b).

Next we assume that (b) holds. Then we see

$$\lim_{p\downarrow p_0}\alpha_p=\lim_{p\downarrow p_0}f'_p(t)=f'_{p_0}(t)=\alpha_{p_0}\,.$$

THEOREM 4. *Let* $0<p_0<\infty$. *If* $H_p(R)$ *is dense in* $H_{p_0}(R)$ *for some* p *with* $p_0<p\leqq\infty$, *then* α_p *is continuous at* p_0.

Proof. Let $\{g_k\}$ be a sequence of functions in $H_p(R)$ such that $\lim_{k\to\infty}\|g_k-f_{p_0}\|_{p_0}=0$. Since $\alpha_r\geqq|g'_k(t)|/\|g_k-g_k(t)\|_r$ for any r with $p_0<r\leqq p$, we see

$$\lim_{r\downarrow p_0}\alpha_r\geqq\lim_{r\downarrow p_0}|g'_k(t)|/\|g_k-g_k(t)\|_r=|g'_k(t)|/\|g_k-g_k(t)\|_{p_0}\,,$$

for any k. Letting $k\to\infty$, we have $\lim_{r\downarrow p_0}\alpha_r\geqq\alpha_{p_0}$.

COROLLARY 2. *If* D *is a regular region* (i.e. *D is bounded by a finite number of disjoint analytic simple closed curves*) *in the extended complex plane, then* α_p *is continuous on* $(0,\infty]$.

Proof. By Lemma 3.4 of Rudin's paper ([11], p. 57), $H_\infty(D)$ is dense in $H_p(D)$ for any $p\in(0,\infty]$.

Remark. It is known that for $0<p<\infty$

$$O_p\subsetneqq\bigcap_{q>p}O_q\,,$$

where O_p denotes the class of all Riemann surfaces R for which $H_p(R)$ contains no functions but the constants ([7], p. 34). Therefore we see that there is a Riemann surface for which α_p is not necessarily continuous.

THEOREM 5. *If* D *is a regular region in the extended complex plane, then* $\lim_{r\to\infty}\|f_r-f_\infty\|_p=0$ *for any* p *with* $0<p<\infty$.

Proof. Since f_∞ is analytic on $\bar{D}$ and $|f_\infty|=1$ on ∂D ([1], [5], [6]), we see that $\|f_\infty\|_p=1$ for $0<p<\infty$. Then, applying Lemma 1 and 4, we have $\lim_{r\to\infty}\|f_r\|_p=1$ for $0<p<\infty$. Since we may assume $1<p<\infty$, the uniform convexity of $H_p(D)$ implies $\lim_{r\to\infty}\|f_r-f_\infty\|_p=0$.

Remark. We do not know whether Theorem 5 is valid or not for general regions.

§ 4. Condition for $\alpha_1=\alpha_\infty$.

THEOREM 6. *If there are r and s such that $0<r<s\leqq\infty$ and $\alpha_r=\alpha_s$, then $\alpha_r=\alpha_\infty$.*

Proof. Since $f_s\in H_r^0$ by Lemma 1, we see

$$\alpha_s=\alpha_r\geqq f_s'(t)=\alpha_s\,.$$

Thus, again by Lemma 1, we have $f_s\in H_\infty^0$, and hence $\alpha_r=\alpha_\infty$.

THEOREM 7. *Let $1<p<\infty$. If $H_p(R)$ is dense in $H_1(R)$ and if $\alpha_p=\alpha_r$ for some r with $p<r\leqq\infty$, then $\alpha_1=\alpha_\infty$.*

Proof. We can regard $H_p(R)$ as a subspace of $L_p(C, (1/2\pi)d\theta)$ as we stated in the remark after Lemma 3. By Hahn-Banach theorem and the conjugate relation between L_p and L_q, where $1/p+1/q=1$, we can find a function $g\in L_q(C, (1/2\pi)d\theta)$ such that $\|g\|_q=\alpha_p$ and

$$f'(t)=\frac{1}{2\pi}\int_C f(e^{i\theta})g(e^{i\theta})d\theta$$

for any $f\in H_p^0$. Applying Lemma 1, we see that $|g|=\alpha_p$ a.e. on C. Since there are $g_k\in H_p(R)$, $k=1, 2, \cdots$, such that $\lim_{k\to\infty}\|g_k-f_1\|_1=0$,

$$\alpha_1=f_1'(t)=\lim_{k\to\infty} g_k'(t)=\lim_{k\to\infty}\frac{1}{2\pi}\int_C g_k(e^{i\theta})g(e^{i\theta})d\theta$$
$$\leqq\lim_{k\to\infty}\alpha_p\|g_k\|_1=\alpha_p\,.$$

Hence, by Theorem 6, we obtain $\alpha_1=\alpha_\infty$.

COROLLARY 3. *Let D be a regular region in the extended complex plane. If there are r and s such that $0<r<s\leqq\infty$ and $\alpha_r=\alpha_s$, then $\alpha_1=\alpha_\infty$.*

Remark. It is known that for $0<p<\infty$

$$\bigcup_{q<p} O_q\subsetneqq O_p\,,$$

where O_p is as we stated in the remark after Theorem 4 ([7], p. 34). Then we see that there is a Riemann surface for which $\alpha_r=0$ if $p\leqq r\leqq\infty$ but $\alpha_r>0$ if $0<r<p$.

§ 5. Another extremal problem.

In this section we consider a similar extremal problem without the restriction $f(t)=0$, that is, consider the problem of maximizing $|f'(t)|$ under the restrictions $f\in H_p(R)$ and $\|f\|_p\leqq 1$. Let H_p^1 denote the unit ball of $H_p(R)$, and we put

$$\beta_p=\sup_{f\in H_p^1}|f'(t)|$$

for $0<p\leqq\infty$. A function $f\in H_p^1$ for which $f'(t)=\beta_p$ (such a function always exists) is called an extremal function for H_p^1. It is evident that $\alpha_p\leqq\beta_p$ for $0<p\leqq\infty$, and it is well known that $\alpha_\infty=\beta_\infty$ ([1]).

We can prove by a similar way the same propositions for this extremal problem as all the theorems and the corollaries before mentioned.

LEMMA 5. $\alpha_2=\beta_2$.

Proof. Let f be the extremal function for H_2^1, and we put $g(z)=f(z)-c$, where $c=f(t)$. Then we see by (5)

$$\begin{aligned}\|g\|_2^2&=\lim_{m\to\infty}\int_{\partial R_m}|g|^2d\mu_m\\ &=\lim_{m\to\infty}\int_{\partial R_m}(|f|^2-\bar{c}f-c\bar{f}+|c|^2)d\mu_m\\ &=\|f\|_2^2-|c|^2\leqq 1,\end{aligned}$$

and hence $\alpha_2=\beta_2$.

Combining Corollary 3, $\alpha_\infty=\beta_\infty$ and Lemma 5, we have the following theorem:

THEOREM 8. *Let D be a regular region in the extended complex plane. Then the following conditions are equivalent:*

(a) $\alpha_1=\alpha_\infty$.

(b) *There are r and s such that $0<r<s\leqq\infty$ and $\alpha_r=\alpha_s$.*

(c) $\beta_1=\beta_\infty$.

(d) *There are r and s such that $0<r<s\leqq\infty$ and $\beta_r=\beta_s$.*

Remark. By Rudin's result ([11], p. 63), the conditions of Theorem 8 are also equivalent to the following condition:

(e) *The critical points of Green's function $G(z,t)$ for D, with pole at t, coincide, including multiplicity, with the zeros of f_∞ except t.*

He also showed that for any ring domain D there is a point $t\in D$ for which $\alpha_1=\alpha_\infty$. And he posed a problem whether there is such a point, if the connectivity of D is greater than 2 ([11], p. 64). The following example, which was given by the author and Suita [9], partially presents an affirmative answer to the problem.

Example. Let k be any positive integer and let

$$E_j=\{z\,;\ |z-e^{i(2\pi j/k)}|\leqq\varepsilon\},$$

for $j=0,1,\cdots,k-1$, where ε is such a small positive number that E_j are pairwise disjoint. Let D be the domain obtained by removing $\bigcup_{j=0}^{k-1}E_j$ from the extended complex plane, and let $t=0$. Then it is easily shown, by the symmetry of D and the uniqueness of $G(z,0)$ and f_∞, that both the critical points of $G(z,0)$ and the zeros of f_∞ except 0 are placed at ∞ with multiplicity $k-1$. Therefore the condition (e) in the previous remark is satisfied, and hence $\alpha_1=\alpha_\infty$.

§ 6. Simply-connected region.

THEOREM 9. *Suppose that R is a simply-connected hyperbolic Riemann surface, then*

(i) *α_p is constant on $(0,\infty]$;*

(ii) *f_p is unique and the same for $0<p\leqq\infty$.*

Proof. Since the problem is conformally invariant, we may assume that $R=U$ and $t=0$. It is easily shown by Cauchy's integral formula that $\alpha_p=1$ and $f_p(z)=z$ for $1\leqq p\leqq\infty$. Let $0<p<1$ and let g be any extremal function for H_p^0. We put

$$h(z)=z(g(z)/B(z))^{\frac{1}{2}p}. \tag{6}$$

where $B(z)$ is the Blaschke product formed by the zeros of g. By the canonical factorization theorem ([3], p. 24, [9], p. 67), we see $h\in H_2^0$. On the other hand

$$|h'(0)|=\lim_{z\to0}|h(z)/z|=|g'(0)/B_1(0)|^{\frac{1}{2}p}\geqq\alpha_p^{\frac{1}{2}p},$$

where $B_1(z)=B(z)/z$. Hence we have that $\alpha_p=1$, $B(z)=z$ and $h(z)=f_2(z)=z$. Thus, by (6), we obtain $g(z)=z$.

Remark. Theorem 9 is not true for the other extremal problem considered in Section 5. In fact, if $R=U$, $t=0$ and $f(z)=1/2(z+1)^2$, then we easily see that $\|f\|_1=1$ and $f'(t)=1$. Since $\beta_p=1$ on $[1,\infty]$ by Theorem 8 and 9, f is an extremal function for H_1^1, which distincts from f_∞. Then, by Lemma 1, $\|f\|_p<1$ for $0<p<1$, since $|f|\neq$const. on C. Thus $\beta_p>\beta_1=\beta_\infty$ and β_p is strictly decreasing on $(0,1)$.

THEOREM 10. *Let D be a regular region in the extended complex plane. If $\alpha_{p_0}=\alpha_1$ for some p_0 with $0<p_0<1$, then D is conformally equivalent to the unit disc U.*

Proof. By Theorem 6 we have $\alpha_1=\alpha_\infty$, and hence the condition (e) in the remark after Theorem 8 is satisfied. Let k be the connectivity of D and we assume $k \geqq 2$. Let $G=G(z, t)$ be Green's function for D, with pole at t, and put $P=G+iH$, where H is the harmonic conjugate of G. Let $t_1, \cdots, t_{k-1}$ be the critical points of G, that is, the zeros of $P'dz$. It is well known that $\bar{D}$ can be completed, by symmetrization, to a closed Riemann surface $\hat{D}$, which is called the double of $\bar{D}$. There is given an involutory, indirectly conformal mapping of $\hat{D}$ onto itself which leaves every point on ∂D fixed, and the image of $z \in \bar{D}$ is denoted by $\tilde{z}$. Let δ be the divisor defined by $\delta = t_1 \cdots t_{k-1} \tilde{t}_1 \cdots \tilde{t}_{k-1} t^{-1} \tilde{t}^{-1}$. If two or more of $t_1 \cdots t_{k-1}$ coincide, we must modify the representation. But nothing in our proof is affected by such a change. Let $\mathcal{L}$ be the complex vector space consisting of all functions meromorphic on $\hat{D}$ which are multiples of δ^{-1}, and $\mathcal{B}$ be that of all Abelian differentials on $\hat{D}$ which are multiples of δ. By Riemann-Roch theorem [12], we see

$$(7) \qquad \dim \mathcal{L} = \dim \mathcal{B} + (2(k-2)+1-(k-1)) = \dim \mathcal{B} + k - 2,$$

since the order of δ is $2(k-2)$ and the genus of $\hat{D}$ is $k-1$. As usual, we can extend P' to a function meromorphic on $\hat{D}$, which is again denoted by P'. For any $\omega \in \mathcal{B}$, $h = \omega/(P'dz) = \text{const.}$ on $\hat{D}$, since h has no poles on $\hat{D}$, and hence $\dim \mathcal{B} = 1$. Then, by (7), we have $\dim \mathcal{L} = k-1$. Therefore there exists a non-constant function $\psi \in \mathcal{L}$. If we put $g_1(z) = (\psi(z) + \overline{\psi(\tilde{z})})/2$ and $g_2(z) = (\psi(z) - \overline{\psi(\tilde{z})})/2i$ for $z \in \bar{D}$, then at least one of them, say g_1, is non-constant on $\bar{D}$. It is evident that g_1 is meromorphic on $\bar{D}$, real-valued on ∂D and multiple of δ_1^{-1}, where δ_1 is the divisor defined by $\delta_1 = t_1 \cdots t_{k-1} t^{-1}$. Let $\phi(z) = (g_1(z)+K)/K$, where K is such a large positive number that $\psi \geqq 0$ on ∂D, and let $f(z) = \phi(z) f_\infty(z)$. So we have

$$\|f\|_1 = \frac{i}{2\pi} \int_{\partial D} |\phi(z) f_\infty(z)| P'(z) dz$$
$$= \frac{i}{2\pi} \int_{\partial D} \phi(z) P'(z) dz = \phi(t) = 1,$$

since $|f_\infty| = 1$ and $\phi \geqq 0$ on ∂D. It is easily shown that $f'(t) = \alpha_1$ and $f(t) = 0$, that is, f is an extremal function for H_1^0. Since ϕ is non-constant on ∂D, we see $\|f\|_{p_0} < 1$ by Lemma 1, and hence $\alpha_{p_0} > \alpha_1$. This contradiction shows $k=1$, and hence D is conformally equivalent to U.

Remark. If D is a regular region in the extended complex plane, then the set of all extremal functions for H_1^0 can be imbedded in $\boldsymbol{R}^{k-1}$ as a convex compact subset with non-empty interior, where k is the connectivity of D ([9]).

REFERENCES

[1] AHLFORS, L. V., Bounded analytic functions. Duke Math., 14 (1947), 1–11.

[2] CLARKSON, J. A., Uniformaly convex spaces. Trans. Amer. Math. Soc., 40 (1936), 396–414.

[3] DUREN, P.L., Theory of H^p spaces. Academic press New York (1970).
[4] FISHER, S.D., Rational functions, H^∞ and H^p on infinitely connected domains. Illinois J. Math., 12 (1968), 513-523.
[5] FISHER, S.D., On Schwarz's lemma and inner functions. Trans. Amer. Math. Soc., 138 (1969), 229-240.
[6] GARABEDIAN, P.R., Schwarz's lemma and Szegö kernel function. Trans. Amer. Math. Soc., 67 (1949), 1-35.
[7] HEINS, M., Hardy classes on Riemann surfaces. Lecture Notes in Math., 98, Springer-Verlag, Berlin and New York (1969).
[8] HOFFMAN, K., Banach spaces of analytic functions. Prentice-Hall, New Jersey (1962).
[9] KOBAYASHI, S. and SUITA, N., On H_1 extremal value. (Japanese) Sūgaku, 26 (1974), 347-349.
[10] PARREAU, M., Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann. Annales de l'Inst. Fourier, 3 (1951), 103-197.
[11] RUDIN, W., Analytic functions of class H_p. Trans. Amer. Math. Soc., 78 (1955), 46-66.
[12] WEYL, H., Die Idee der Riemannschen Fläche. Teubner, Stuttgart (1955).

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.