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A NOTE ON THE TRANSFINITE DIAMETER

To Professor Yûsaku Komatu on the occasion of his 60th birthday

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In this note, we shall try to extend the notion of the transfinite diameter taken with respect to a symmetric kernel to the case when a kernel is not always symmetric. In a locally compact Hausdorff space, let $K(P, Q)$ be a continuous function in P and Q, $+\infty$ for $P=Q$, finite for $P\neq Q$ and symmetric: $K(P, Q)=K(Q, P)$. For any compact set F of Ω containing an infinite number of points, put

$$W_n(F)=\inf \frac{\sum_{i<j} K(P_i, P_j)}{\binom{n}{2}},$$

where inf is taken with respect to pairs of n distinct points $P_1, P_2, \cdots, P_n$ of F. Owing to the hypothesis for the function K to be continuous, the inf is attained by a pair of n distinct points $Q_1, Q_2, \cdots, Q_n$ of F, and so the inf can be replaced by min. K being symmetric, we have

$$W_n(F)=\min \frac{\sum_{i\neq j} K(P_i, P_j)}{n(n-1)}.$$

An important fact is that $W_n(F)$ increases with n. Any compact set F is said to be of K-transfinite diameter zero if

$$W(F)=\lim_{n\to+\infty} W_n(F)=+\infty,$$

and said to be of K-transfinite diameter positive if $W(F)<+\infty$.

The notion of the transfinite diameter was obtained for the first time by Fekete for sets on the plane (to see [1]), and next by Pólya-Szegö for sets in the ordinary space (to see [6]). According to Pólya-Szegö, for any compact set F of the m-dimensional Euclidean space R^m $(m\geqq 3)$ containing an infinite number of points and n variable points $P_1, P_2, \cdots, P_n$ of F, put

$$\{D_n(F)\}^{\alpha-m}=W_n(F)=\min \frac{\sum_{i<j} \overline{P_iP_j}^{\alpha-m}}{\binom{n}{2}} \qquad (0<\alpha<m).$$

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Then, $D_n(F)$ decreases when n increases. When we put

$$\lim_{n\to+\infty} D_n(F)=D(F) \quad \text{and} \quad \lim_{n\to+\infty} W_n(F)=W(F)\,,$$

we have naturally

$$0\leqq D(F)<+\infty \quad \text{and} \quad 0<W(F)\leqq+\infty\,.$$

$D(F)$ is called the α-ordered transfinite diameter of F. According to Pólya-Szegö and Frostman, $W(F)$ is equal to the minimum of the α-ordered energy integral of positive measures μ supported by F with total mass 1 (to see [6], p. 15 and [3], p. 46):

$$W(F)=\min_{\mu}\iint \overline{PQ}^{\alpha-m}d\mu(Q)d\mu(P)\,.$$

This fact is assured to be also valid in the case of $K(P,Q)$ and $W(F)$ as stated on the beginning.

The following theorem is well-known as the Evans' theorem (to see [2]): Given any compact set F in R^m ($m\geqq3$) of Newtonian transfinite diameter zero (equivalent to say "of Newtonian capacity zero"), there exists a positive measure μ supported by F with total mass 1 whose potential

$$U^{\mu}(P)=\int \overline{PQ}^{2-m}d\mu(Q)$$

is $+\infty$ at each point P of F. This theorem is assured to be also valid for the potential taken with respect to a kernel $K(P,Q)$ as stated on the beginning (to see [5]).

The hypothesis of continuity and symmetricity for a kernel K seems to play an essential role in the statements so far discussed on the transfinite diameter, the capacity and the Evans' theorem. In this note we are going to introduce a notion of the transfinite diameter taken with respect to a non-symmetric kernel and to extend the Evans' theorem.

In a locally compact Hausdorff space Ω, let $K(P,Q)$ be a lower semi-continuous function in P and Q, may be $+\infty$ for $P=Q$, always finite for $P\neq Q$ and bounded from above for P and Q belonging to disjoint compact sets respectively. It is not assumed for K to be symmetric. For any compact set F of Ω and a non-negative number t, consider the quantity*)

$$W_n(F)=\inf\frac{\sum_{i<j}K(P_i,P_j)+t\sum_{i>j}K(P_i,P_j)}{n(n-1)}\,,$$

where inf is taken with respect to pairs of n variable points $P_1,P_2,\cdots,P_n$ of F, admitted to be overlapping. $W_n(F)$ will be finite or positively infinite. Owing

*) Nakai has studied the transfinite diameter in his paper (to see [4], p. 222). His idea is the case of $t=0$ in the quantity. The Evans' theorem is obtained there, but no relation between the transfinite diameter and the capacity is discussed.

to the hypothesis for the function K to be lower semi-continuous, the inf is attained by a pair of n points $Q_1, Q_2, \cdots, Q_n$ of F which may be overlapping, and so the inf can be replaced by min. $W_n(F)$ coincided with one as stated on the beginning if K is symmetric and $t=1$. Then, there holds

THEOREM 1. *For any compact set F of Ω and a non-negative number t, $W_n(F)$ increases with n.*

Proof. First of all, we should like to show that the value of $W_n(F)$ is really attained by a pair $(Q_1, Q_2, \cdots, Q_n)$ of n points of F. Let $(P_{1k}, P_{2k}, \cdots, P_{nk})$ be a sequence of pairs of n points of F, admitted to be overlapping, such that

$$\frac{\sum_{i<j} K(P_{ik}, P_{jk}) + t\sum_{i>j} K(P_{ik}, P_{jk})}{n(n-1)} \downarrow W_n(F)$$

when $k\to+\infty$. Let $Q_1, Q_2, \cdots$, and Q_n be the limiting points of $P_{1k}, P_{2k}, \cdots$ and P_{nk} respectively, if necessary, by extracting their suitable subsequences. The function K being lower semi-continuous and t being non-negative, we have

$$\begin{aligned} W_n(F) &\leqq \frac{\sum_{i<j} K(Q_i, Q_j) + t\sum_{i>j} K(Q_i, Q_j)}{n(n-1)} \\ &\leqq \varliminf_{k\to+\infty} \frac{\sum_{i<j} K(P_{ik}, P_{jk}) + t\sum_{i>j} K(P_{ik}, P_{jk})}{n(n-1)} \\ &= W_n(F), \end{aligned}$$

and so

$$W_n(F) = \frac{\sum_{i<j} K(Q_i, Q_j) + t\sum_{i>j} K(Q_i, Q_j)}{n(n-1)},$$

whose right hand side will be denoted by

$$\frac{v_n}{n(n-1)}.$$

Although the pair $(Q_1, Q_2, \cdots, Q_n)$ outght to be written strictly by $(Q_1^n, Q_2^n, \cdots, Q_n^n)$, we shall go ahead without doing so, since n is fixed. Some of $Q_1, Q_2, \cdots$ and Q_n may be overlapping. Then, we have n following inequalities:

$$\begin{aligned} v_n &= \sum_{j=2}^{n} K(Q_1, Q_j) + t\sum_{i=2}^{n} K(Q_i, Q_1) + \sum_{\substack{i<j \\ i,j\neq 1}} K(Q_i, Q_j) + t\sum_{\substack{i>j \\ i,j\neq 1}} K(Q_i, Q_j) \\ &\geqq \sum_{j=2}^{n} K(Q_1, Q_j) + t\sum_{i=2}^{n} K(Q_i, Q_1) + v_{n-1}, \end{aligned}$$

$$\begin{aligned} v_n &= \sum_{j=3}^{n} K(Q_2, Q_j) + t\sum_{i=3}^{n} K(Q_i, Q_2) + K(Q_1, Q_2) + tK(Q_2, Q_1) \\ &\quad + \sum_{\substack{i<j \\ i,j\neq 2}} K(Q_i, Q_j) + t\sum_{\substack{i>j \\ i,j\neq 2}} K(Q_i, Q_j) \end{aligned}$$

$$\geqq \sum_{j=3}^{n} K(Q_2, Q_j) + t\sum_{i=3}^{n} K(Q_i, Q_2) + K(Q_1, Q_2) + tK(Q_2, Q_1) + v_{n-1},$$

$$v_n \geqq \sum_{j=4}^{n} K(Q_3, Q_j) + t\sum_{i=4}^{n} K(Q_i, Q_3)$$

$$+K(Q_1, Q_3) + K(Q_2, Q_3) + tK(Q_3, Q_1) + tK(Q_3, Q_2) + v_{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$v_n \geqq K(Q_{n-1}, Q_n) + tK(Q_n, Q_{n-1}) + \sum_{i=1}^{n-2} K(Q_i, Q_{n-1}) + t\sum_{j=1}^{n-2} K(Q_{n-1}, Q_j) + v_{n-1}$$

and finally

$$v_n \geqq \sum_{i=1}^{n-1} K(Q_i, Q_n) + t\sum_{j=1}^{n-1} K(Q_n, Q_j) + v_{n-1}.$$

Summing up these inequalities, we have

$$nv_n \geqq v_n + v_n + nv_{n-1}.$$

Thus, we have

$$W_{n-1}(F) \leqq W_n(F).$$

DEFINITION. Given any compact set F of Ω and a non-negative number t, F is said to be of (K, t)-transfinite diameter zero if

$$W(F) = \lim_{n\to+\infty} W_n(F) = +\infty,$$

and said to be of (K, t)-transfinite diameter positive if $W(F) < +\infty$.

Consider the potential and the energy integral of positive measures μ taken with respect to a kernel K and its adjoint kernel $\check{K}$:

$$K(P, \mu) = \int K(P, Q)d\mu(Q),$$

$$K(\mu, P) = \int \check{K}(P, Q)d\mu(Q) = \int K(Q, P)d\mu(Q)$$

and

$$K(\mu, \mu) = \iint K(P, Q)d\mu(Q)d\mu(P) = \iint \check{K}(P, Q)d\mu(Q)d\mu(P).$$

Then, there holds

THEOREM 2. *Let t be a non-negative number. Given any compact set F of Ω of (K, t)-transfinite diameter zero, there exists a positive measure μ supported by F with total mass* 1 *such that*

$$K(P, \mu) + tK(\mu, P) = +\infty$$

and

$$K(\mu, P) + tK(P, \mu) = +\infty$$

at each point P of F.

Proof. Once for all, denote by $(Q_1^n, Q_2^n, \cdots, Q_n^n)$ a pair of n points of F where the value of $W_n(F)$ is attained. They are admitted to be overlapping. Then, we have for any point P of F

$$\begin{aligned} W_n(F) &\leqq W_{n+1}(F) \\ &\leqq \frac{\sum_{i<j} K(Q_i^{n+1}, Q_j^{n+1}) + t\sum_{i>j} K(Q_i^{n+1}, Q_j^{n+1})}{(n+1)n} \\ &\leqq \frac{\sum_{j=1}^{n} K(P, Q_j^n) + t\sum_{i=1}^{n} K(Q_i^n, P) + \sum_{i<j} K(Q_i^n, Q_j^n) + t\sum_{i>j} K(Q_i^n, Q_j^n)}{(n+1)n} \\ &\leqq \frac{\sum_{j=1}^{n} K(P, Q_j^n) + t\sum_{i=1}^{n} K(Q_i^n, P) + n(n-1)W_n(F)}{(n+1)n}. \end{aligned}$$

Accordingly, there holds

$$2W_n(F) \leqq \frac{1}{n}\left\{\sum_{i=1}^{n} K(P, Q_i^n) + t\sum_{i=1}^{n} K(Q_i^n, P)\right\},$$

and similarly holds

$$2W_n(F) \leqq \frac{1}{n}\left\{\sum_{i=1}^{n} K(Q_i^n, P) + t\sum_{i=1}^{n} K(P, Q_i^n)\right\}.$$

Let k be any positive number and $W_n(F) > 2^k$ for some large number n. Let μ_k be the measure with the mass $n^{-1}\cdot 2^{-k}$ at each point Q_i^n. It is a positive measure supported by F with total mass 2^{-k}. Evidently, we have

$$2 \leqq K(P, \mu_k) + tK(\mu_k, P)$$

and

$$2 \leqq K(\mu_k, P) + tK(P, \mu_k)$$

at each point P of F. Then, for the measure

$$\mu = \sum_{k=1}^{+\infty} \mu_k,$$

there hold

$$K(P, \mu) + tK(\mu, P) = +\infty$$

and

$$K(\mu, P) + tK(P, \mu) = +\infty$$

at each point P of F.

THEOREM 3. *Let F be any compact set of Ω and t a positive number. F is of (K, t)-transfinite diameter positive if and only if there exist positive measures of finite K-energy supported by F.*

Proof. Let μ be a positive measure of finite K-energy supported by F with total mass 1. For n arbitrary points $P_1, P_2, \cdots, P_n$ of F, we have

$$W_n(F) \leqq \frac{\sum_{i<j} K(P_i, P_j) + t \sum_{i>j} K(P_i, P_j)}{n(n-1)},$$

hence, regarding the right hand side as a function in n variable points $P_1, P_2, \cdots, P_n$ and intergrating the inequality by the positive measure

$$d\mu(P_1) d\mu(P_2) \cdots d\mu(P_n)$$

whose total mass is one, we have

$$W_n(F) = \frac{\binom{n}{2} K(\mu, \mu) + t \binom{n}{2} K(\mu, \mu)}{n(n-1)} = \frac{(1+t) K(\mu, \mu)}{2} < +\infty$$

and so

$$2W(F) \leqq (1+t) K(\mu, \mu) < +\infty .$$

In the next, suppose that $W(F) < +\infty$. We are going to find a positive measure μ of finite K-energy supported by F with total mass 1. Take n points $Q_1, Q_2, \cdots, Q_n$ of F such that

$$W_n(F) = \frac{\sum_{i<j} K(Q_i, Q_j) + t \sum_{i>j} K(Q_i, Q_j)}{n(n-1)} .$$

Let μ_n be the measure with $1/n$ at each Q_i $(i=1, 2, \cdots, n)$. It is a positive measure supported by F with total mass 1. Suppose that $\{\mu_n\}$ converges vaguely to a measure μ, if necessary, by extracting its suitable subsequence. Then, μ is a positive measure supported by F with total mass 1. Let $\{f_k(P, Q)\}$, $k=1, 2, \cdots$, be a sequence of finite and continuous functions that increases monotoneously to $K(P, Q)$ and C a positive number such that $f_1(P, Q) + C > 0$ for all the points P and Q of F. Putting

$$K'(P, Q) = K(P, Q) + C \quad \text{and} \quad f'_k(P, Q) = f_k(P, Q) + C ,$$

we have

$$\begin{aligned}
W_n(F) + \frac{1+t}{2} C &= \frac{\sum_{i<j} K'(Q_i, Q_j) + t \sum_{i>j} K'(Q_i, Q_j)}{n(n-1)} \\
&\geqq \min(1, t) \frac{\sum_{i \neq j} K'(Q_i, Q_j)}{n(n-1)} \\
&\geqq \min(1, t) \frac{\sum_{i \neq j} f'_k(Q_i, Q_j)}{n(n-1)} \\
&\geqq \min(1, t) \frac{\sum_{i,j} f'_k(Q_i, Q_j) - n \cdot \max_F f'_k(P, P)}{n(n-1)} \\
&= \min(1, t) \frac{n}{n-1} \left\{ \sum_{i,j} f'_k(Q_i, Q_j) \frac{1}{n^2} - \frac{1}{n} \max f'_k(P, P) \right\} \\
&= \min(1, t) \frac{n}{n-1} \left\{ \iint f'_k(P, Q) d\mu_n(Q) d\mu_n(P) - \frac{1}{n} \max_F f'_k(P, P) \right\}.
\end{aligned}$$

Fixing k and making $n\to+\infty$, we have

$$W_n(F)+\frac{1+t}{2}C\geqq\min(1,t)\iint f'_k(P,Q)d\mu(Q)d\mu(P),$$

hence, making $k\to+\infty$, we have

$$W(F)+\frac{1+t}{2}C\geqq\min(1,t)\iint K'(P,Q)d\mu(Q)d\mu(P),$$

that is,

$$K(\mu,\mu)\leqq\frac{2W(F)+(1+t)C}{2\min(1,t)}-C<+\infty.$$

DEFINITION. Any compact set F of Ω is said to be of K-capacity positive if there exist positive measures μ supported by F whose potential $K(P,\mu)$ is bounded from above on all the compact sets of Ω. Owing to the hypothesis for a kernel K, a compact set F is of K-capacity zero if and only if

$$\sup_F K(P,\mu)=+\infty$$

for every positive measure μ supported by F.

DEFINITION. A kernel K is said to satisfy the continuity principle if there holds the following property: Given any positive measure μ with compact support F and its potential $K(P,\mu)$, every continuous point of $K(P,\mu)$ as a function on F is also a continuous point of $K(P,\mu)$ as a function in Ω.

The following is an important fact in that it expresses a relation between the transfinite diameter and the capacity.

PROPOSITION. *Any compact set F of K-capacity positive always supports positive measures of finite K-energy. The converse is also correct if a kernel K satisfies the continuity principle.*

In fact, given any compact set F of K-capacity positive, take a positive measure μ supported by F whose potential $K(P,\mu)$ is bounded from above on all the compact sets of Ω. It is evident that μ is of finite K-energy. Conversely, suppose that a kernel K satisfies the continuity principle and that, given any compact set F of Ω, a positive measure μ supported by F is of finite K-energy. Then, the set

$$\{P;\ K(P,\mu)=+\infty\}$$

being of μ-mass zero, there exists a restricted measure μ' ($\neq 0$) of μ such that $K(P,\mu)$ is finite and continuous as a function on the support of μ'. By the lower semi-continuity of the potential and the continuity principle, it is easily seen that $K(P,\mu')$ is finite and continuous at each point of the support of μ'. Hence, by the hypothesis for a kernel K, it is also seen that $K(P,\mu')$ is bounded from above on all the compact sets of Ω.

When a kernel K satisfies the continuity principle, any compact set F of K-capacity zero supports no positive measures of finite K-energy. Therefore, such a compact set F is always of (K, t)-transfinite diameter zero for any positive number t. Thus, we can extend the Evans' theorem in the following form.

THEOREM 4. *Suppose that a kernel K satisfies the continuity principle. Let F be any compact set of K-capacity zero of Ω. Given any positive number t, there exists a positive measure μ supported by F (naturally depending upon t) such that*

$$K(P, \mu)+tK(\mu, P)=+\infty$$

and

$$K(\mu, P)+tK(P, \mu)=+\infty$$

at each point P of F.

QUESTION. Is the above theorem also valid for $t=0$?

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